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Resource-Allocation Mechanism: Game-Theory Analysis

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Abstract: Recently, owing to the trend of cross-disciplinary research in various fields, it is imperative to optimize the resource-allocation performance by analyzing allocation behavior and strategies from different perspectives as well as based on ideas from various fields. There are many decisive roles, such as changes in allocation behavior, related allocation methods, and the interaction and work effectiveness of strategies, to be implemented. By using game-theory analysis under resource-allocation procedures, in this article, we analyzed, constructed, simulated, and derived an efficient resource-allocation mechanism. Additionally, we generalized a power index by evaluating the operators and their activity levels. This article adopted the axioms of level completeness, criterion for weighted circumstances property, level synchronization, pure excess equal symmetry, and specific consonance to offer characterizations to assess the related rationality and accuracy of the proposed power index. Based on the above discussion, different from the rule of thumb, expert meeting, or other existing concepts, in this research game theory is used to provide an alternative guide for resource-allocation procedures by the optimal or equilibrium state established.

Keywords: operator; active level; power index; resource-allocation processes

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1. Introduction

Changes and relevant adjustments related to allocation behavior and allocating strategies have a significant impact on resource management as a whole. For a long time, it has been the goal of researchers to understand how to properly improve the allocations and operations of resource management. The types of resources owned by an operational organization or a system are diverse, and they take different forms, such as human resources, energy, capital, equipment, brand image, public relations, etc.

Appropriate resource allocation and utilization are essential to the sustainable operations of the organization or system. However, the maximized utilization of resources does not necessarily generate the greatest efficiency. As resources take various forms, it is important to effectively coordinate the operators of the operational organization or system to adopt corresponding strategies for all resources, so that limited resources can generate the most balanced productivity.

In the same spirit, it is very helpful to discuss resource allocation from various perspectives of different fields, and suggest concepts for allocation that provide a frame of reference that differs from what has come before it. An efficient resource allocation mechanism enables the proper use of limited resources to the most needed links and to achieve the most efficient objectives.

Game theory has been applied to construct and analyze all procedures with interactive behavior in various mathematical fields and to, then, produce optimal or equilibrium results with correctness, rationality, feasibility, and acceptability at the same time. For example, the non-emptiness of the core concepts of game theory have been always analyzed by applying duality results in the field of operational research. Dynamic procedures for solution concepts have been always analyzed by applying the fixed point results in the field of optimization.

The stability for solution concepts has been analyzed by the geometric results in the field of convex analysis. Of course, this also includes the establishment and analysis for how to manage the allocation mechanism in the interactive procedure, such as the distribution of resource management and the proportion allocation of strategy implementation. Therefore, game-theory solutions have been widely used in many fields, such as medical engineering, environmental sustainability, management science, economic analysis, and strategy formulation.

For example, Shapley [1] introduced the Shapley value to investigate the utility-allocating behavior by summarizing the total participation expected value for each operator. Hart and Mas-Colell [2] and Maschler and Owen [3] adopted the self-reduction and the properties of efficiency, symmetry, covariance and standardness for games to present the rationality of the Shapley value. Ransmeier [4] introduced the equal allocation non-separable costs (EANSC) to distribute the optimum yield for dams operated by the Tennessee Valley Authority. Under the allocating notion of the EANSC, agents first receive its marginal contributions and further allocate the remaining utility equally. Moulin [5] adopted the complement reduction and the properties of equal treatment for equal, efficiency, and zero-independence to show that the EANSC is a stable rule as a sharing resource. The core is the collection of related payoffs satisfying efficient and coalitional rationality under transferable-utility systems. Peleg [6] applied the max-reduction of Davis and Maschler [7] to characterize the allocation notion of the core. Hwang and Shih [8] analyzed the equilibrium of an exchange economy with the same number of agents and commodities.

Different from the EANSC, Hsieh and Liao [9] proposed the pseudo equal allocation non-separable costs (PEANSC) by simultaneously applying the notions of EANSC and the individual marginality. Under the allocating notion of the PEANSC, agents first receive its individual marginal contributions, and further allocate the remaining utility equally. Inspired by Moulin [5], they also applied a specific reduced circumstance to present that the PEANSC is a stable power index matching the properties of efficiency and equal treatment for equal and zero-independence.

Under *traditional transferable usability* (TU) circumstances, a resource mapping might always be defined by attending to whole subsets among all operators. This signifies that the choices available for each operator are either to operate fully under an operational procedure or not to operate at all. In real life, however, control, distribution, regulation, and simulation always vary relatively to each other in response to the rapidly converting interactions among environments, individuals, and groups.

Each operator can participate at a certain number of active levels, and therefore their abilities might be different. Hence, a *multi-choice TU circumstance* is generalized to be a reasonable spread of a traditional TU circumstance in which each operator can take different active levels. Several power indexes have been proposed under multi-choice TU circumstances. For example, by determining the overall influences for a specific operator under multi-choice TU circumstances, Hwang and Liao [10] proposed an extended core concept by adopting duplicated relations among operators and active levels; Liao [11,12] proposed two extensions of the EANSC by considering the maximal contributions of all operators among its active levels and the duplicated relations among operators and its active levels; Nouweland et al. [13] considered an extended Shapley value [1] by applying replicated relations among operators and the active levels.

The effects or importance utilized by operators might differ depending upon various subjective and objective characteristics, such as the scale of a constituency exhibited by a member of the congress, contributions from a member of a university, and the bargaining power of the financial personnel. In addition, the lack of symmetry might result in various bargaining abilities for different operators when modeled. In line with the previous explanations, one would desire that arbitrary resources might be allocated among operators and their activity levels proportionally to their *weights*.

