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Generalized Fuzzy Soft Power Bonferroni Mean Operators and Their Application in Decision Making

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Abstract: In decision-making process, decision-makers may make different decisions because of their different experiences and knowledge. The abnormal preference value given by the biased decision-maker (the value that is too large or too small in the original data) may affect the decision result. To make the decision fair and objective, this paper combines the advantages of the power average (PA) operator and the Bonferroni mean (BM) operator to define the generalized fuzzy soft power Bonferroni mean (GFSPBM) operator and the generalized fuzzy soft weighted power Bonferroni mean (GFSWPBM) operator. The new operator not only considers the overall balance between data and information but also considers the possible interrelationships between attributes. The excellent properties and special cases of these ensemble operators are studied. On this basis, the idea of the bidirectional projection method based on the GFSWPBM operator is introduced, and a multi-attribute decision-making method, with a correlation between attributes, is proposed. The decision method proposed in this paper is applied to a software selection problem and compared to the existing methods to verify the effectiveness and feasibility of the proposed method.

Keywords: generalized fuzzy soft sets; power average operator; Bonferroni mean operator; bidirectional projection; multi-attribute decision making

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1. Introduction

1.1. Research Background

Since the decision-making problem exists in every field of life, it has always been paid close attention by the majority of scholars. As the social environment becomes increasingly complex, more and more factors are involved in the decision-making problems, which leads to the uncertainty, hesitation and fuzziness of decision makers (DMs) when they give evaluation opinions. Therefore, fuzzy set theory, which is used to fit people's fuzzy opinions, has become one of the most commonly used tools to solve decision-making problems [1–3]. At the same time, as a more general form of fuzzy sets, soft sets have also been paid close attention by many scholars, and have been applied to uncertain decision-making problems in various fields [4–8]. The concepts of fuzzy soft sets (FSS) [9] and generalized fuzzy soft sets (GFSS) [10] are also proposed. In recent years, an increasing number of scholars have used generalized fuzzy soft sets to express people's fuzzy views in order to solve decision-making problems [5–8,11,12]. Considering the complexity of practical problems, DMs with different backgrounds, and different levels of professional knowledge and experience, are often required to participate in the decision-making process, and different DMs often give different decision-making opinions. Therefore, in order to obtain a comprehensive opinion that is accepted by everyone, it is necessary to aggregate different opinions or carry out corresponding operation. At present, scholars have studied various forms of operator (such as the power average (PA) [13] operator, the Bonferroni mean (BM) [14] operator and the power Bonferroni mean (PBM) [15] operator) to carry out corresponding operations on different opinions in order to obtain as accurate a decision scheme as possible. These operators effectively integrate the information of individual DMs into the overall information, and better consider the possible correlation between different attribute variables.The existing literature on GFSS integration methods is mostly proposed under the condition that the attributes are independent of each other. In fact, there may be different degrees of correlation between different attributes. The PA operator and BM operator can solve these problems. Among them, the PA operator can determine the attribute weights according to the support relationship between the attributes to reduce the influence of the biased decision-maker's abnormal preference value on the decision results, and the BM operator can fully consider the correlation between the attributes. However, there is no research on integrating GFSS using PBM operators. This paper proposes the generalized fuzzy soft power Bonferroni mean (GFSPBM) operator and the generalized fuzzy soft weighted power Bonferroni mean (GFSWPBM) operator by combining the advantages of the PA operator and BM operator. The new operator proposed in this paper can enrich the integration method of GFSS, expand the application field of PBM operator, and provide a new method for multi-attribute decision-making problems. The decision-making method proposed in this paper can be applied to fields such as supplier selection evaluation, product program selection evaluation, and recommendation of talent introduction in the human resources department.