Weights inevitably arise in the topic of resource allocation. For example, one might be dealing with an issue of resource distribution among investment projects, and the weights could be assigned to the profitability of the alternative choices of all projects. In the topic of distributing travel costs among several visited areas, the weights might be the number of days spent at each place (cf. Shapley [1]). In addition, the *symmetry* property is always applied to evaluate the consideration or importance of relationships among operators of a circumstance by considering the resource allocating rules or power indexes. Thus, many solution concepts have been analyzed by means of symmetry properties, such as the EANSC and the Shapley value [1]. Therefore, it is reasonable that the notion of symmetry should be generalized under weighted consideration.

These statements mentioned above generalize one motivation:

- whether the existing outcomes due to the PEANSC could be generalized to analyze, construct, derive, and simulate the most efficient optimal or balanced allocation mechanism by simultaneously considering multi-choice behavior and weights.

The article is dedicated to resolving the above motivation. The outcomes of this article are as follows.

- Inspired by Hwang and Liao [14], we consider a generalized analogue of the PEANSC by evaluating the operators, its active levels and weights in Section 2, which we name as the *multi-choice weighted-individual index* (MWII).
- By extending the properties due to Hsieh and Liao [9], the axioms of *consonance*, *criterion for weighted circumstances property*, *level completeness*, *level synchronization*, and *pure excess equal symmetry* are proposed to assess the related rationality and accuracy of the MWII in Section 3.
- Combined with constructed game-theory analysis related to resource-allocation procedures, in this article, we further analyze, evaluate, demonstrate, and verify the accuracy, applicability, feasibility, plausibility, and validity of the allocation mechanism in Section 4. More applications, comparisons, and related explanations are presented throughout this article.

2. Preliminaries

Let UO be the universe of operators, for instance, the collection formed by the total operators throughout the Earth. $x \in UO$ is a operator of UO, for example, an operator of the Earth. For $x \in UO$ and $v_x \in \mathbb{N}$, $V_x = \{0, 1, \cdots, v_x\}$ is the active level collection of operator x and $V_x^+ = V_x \setminus \{0\}$, where 0 means no operation. Suppose that $O \subseteq UO$ is the largest collection of the total operators of an interactive procedure in UO, for instance, the total operators of a country. Let $V^O = \prod_{x \in O} V_x$ be the product collection of the active level collections of the total operators of O. For all $H \subseteq O$, we define $\kappa^H \in P^O$ as the vector with $\kappa_x^H = 1$ if $x \in H$ and $\kappa_x^H = 0$ if $x \in O \setminus H$. Let 0_O be the zero vector in $\mathbb{R}^O$.

A *multi-choice TU circumstance* is a triple (O, v, R), where $O \neq \emptyset$ is a finite collection of operators, $v = (v_x)_{x \in O}$ is the vector that displays the number of total active levels for every operator, and $R : V^O \to \mathbb{R}$ is a resource mapping with $R(0_O) = 0$, which appoints to each $\eta = (\eta_x)_{x \in O} \in P^O$ the resource that the operators can arise when every operator x operates at the active level η_x. Since $v \in \mathbb{R}^O$ is fixed throughout this article, one would write (O, R) rather than (O, v, R).

Let Θ be the class of all multi-choice TU circumstances. Given $(O, R) \in \Theta$ and $\eta \in P^O$, one would write $N(\eta) = \{x \in O | \, \eta_x \neq 0\}$, η_H to be the restriction of η at H for each $H \subseteq O$ and $\|\eta\| = \sum_{x \in O} \eta_x$.

Given $(O, R) \in \Theta$, let $P^O = \{(x, k_x) \mid x \in O, k_x \in V_x^+\}$. A *power index* on Θ is a mapping ρ appointing to every $(O, R) \in \Theta$ element,

$$\rho(O, R) = \Big(\rho_{x,k_x}(O, R)\Big)_{(x,k_x) \in P^O} \in \mathbb{R}^{P^O}.$$

Here, $\rho_{x,k_x}(O,R)$ is the source or the payoff of the operator x when it operates at the level of k_x in (O,R). For convenience, one would define that $\rho_{x,0}(O,R) = 0$ for every $x \in O$.

As mentioned earlier, the weights involuntarily rise in the topic of resource allocation. For example, one might be dealing with an issue of resource distribution among investment projects, and the weights could be assigned to the profitability of the alternative choices of all projects. Weights are also contained in agreements approved by the holders of townhouses and used to allocate the cost of building or maintaining common facilities.

Another application is data or patent pooling among trading corporations where the scale of the trading companies inspected, for example by its market shares, are fundamental weights. If $w : \cup_{x \in UO} V_x^+ \to \mathbb{R}^+$ is a positive map, then w is said to be a *weight map for levels*. Given $(O,R) \in \Theta$ and a weight map for the levels w and $\eta \in P^O$, we define that $\|\eta\|_w = \sum_{x \in O} \sum_{k_x=1}^{\eta_x} w(k_x)$.

Subsequently, a multi-choice analogue of the PEANSC is defined as follows.