1.2. Literature Review

In 1965, Zadeh [16] proposed the fuzzy set theory. This theory regards the object to be investigated and the fuzzy concept reflecting the object as a certain fuzzy set, establishes an appropriate membership function for this, and analyzes the fuzzy object through the related operations and transformations of the fuzzy set. In 1999, Molodtsov [4] introduced soft set theory, which is a mathematical tool for solving uncertain problems which can be widely used in economics, engineering, physics and other fields. Maji et al. further studied the theory of soft sets. They defined the operation of soft sets [17], applied the theory of soft sets to the solution of mathematical decision-making problems [18], and combined fuzzy sets with soft sets to introduce FSS [9]. In order to understand the influence of decision makers' cognition on the effectiveness of information provided by them, Majumdar [10] further proposed GFSS on the basis of fuzzy soft sets. In recent years, the theoretical research and application exploration of generalized fuzzy soft sets have attracted the attention of many scholars. Among them, Chen et al. [5] applied the Bonferroni mean operator to GFSS and proposed the generalized fuzzy soft set Bonferroni mean (GFSSBM) operator, which solved the problem of group decision-making under limited cognition by decision-makers. Dey and Pal [11] introduced the concept of generalized multi-fuzzy soft sets and applied it to decision-making problems. Agarwal et al. [12] extended the intuitionistic soft set (IFSS) to the intuitionistic fuzzy set (IFS) and defined the generalized intuitionistic fuzzy soft set (GIFSS), which can provide a given standard evaluation and the host's evaluation of the data. Xu et al. [6,7] combined the extreme learning machine and GFSS to establish an ensemble credit scoring model. Li et al. [8] combined GFSS with hesitant fuzzy sets and proposed a generalized hesitant fuzzy soft set (GHFSS). Due to the dynamic development of multi-criteria assessment methods, fuzzy criteria are being increasingly considered by scholars. Bazzocchi et al. [19] proposed a method to prioritize space debris by using multi-criteria decision-making (MCDM) methods and fuzzy logic. Dong et al. [20] propose a new fuzzy best–worst method (BWM) based on triangular fuzzy numbers for MCDM; this method is very useful in solving multi-attribute decision-making problems in a fuzzy environment. Thakur et al. [21] depict an MCDM issue and offer the means of the VIKOR approach inside the pythagorean fuzzy system. The information fusion operator of fuzzy numbers is a useful tool to integrate all input-independent variables into the composite total value. The existing literature on the integration methods and applications of GFSS are mostly proposed when the attributes are independent of each other. In many actual decision-making problems, the degree of correlation between different attributes may be different, such as complementarity, redundancy, and preference relationships. The BM operator, proposed by Bonferroni [14] in 1950, is a mean-type ensemble operator, which can effectively find the interrelationship between the variables we entered and aggregate multiple input variables into one variable, is a bounded ensemble operator. In recent years, the BM operator has been widely used in different multi-attribute decision-making problems. For example, Wei et al. [22] studied uncertain linguistic BM operators. The generalized BM operator was proposed by Yager [23]. Intuitionistic fuzzy BM operator is defined by Xu and Yager [24]. Liu and Zhang [8] defined four kinds of intuitionistic uncertain linguistic arithmetic Bonferroni mean (IULABM) operators. In addition, the PA [13] operator is also an integrated operator that can capture the correlation between existing data and cognitive information. It considers the support relationship between the input data to calculate the weight of the attribute, which can effectively reduce anomalies. The impact of data on decision-making results makes the processing of decision-making information more objective and fair and, therefore, has received widespread attention. For example, Liu et al. [25] proposed the generalized neutrosophic number weighted power average (GNNWPA) operator to solve the multi-attribute decision-making (MAGDM) problem. A PA integration operator is applied to an intuitionistic fuzzy number (IFN) environment by Xu [26]. To comprehensively utilize the advantages of the BM operator and the PA operator, He et al. [15] combined the PA operator with the BM operator and proposed the PBM operator. Now, PBM operators are used in various fuzzy environments, such as hesitant fuzzy sets [15], intuitionistic fuzzy sets [27,28], interval-value intuitionistic fuzzy sets [29], and linguistic intuitionistic fuzzy sets [30]. However, to the best of our knowledge, there is no research on how to use the PBM operator to integrate GFSS. Therefore, to enrich the GFSS integration method and expand the application field of the PBM operator, this paper will study the GFSS integration method based on the PBM operator and propose two new GFSS integration operators, namely, the GFSPBM Operator and GFSWPBM operator. In this paper, we discuss and study the excellent properties of operators carefully. On this basis, the idea of the bidirectional projection method is introduced, and a new GFSS multi-attribute decision-making method is given. The combination of different operators and different environments will form new environment operators. The environmental operator has the advantages of its component operator. We classify the environmental operator according to the difference in the component operator as follows