Definition 1. *The multi-choice weighted-individual index (MWII) of multi-choice circumstances, $\overline{\Phi^w}$, is the power index on Θ that associates to each $(O,R) \in \Theta$, each weight map for the levels w, each operator $x \in O$, and each $k_x \in V_x^+$ for the source or the payoff*

$$\overline{\Phi^w}_{x,k_x}(O,R) = \Phi_{x,k_x}(O,R) + \frac{w(k_x)}{\|v\|_w} \cdot \Big[R(v) - \sum_{t \in O} \sum_{q=1}^{v_t} \Phi_{t,q}(O,R)\Big],$$

where $\Phi_{x,k_x}(O,R) = R\big(k_x, 0_{O\setminus\{x\}}\big) - R\big(k_x - 1, 0_{O\setminus\{x\}}\big)$ is the individual-level variation *of the operator x from its active level $k_x - 1$ to k_x. By applying the power index $\overline{\Phi^w}$, operators obtain individual-level variations under the corresponding levels and further allocate the rest of the resources proportionally by weight for the levels.*

To elaborate how the notion of multi-choice TU circumstance and the MWII could be used and to ensure its meaning more clearly and visibly, an example on resource-allocation procedures is offered as follows.

Example 1. *As mentioned in the Introduction, the types of resources owned by an information technology company are diverse, and they take different forms, such as human resources, capital, equipment, and brand image. On the other hand, there are many operational departments in an information technology company, for example, the board of directors is in charge of the business direction of the company; the risk management department evaluates operational risks; the market research department analyzes current market trends; the marketing department develops marketing programs for the target consumers; the accounting department controls and manages the capital operations for the company; the production department is responsible for product manufacturing and quality-related tasks; the human resources department recruits personnel and expands or downsizes the work force according to the company needs; and the design department proposes new product design concepts or updates and revises designs as needed.*

Thus, we assume that O is the set of all operational departments. The operating strategy of each department is not set in stone, and there are different operational strategies in response to different situations. That is, each department $x \in O$ will have different operational strategies v_x; moreover, the operational strategies between departments also affect one another as a result of different situations.

For example, the market research department proposes popular trends and the needs of information products; then the design department proposes product design concepts and a products mix; the marketing department develops relevant sales strategies and market outlooks; the human resources department, the production department, and accounting department carry out rightsizing, output capacity evaluation, and capital allocation; and finally, the plan is submitted to the board of directors to decide on the business direction and make necessary adjustments.

In other words, each department will interact within the context of the situation and evaluate and suggest different implementation strategies; as a result, there will be different combination of strategies and corresponding advantages. That is, each department will intersect with other departments for different situations, and adopt different operational strategies $\eta_x \in V_x$ for different situations. Thus, a function C can be used to evaluate the benefits of the operational strategy $\eta \in P^O$ taken by all operational departments (i.e., $R(\eta)$).

Therefore, the resource-allocation procedure of an information technology company can be regarded as a multi-choice circumstance (O, R). However, the importance of different departments is certainly different from the nature of its operations. For example, in fund-raising activities, the marketing department is certainly more important than other departments, and the board of directors is more important than other departments in policy-enforcement meetings. It is reasonable that one could use a function w to weight each operational strategy of each department against different scenarios.

Thus, it is important to effectively coordinate departments to adopt corresponding strategies for all resources so that limited resources can generate the most balanced productivity. By applying the characterizations of the MWII, all the resources and utilities are collected and evaluated under a mode where operators exert multiple actions. We expect that an assessment model for the resource-allocation environment will eventually be developed by combining real-world situations with the game-theory results of the MWII.

In the following, a numerical example is provided.

Example 2. *Let $(O, R) \in \Theta$ be a resource-allocation procedure with operator collection $O = \{x, q, k\}$, active level vector $v = (2, 1, 1)$, and weights $w(2_x) = 1$, $w(1_x) = 3$, $w(1_q) = 2$, $w(1_k) = 4$. Let $R(2, 1, 1) = 5$, $R(2, 1, 0) = 6$, $R(2, 0, 1) = 5$, $R(2, 0, 0) = 9$, $R(1, 1, 1) = 4$, $R(1, 1, 0) = 2$, $R(1, 0, 1) = -7$, $R(1, 0, 0) = -1$, $R(0, 1, 1) = 3$, $R(0, 1, 0) = 2$, $R(0, 0, 1) = -3$, and $R(0, 0, 0) = 0$ be the resource where the operators can arise under entire operational situations. By Definition 1,*

$$\begin{array}{rclrcl} \Phi_{x,2}(O,R) &=& 10, & \Phi_{x,1}(O,R) &=& -1, \\ \Phi_{q,1}(O,R) &=& 2, & \Phi_{k,1}(O,R) &=& -3, \\ \overline{\Phi^{w}_{x,2}}(O,R) &=& 9.7, & \overline{\Phi^{w}_{x,1}}(O,R) &=& -1.9, \\ \overline{\Phi^{w}_{q,1}}(O,R) &=& 1.4, & \overline{\Phi^{w}_{k,1}}(O,R) &=& -4.2. \end{array}$$

It is clear to compute the source or the payoff of each operator when it operates at a specific level in (O, R). For instance, the source or the payoff of operator x is $\Phi_{x,2}(O, R) = 9.7$ when x operates at the level 2_x in (O, R).

3. Game-Theory Characterizations

To evaluate the rationality of the MWII, this section demonstrates that there exists a specific reduction that could be used to characterize the MWII. Further axioms and related interpretations are needed. Let ρ be a power index on Θ.

- ρ matches the *level completeness (LCOM)* if $\sum_{x \in O} \sum_{q=1}^{v_x} \rho_{x,q}(O, R) = R(v)$ for every $(O, R) \in \Theta$.
- ρ matches the *criterion for the weighted circumstances property (CWCP)* if $\rho(O, R) = \overline{\Phi^w}(O, R)$ for every $(O, R) \in \Theta$ with $|O| \leq 2$ and for every weight map for levels w.
- Given $(O, R) \in \Theta$ and $(x, k_x) \in P^O$, we define the *pure excess* for the level k_x to be $e^{\rho}_{k_x}(O, R) = \rho_{x,k_x}(O, R) - [R(k_x, 0_{O\setminus\{x\}}) - R(k_x - 1, 0_{O\setminus\{x\}})]$. ρ matches the *pure excess equal symmetry (PEES)* for all $(O, R) \in \Theta$ with $e^{\rho}_{k_x}(O, R) = e^{\rho}_{k_h}(O, R)$ and $R(k_x, 0, \eta_{O\setminus\{x,h\}}) - R(k_x - 1, 0, \eta_{O\setminus\{x,h\}}) = R(0, k_h, \eta_{O\setminus\{x,h\}}) - R(0, k_h - 1, \eta_{O\setminus\{x,h\}})$ for all $\eta \in P^O$, and for some $(x, k_x), (h, k_h) \in P^O$, it holds that $\rho_{x,k_x}(O, R) = \rho_{h,k_h}(O, R)$.
- ρ matches the *level synchronization (LSYN)* for all $(O, R), (O, Q) \in \Theta$ with $R(\eta) = Q(\eta) + \sum_{x \in N(\eta)} \mu_{x,\eta_x}$ for some $\mu \in \mathbb{R}^{P^O}$, and for all $\eta \in P^O$, it holds that $\rho(O, R) = \rho(O, Q) + \mu$.

LCOM asserts that all operators allocate whole resource entirely when all operators operate at all levels in a circumstance. CWCP is a self-sufficient situation if there exists only one operator in the circumstance; however, if there are two operators in the circumstance, each of them first takes back what they could have done alone, and, at the end of the circumstance, they allocate the rest of profits and losses. PEES asserts that the values of the two active levels should be coincident if the pure excess of these two levels are equal. LSYN could be treated as an extreme weakness of the *additivity*. In Section 4, the interaction between the above game-theory analysis and allocation procedures for resources is presented in detail. The following results show the MWII matches the LCOM, CWCP, PEES, and LSYN.

Lemma 1. *The MWII matches the LCOM and CWCP.*

Proof. For all $(O,R) \in \Theta$,

$$\begin{aligned}\sum_{x\in O}\sum_{q=1}^{v_x} \overline{\Phi^w}_{x,q}(O,R) &= \sum_{x\in O}\sum_{q=1}^{v_x} \Phi_{x,k_x}(O,R) + \frac{w(k_x)}{\|v\|_w}\cdot\Big[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t}\Phi_{t,q}(O,R)\Big]\\ &= \sum_{x\in O}\sum_{q=1}^{v_x} \Phi_{x,k_x}(O,R) + \frac{\|v\|_w}{\|v\|_w}\cdot\Big[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t}\Phi_{t,q}(O,R)\Big]\\ &= R(v).\end{aligned}$$

Thus, the MWII matches the LCOM. The MWII matches the CWCP by applying the definitions of CWCP and the MWII. □

Lemma 2. *The MWII matches the PEES.*

Proof. Let $(O,R) \in \Theta$, w be the weight map for the levels and $x,h \in O$. Assume that $e^{\overline{\Phi^w}}_{k_x}(O,R) = e^{\overline{\Phi^w}}_{k_h}(O,R)$ and $R(k_x,0,\eta_{O\setminus\{x,h\}}) - R(k_x-1,0,\eta_{O\setminus\{x,h\}}) = R(0,k_h,\eta_{O\setminus\{x,h\}}) - R(0,k_h-1,\eta_{O\setminus\{x,h\}})$ for all $\eta \in P^O$. Thus, $\Phi_{x,k_x}(O,R) = R(k_x,0_{O\setminus\{x\}}) - R(k_x-1,0_{O\setminus\{x\}}) = R(k_h,0_{O\setminus\{h\}}) - R(k_h-1,0_{O\setminus\{h\}}) = \Phi_{h,k_h}(O,R)$, and

$$\begin{aligned}\overline{\Phi^w}_{x,k_x}(O,R) &= \Phi_{x,k_x}(O,R) + \frac{w(k_x)}{\|v\|_w}\cdot[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t}\Phi_{t,q}(O,R)]\\ &= \Phi_{x,k_x}(O,R) + \frac{w(k_x)}{\|v\|_w}\cdot[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t}(R(k_x,0_{O\setminus\{x\}}) - R(k_x-1,0_{O\setminus\{x\}}))]\\ &= \Phi_{x,k_x}(O,R) + e^{\overline{\Phi^w}}_{k_x}(O,R)\\ &= \Phi_{h,k_h}(O,R) + e^{\overline{\Phi^w}}_{k_h}(O,R)\\ &= \Phi_{h,k_h}(O,R) + \frac{w(k_h)}{\|v\|_w}\cdot[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t}(R(k_x,0_{O\setminus\{x\}}) - R(k_x-1,0_{O\setminus\{x\}}))]\\ &= \Phi_{h,k_h}(O,R) + \frac{w(k_h)}{\|v\|_w}\cdot[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t}\Phi_{t,q}(O,R)]\\ &= \overline{\Phi^w}_{h,k_h}(O,R).\end{aligned}$$