(1) PA operator class [25,26]: This can reduce the influence of biased decision-makers' abnormal preference values on decision-making results;
(2) BM operator class [22,24,31]: This can fully consider the interrelationship between attributes;
(3) PBM operator class [15,27–30]: This can effectively reduce the influence of abnormal data on decision-making results and fully consider the correlation between attributes.

In decision-making problems, when we need to eliminate the influence of abnormal data on the decision-making results, we can use operators of the PA operator class. When we need to consider the correlation between attributes, we can use operators of the BM operator class. When we need to eliminate the influence of abnormal data on decision results, but also consider the correlation between attributes, we can use operators of the PBM operator class. When we need to consider the uncertainty of fuzzy evaluation information, and we also want to reduce the influence of abnormal preference values on decision results, and consider the correlation between attributes. At this time, the GFSPBM operator proposed in this paper can be used. Our arrangement for the rest of this paper is as follows. To make our discussion easier, we first introduce the definitions of GFSS, PA operator, BM operator and PBM operator in Section 2. In Section 3, we proposed the GFSPBM operator and the GFSWPBM operator and carefully analyze and discuss the excellent properties of these operators. In Section 4, we introduce the idea of a bidirectional projection method and provide a multi-attribute decision-making method based on the GFSWPBM operator. In Section 5, we provide a practical application example of software selection, which shows the feasibility of our decision-making method. We compare and analyze different operators to verify the applicability of our proposed method, and analyze the sensitivity of the decision-making process. In the end, the conclusion is given in Section 6.

2. Preliminaries

Definition 1. (Ref. [10]) *Suppose $U = \{x_1, x_2, \ldots, x_m\}$ is the universal collection of elements, $E = \{e_1, e_2, \ldots, e_n\}$ is the universal set of parameters, $\widetilde{F}(U)$ is the set of all fuzzy soft sets over U. The pair $(\widetilde{F}, E)$ is called soft universe. Let $F : E \to I^U$. Suppose γ is the fuzzy subset of E, that is $\gamma : E \to I = [0,1]$, where I^U is the set of all fuzzy subsets of U. Suppose F_γ is the mapping $F_\gamma : E \to I^U \times I$. Define mapping as $F_\gamma(e) = (F(e), \gamma(e))$, where $F(e) \in I^U$. Then, we call F_γ a generalized fuzzy soft set over the soft universe $(\widetilde{F}, E)$.*

At this point, every parameter e_j, $F_\gamma(e_j) = (F(e_j), \gamma(e_j))$ can not only express the attribution degree of the elements of U in $F(e_j)$, but also the degree of possibility of this attribution degree, which is represented by $\gamma(e_j)$. So, $F_\gamma(e_j)$ can be expressed as

$$F_\gamma(e_j) = \left(\left\{ \frac{x_1}{F(e_j)(x_1)}, \frac{x_2}{F(e_j)(x_2)}, \ldots, \frac{x_m}{F(e_j)(x_m)} \right\}, \gamma(e_j) \right) \tag{1}$$

where $F(e_j)(x_1), F(e_j)(x_2), \ldots, F(e_j)(x_m)$ express the degrees of belongingness and $\gamma(e_j)$ indicate the degree of possibility of such belongingness.