Thus, the MWII matches the PEES. □

Lemma 3. *The MWII matches the LSYN.*

Proof. Let $(O,R),(O,Q) \in \Theta$ with $R(\eta) = Q(\eta) + \sum_{t \in N(\eta)} \mu_{t,\eta_t}$ for some $\mu \in \mathbb{R}^{P^O}$ and for all $\eta \in P^O$. Thus, for all $(x,k_x) \in P^O$,

$$\begin{aligned} & \overline{\Phi^w}_{x,k_x}(O,Q) \\ = \;& \Phi_{x,k_x}(O,Q) + \tfrac{w(k_x)}{\|v\|_w} \cdot [Q(v) - \sum_{s \in O} \sum_{q=1}^{v_s} \Phi_{s,q}(O,Q)] \\ = \;& (Q(k_x, 0_{O \setminus \{x\}}) - Q(k_x - 1, 0_{O \setminus \{x\}})) + \tfrac{w(k_x)}{\|v\|_w} \cdot [Q(v) - \sum_{s \in O} \sum_{q=1}^{v_s} (Q(q, 0_{O \setminus \{s\}}) - Q(q-1, 0_{O \setminus \{s\}}))] \\ = \;& (R(k_x, 0_{O \setminus \{x\}}) - R(k_x - 1, 0_{O \setminus \{x\}})) + \mu_{x,\eta_x} + \tfrac{w(k_x)}{\|v\|_w} \cdot [R(v) - \sum_{s \in O} \sum_{q=1}^{v_s} (R(q, 0_{O \setminus \{s\}}) - R(q-1, 0_{O \setminus \{s\}}))] \\ = \;& \Phi_{x,k_x}(O,R) + \mu_{x,\eta_x} + \tfrac{w(k_x)}{\|v\|_w} \cdot [R(v) - \sum_{s \in O} \sum_{q=1}^{v_s} \Phi_{s,q}(O,R)] \\ = \;& \overline{\Phi^w}_{x,k_x}(O,R) + \mu_{x,\eta_x}. \end{aligned}$$

Thus, the MWII matches the LSYN. □

A generalized analogue of the reduction defined by Hsieh and Liao [9] is provided as follows. Given $(O,R) \in \Theta$, $S \subseteq O$ and a power index ρ, the *reduced circumstance* (S, R^ρ_S) related to S and ρ is defined by for all $\eta \in V^S$,

$$R^\rho_S(\eta) = \begin{cases} 0 & \eta = 0_S, \\ R(\eta_x, 0_{O \setminus \{x\}}) & S \geq |2|,\ N(\eta) = \{x\} \text{ for some } x, \\ R\big(\eta, v_{O \setminus S}\big) - \sum_{x \in O \setminus S} \sum_{q=1}^{v_x} \rho_{x,q}(O,R) & \text{otherwise.} \end{cases}$$

The *consonance* (consistency) axiom could be interpreted informally as follows. Let ρ be a power index on Θ. For any two-operators coupled under a circumstance, one would introduce a "reduced circumstance" among them by evaluating the amounts remaining after the rest of the operators are rendered and the effects assigned from ρ. Then, ρ is said to be *consonant* if it yields the same effects as in the original circumstance when it is used to arbitrary reduce the circumstances. Thus, a power index ρ matches the *consonance (CSE)* if $\rho_{x,k_x}(O,R) = \rho_{x,k_x}(S, R^\rho_S)$ for all $(O,R) \in \Theta$ with $|O| \geq 3$, for all $S \subseteq O$ with $|S| = 2$, and for all $(x,k_x) \in P^S$. Some outcomes related to the CSE property are introduced as follows.

Lemma 4. *The MWII matches the CSE.*

Proof. Let $(O,R) \in \Theta$ with $|O| \geq 3$ and $S \subseteq O$ with $|S| = 2$. Assume that $S = \{x,h\}$. Thus, for all $(x,k_x) \in P^S$,

$$\overline{\Phi^w}_{x,k_x}(S, R^{\overline{\Phi^w}}_S) = \Phi_{x,k_x}(S, R^{\overline{\Phi^w}}_S) + \frac{w(k_x)}{\|v_S\|_w} \cdot \Big[R^{\overline{\Phi^w}}_S(v_S) - \sum_{t \in S} \sum_{q=1}^{v_t} \Phi_{t,q}(S, R^{\overline{\Phi^w}}_S) \Big]. \quad (1)$$

Further, for all $k_x \in V^+_x$,

$$\begin{aligned} \Phi_{x,k_x}(S, R^{\overline{\Phi^w}}_S) &= R^{\overline{\Phi^w}}_S(k_x, 0) - R^{\overline{\Phi^w}}_S(k_x - 1, 0) \\ &= R\big(k_x, 0_{O \setminus \{x\}}\big) - R\big(k_x - 1, 0_{O \setminus \{x\}}\big) \quad \text{(by definition of } R^{\overline{\Phi^w}}_S) \\ &= \Phi_{x,k_x}(O,R). \end{aligned} \quad (2)$$

Hence, by Equations (1) and (2) and the definitions of $R_S^{\overline{\Phi^w}}$ and $\overline{\Phi^w}$,

$$\begin{aligned}
\overline{\Phi^w}_{x,k_x}(S,R_S^{\overline{\Phi^w}}) &= \Phi_{x,k_x}(O,R) + \tfrac{w(k_x)}{\|v_S\|_w} \cdot \Big[R_S^{\overline{\Phi^w}}(v_S) - \sum_{t\in S}\sum_{q=1}^{v_t} \Phi_{t,q}(O,R)\Big] \\
&= \Phi_{x,k_x}(O,R) + \tfrac{w(k_x)}{\|v_S\|_w} \cdot \Big[R(v) - \sum_{t\in O\setminus S}\sum_{q=1}^{v_t} \overline{\Phi^w}_{t,q}(O,R) - \sum_{t\in S}\sum_{q=1}^{v_t} \Phi_{t,q}(O,R)\Big] \\
&= \Phi_{x,k_x}(O,R) + \tfrac{w(k_x)}{\|v_S\|_w} \cdot \Big[\sum_{t\in S}\sum_{q=1}^{v_t} \overline{\Phi^w}_{t,q}(O,R) - \sum_{t\in S}\sum_{q=1}^{v_t} \Phi_{t,q}(O,R)\Big] \\
&= \Phi_{x,k_x}(O,R) + \tfrac{w(k_x)}{\|v_S\|_w} \cdot \Big[\tfrac{\|v_S\|_w}{\|v\|_w} \cdot \big[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t} \Phi_{t,q}(O,R)\big]\Big] \\
&= \Phi_{x,k_x}(O,R) + \tfrac{w(k_x)}{\|v\|_w} \cdot \Big[R(v) - \sum_{t\in O}\sum_{q=1}^{v_t} \Phi_{t,q}(O,R)\Big] \\
&= \overline{\Phi^w}_{x,k_x}(O,R).
\end{aligned}$$

Similarly, $\overline{\Phi^w}_{h,k_h}(S,R_S^{\overline{\Phi^w}}) = \overline{\Phi^w}_{h,k_h}(O,R)$ for each $k_h \in V_h^+$. Thus, the MWII matches the CSE. □

Lemma 5. *A power index matches LCOM if it matches the CWCP and CSE.*

Proof. Let ρ be a power index on Θ matching the CWCP and CSE and $(O,R) \in \Theta$. It is trivial for $|O| \le 2$ by CWCP. Suppose that $|O| \ge 3$. Let $z \in O$ and consider the reduced circumstance $\big(\{z\}, R^{\rho}_{\{z\}}\big)$. Thus,

$$R^{\rho}_{\{z\}}(v_z) = R(v) - \sum_{x\in O\setminus\{z\}}\sum_{q=1}^{v_x} \rho_{x,q}(O,R).$$

Since ρ matches the CSE, $\rho_{z,k_z}(O,R) = \rho_{z,k_z}\big(\{z\}, R^{\rho}_{\{z\}}\big)$ for each $k_z \in V_z$. In particular, $\rho_{z,v_z}(O,R) = \rho_{z,v_z}\big(\{z\}, R^{\rho}_{\{z\}}\big)$. In addition, by the CWCP of ρ, $\sum_{q=1}^{v_z} \rho_{z,q}(O,R) = R^{\rho}_{\{z\}}(v_z)$. Hence, $\sum_{x\in O}\sum_{q=1}^{v_x} \rho_{x,q}(O,R) = R(v)$, i.e., ρ matches the LCOM. □

Inspired by Hart and Mas-Colell [2], the following outcome shows that the MWII could be characterized by applying criteria for the weighted circumstances property and consonance.

Theorem 1. *A power index ρ matches the CWCP and CSE if and only if $\rho = \overline{\Phi^w}$.*

Proof. Based on Lemmas 1 and 4, $\overline{\Phi^w}$ matches the CSE. Clearly, $\overline{\Phi^w}$ matches the CWCP.

To present uniqueness, assume that ρ matches the CWCP and CSE on Θ. Based on Lemma 5, ρ also matches the LCOM. Let $(O,R) \in \Theta$. By the CWCP of ρ, $\rho(O,R) = \overline{\Phi^w}(O,R)$ if $|O| \le 2$. The condition $|O| > 2$: Let $x \in O$ and $S = \{x,q\}$ for some $q \in O \setminus \{x\}$. Then,

$$\begin{aligned}
\Phi_{x,k_x}(S,R_S^{\rho}) &= R_S^{\rho}(k_x,0) - R_S^{\rho}(k_x-1,0) \\
&= R(k_x,0_{O\setminus\{x\}}) - R(k_x-1,0_{O\setminus\{x\}}) \quad \text{(by definition of } R_S^{\rho}) \\
&= R_S^{\rho}(k_x,0) - R_S^{\rho}(k_x-1,0) \quad \text{(by definition of } R_S^{\overline{\Phi^w}}) \\
&= \Phi_{x,k_x}(S,R_S^{\overline{\Phi^w}}).
\end{aligned} \tag{3}$$

For all $k_x \in V_x^+$,

$$\begin{array}{rll} & \rho_{x,k_x}(O,R) - \overline{\Phi^w}_{x,k_x}(O,R) & \\ = & \rho_{x,k_x}(S,R_S^\rho) - \overline{\Phi^w}_{x,k_x}(S,R_S^{\overline{\Phi^w}}) & \text{(by CSC of } \rho \text{ and } \overline{\Phi^w}) \\ = & \overline{\Phi^w}_{x,k_x}(S,R_S^\rho) - \overline{\Phi^w}_{x,k_x}(S,R_S^{\overline{\Phi^w}}) & \text{(by CWCP of } \rho \text{ and } \overline{\Phi^w}) \\ = & \frac{w(k_x)}{\|v_S\|_w}\Big[R_S^\rho(v_S) - R_S^{\overline{\Phi^w}}(v_S)\Big]. & \text{(by equation (3) and CWCP of } \rho \text{ and } \overline{\Phi^w}) \end{array} \tag{4}$$

Similarly, for all $k_q \in V_q^+$,

$$\rho_{q,k_q}(O,R) - \overline{\Phi^w}_{q,k_q}(O,R) = \frac{w(k_q)}{\|v_S\|_w}\Big[R_S^\rho(v_S) - R_S^{\overline{\Phi^w}}(v_S)\Big]. \tag{5}$$

By Equations (4) and (5),

$$\rho_{x,k_x}(O,R) - \overline{\Phi^w}_{x,k_x}(O,R) = \frac{w(k_q)}{w(k_x)}\Big[\rho_{j,k_q}(O,R) - \overline{\Phi^w}_{j,k_q}(O,R)\Big].$$

This implies that $\rho_{x,k_x}(O,R) - \overline{\Phi^w}_{x,k_x}(O,R) = \Delta$ for all (x,k_x). It remains to demonstrate that $\Delta = 0$. By LCOM of ρ and $\overline{\Phi^w}$,

$$0 = R(v) - R(v) = \sum_{x \in O}\sum_{k_x=1}^{v_x}\Big[\rho_{x,k_x}(O,R) - \overline{\Phi^w}_{x,k_x}(O,R)\Big] = \frac{\Delta \cdot \|v\|_w}{w(k_x)}\Big[\rho_{x,k_x}(O,R) - \overline{\Phi^w}_{x,k_x}(O,R)\Big].$$

That is, $\Delta = 0$. □

Inspired by Maschler and Owen [3] and Moulin [5], the following result shows that the MWII could be characterized using the level completeness, pure excess equal symmetry, level synchronization, and consonance.

Lemma 6. *A power index ρ matches the CWCP if it matches the LCOM, PEES, and LSYN.*

Proof. Suppose that a power index ρ matches the LCOM, PEES, and LSYN. Given $(O,R) \in \Theta$ with $O = \{x,h\}$ for some $x \neq h$. We define a circumstance (O,Q) as for all $\eta \in P^O$,

$$Q(\eta) = R(\eta) - \sum_{t \in N(\eta)}\sum_{k_t=1}^{\eta_t}[(R(k_t,0) - R(k_t-1,0)) - \rho_{t,k_t}(O,D)].$$

Clearly, $Q(k_x,0) - Q(k_x-1,0) = \rho_{x,k_x}(O,Q)$ and $Q(0,k_h) - Q(0,k_h-1) = \rho_{h,k_h}(O,Q)$. Further,

$$e_{k_x}^\rho(O,Q) = \rho_{x,k_x}(O,Q) - (Q(k_x,0) - Q(k_x-1,0)) = 0$$

and

$$e_{k_h}^\rho(O,Q) = \rho_{h,k_h}(O,Q) - (Q(0,k_h) - Q(0,k_h-1)) = 0.$$

By the PEES of ρ, $\rho_{x,k_x}(O,Q) = \rho_{h,k_h}(O,Q)$. By the LCOM of ρ,

$$Q(v) = \sum_{k_x=1}^{v_x}\rho_{x,k_x}(O,Q) + \sum_{k_h=1}^{v_h}\rho_{h,k_h}(O,Q) = \|v\| \cdot \rho_{x,k_x}(O,Q) \tag{6}$$

for each $k_x \in V_x^+$. Thus, by equation (6) and the definition of Q, $\rho_{x,k_x}(O,Q) = \frac{Q(v)}{\|v\|}$. By the LSYN of ρ, $\rho_{x,k_x}(O,R) = \overline{\Phi^w}_{x,k_x}(O,R)$. Hence, ρ contains the CWCP. □

Theorem 2. *A power index ρ matches the LCOM, PEES, LSYN, and CSE if and only if $\rho = \overline{\Phi^w}$.*

Proof. Based on Lemmas 1–4, $\overline{\Phi^w}$ matches the LCOM, PEES, LSYN, and CSE. The rest of the proofs can be completed by applying Theorem 1 and Lemma 6. □

4. Discussion and Concluding Remarks

By investigating the rapidly changing interactions, we applied game-theory analysis to evaluate the accuracy, feasibility, and plausibility of the allocation mechanism for resources by investigating the questions: "how is the mechanism formed," "why does one apply the mechanism," "is such a mechanism precise," and "how effective is such a mechanism?"

In Sections 2 and 3, we presented that the main advantages of the MWII and the related axiomatic outcomes are that the MWII of a multi-choice TU circumstance exactly exists and produces an accurate influence for a specific operator operating at a specific active level different from the general notion with multi-choice TU circumstances, which generates a kind of global influence for a specific operator by summarizing the related variations of the operator among all its active levels. We expect that the MWII could exactly provide a "balanced outcome" or "optimal outcome" under resource-allocating procedures.

To elaborate how the MWII could be performed and clarify its meaning, we further explored the interactions between game-theory axioms and resource-allocation procedures.

1. *Level completeness*: Balanced resource-allocation procedures exploit environmental resources. Thus, a useful resource-allocation procedure should match the level completeness.
2. *Criterion for weighted circumstances property*: Operators have the characteristics of actions. Interactions among operators are always derived from two-operator interactions followed by coalitional interactions. Accordingly, a useful resource-allocation procedure should match the criteria for the weighted circumstance properties.
3. *Pure excess equal symmetry*: If two arbitrary operators are equally pure in excess to the overall situation after the action of operator grouping, the related influence among these two operators on the overall situation should also be coincident. Thus, a useful resource-allocation procedure should match the pure excess equal symmetry.
4. *Level synchronization*: Balanced resource-allocation procedures wherein each operator is applied with the right proportion to accomplish the target action, rather than the amount (small or large) depending upon the target action, should accomplish the most balanced effect according to the proportionality standard. Therefore, a useful resource-allocation procedure should match the level synchronization.
5. *Consonance*: Balanced resource-allocation procedures are certified through a continuous iterative synergistic procedure and should yield consistent benefits. Thus, a useful resource-allocation procedure should match the consonance.

Employing related definitions, examples, and statements from Section 2, the framework of resource-allocation procedures could be developed as a multi-choice TU circumstance. By applying Theorems 1 and 2, the MWII is the only mechanism satisfying the level completeness, criterion for weighted circumstances, pure excess equal symmetry, level synchronization, and consonance. Based on the interpretations 1–5 in this section, the level completeness, criterion for weighted circumstances, pure excess equal symmetry, level synchronization, and consonance should be seen as essential characteristics under the topic of resource-allocation procedures. Thus, the MWII might be evaluated to be a useful allocation mechanism under the topic of resource-allocation procedures.

In sum, the purpose of this article is to introduce different analyses for resource-allocation procedures.

- In traditional TU circumstances, power indexes only considered either non-participation or the participation of all operators. As mentioned in the Introduction, however, it is reasonable that each operator might have different active levels to operate. Thus, different from the PEANSC, due to Hsieh and Liao [9], on traditional TU circumstances, the MWII is proposed to analyze the distribution mechanism under multi-choice behavior and weights simultaneously.
- In many real-world situations, one might focus on the related outputs or payoffs of operators under specific active levels. By attending to real-world conditions, the MWII of this article is proposed to evaluate the related effects on multi-choice TU circumstances by focusing on a specific operator operating at a specific active level. In

the context of multi-choice TU circumstances, Hwang and Liao [10], Liao [11,12], and Nouweland et al. [13] adopted several allocation concepts to determine several kinds of global influence for a specific operator by summarizing the related variations of the operator among all the active levels.

- Inspired by the core of traditional TU circumstances, Hwang and Liao [10] proposed a *replicated core* to determine a kind of global influence for a specific operator by evaluating the *replicated level sets* of operators. Differing from the solution concept due to Liao [12], in this article, we focused on the PEANSC, and considered the power index, the reduction, and several axioms by applying the operators and the active levels. The other major difference is the fact that we offer related axioms of pure excess equal symmetry and level synchronization to analyze the MWII introduced in this article. Related axioms of pure excess equal symmetry and level synchronization were not present in Hwang and Liao [10].
- Inspired by the EANSC in traditional TU circumstances, Liao [11] defined the *maximal EANSC* to determine a kind of global influence for a specific operator by evaluating the *maximal marginal values* of operators among the total active levels. Differing from the solution concept due to Liao [11], we focused on the PEANSC and considered the power index, the reduction, and several axioms by applying the operators and the active levels. The other major difference is the fact that we offer related axioms of pure excess equal symmetry, and level synchronization to analyze the MWII introduced in this article. Related axioms of pure excess equal symmetry and level synchronization were not present in Liao [11].
- Inspired by the EANSC in traditional TU circumstances, Liao [12] proposed the *duplicate EANSC* to determine a kind of global influence for a specific operator by evaluating the *duplicated relations* of operators among the total levels. Differing from the solution concept due to Liao [12], we focused on the PEANSC and considered the power index, the reduction, and several axioms by applying the operators and the active levels. The other major difference is the fact that we offer related axioms of pure excess equal symmetry and level synchronization to analyze the MWII introduced in this article. Related axioms of pure excess equal symmetry and level synchronization were not present in Liao [12].
- Inspired by the Shapley value in traditional TU circumstances, Nouweland et al. [13] proposed the *multi-choice Shapley value* to determine a kind of global influence for a specific operator by evaluating the *replicated games* due to the operators and levels. Differing from the solution concept due to Nouweland et al. [13], we focused on the PEANSC and considered the power index, the reduction, and several axioms by applying the operators and the active levels. The other major difference is the fact that we offer related axioms of pure excess equal symmetry and level synchronization to analyze the MWII introduced in this article. Related axioms of pure excess equal symmetry and level synchronization were not present in Nouweland et al. [13].

- To relate the rationality and accuracy of the MWII, we provide two characterizations. These characterizations are multi-choice analogues of the axiomatic results from Hsieh and Liao [9].
- Combined with the constructed game-theory analysis related to resource-allocation procedures, we further analyzed, evaluated, demonstrated, and verified the accuracy, applicability, feasibility, plausibility, and validity of the MWII under resource-allocation procedures by applying examples and related interpretations.

By attending to the operators and active levels, Hwang and Liao [14] proposed an extended Shapley value [1] and related axiomatic outcomes under fuzzy TU circumstances. Inspired by Hwang and Liao [14], the MWII is proposed in this article by attending to the operators, the active levels, and weights. One should also compare our works with the outcomes of Hwang and Liao [14]. There are several major differences:

- The MWII and related outcomes are introduced initially.
- The solution concept due to Hwang and Liao [14] is introduced by considering fuzzy behavior. However, the active levels of all operators might be not ambiguous. Therefore, the MWII of this article is proposed by considering multi-choice behavior. In addition, the notion of weights does not appear in Hwang and Liao [14].
- The solution concept due to Hwang and Liao [14] is an extension of the Shapley value. The MWII of this article is an extension of the PEANSC.

The results introduced in this article generated two motivations:

- whether further game-theory outcomes could be applied to analyze, construct, derive, evaluate, and simulate the most efficient optimal or balanced mechanisms under resource-allocation procedures; and
- whether further traditional solution concepts could be generated to analyze, construct, derive, evaluate, and simulate the most efficient optimal or balanced allocation mechanism by considering multi-choice behavior and the weights.

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