Example 1. *We assume $U = \{x_1, x_2, x_3\}$ is a collection of three car under consideration. Let $E = \{e_1, e_2, e_3\}$ be a collection of qualities: $e_1 = good\ appearance$, $e_2 = cheap$, $e_3 = good\ performance$. Let $\gamma : E \to I = [0,1]$ be defined as follows: $\gamma(e_1) = 0.1$, $\gamma(e_2) = 0.4$, $\gamma(e_3) = 0.6$.*

We defined a function $F_\gamma : E \to I^U \times I$ as follows: $F_\gamma(e_1) = (\{\frac{x_1}{0.7}, \frac{x_2}{0.4}, \frac{x_3}{0.2}\}, 0.6)$, $F_\gamma(e_2) = (\{\frac{x_1}{0.1}, \frac{x_2}{0.2}, \frac{x_3}{0.9}\}, 0.4)$, $F_\gamma(e_3) = (\{\frac{x_1}{0.8}, \frac{x_2}{0.5}, \frac{x_3}{0.2}\}, 0.6)$.

Then, F_γ is a GFSS above (U, E). This membership matrix of F_γ can be written as $F_\gamma = \begin{pmatrix} 0.7 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.9 & 0.4 \\ 0.8 & 0.5 & 0.2 & 0.6 \end{pmatrix}$. The j th row vector of the matrix indicate $F_\gamma(e_j)$, the j th column vector of the matrix indicate x_j, the last column of the matrix indicate the values of γ.

Combined with the operation defined by Chen [5], we have made the following concise definition of the GFSS operation

Definition 2. *Let $F_\gamma(e_j) = \left(\left\{ \frac{x_1}{F(e_j)(x_1)}, \frac{x_2}{F(e_j)(x_2)}, \ldots, \frac{x_m}{F(e_j)(x_m)} \right\}, \gamma(e_j) \right), (j = 1, 2)$ be two GFSSs over (U, E), their operations are defined as follows:*

$$\begin{aligned}
&(1) \quad F_\gamma(e_1) \oplus F_\gamma(e_2) \\
&= \left(\left\{ \frac{x_1}{F(e_1)(x_1) + F(e_2)(x_1)}, \ldots, \frac{x_m}{F(e_1)(x_m) + F(e_2)(x_m)} \right\}, \gamma(e_1) + \gamma(e_2) \right) \\
&(2) \quad F_\gamma(e_1) \otimes F_\gamma(e_2) \\
&= \left(\left\{ \frac{x_1}{F(e_1)(x_1) \times F(e_2)(x_1)}, \ldots, \frac{x_m}{F(e_1)(x_m) \times F(e_2)(x_m)} \right\}, \gamma(e_1) \times \gamma(e_2) \right) \\
&(3) \quad \lambda F_\gamma(e_1) = \left(\left\{ \frac{x_1}{\lambda F(e_1)(x_1)}, \frac{x_2}{\lambda F(e_1)(x_2)}, \ldots, \frac{x_m}{\lambda F(e_1)(x_m)} \right\}, \quad \lambda\gamma(e_j) \right) \\
&(4) \quad F_\gamma(e_1)^\lambda = \left(\left\{ \frac{x_1}{F(e_1)(x_1)^\lambda}, \frac{x_2}{F(e_1)(x_2)^\lambda}, \ldots, \quad \frac{x_m}{F(e_1)(x_m)^\lambda} \right\}, (\gamma(e_j))^\lambda \right)
\end{aligned}$$

Definition 3. (Ref. [32]) *Let $F_\gamma(e_j), (j = 1, 2)$ be two GFSSs over (U, E), the degree of support between them is defined as*

$$\mathrm{Sup}(F_\gamma(e_1), F_\gamma(e_2)) = 1 - d(F_\gamma(e_1), F_\gamma(e_2)) \tag{2}$$

where

$$d(F_\gamma(e_1), F_\gamma(e_2)) = \frac{1}{m+1}\left[\left(\sum_{i=1}^{m}|F(e_1)(x_i) - F(e_2)(x_i)|\right) + |\gamma(e_1) - \gamma(e_2)|\right] \quad (3)$$

Definition 4. (Ref. [13]) *Suppose $X = (x_1, x_2, \ldots, x_n)$ is a group of nonnegative real numbers, $x_i \in [0,1], i = 1,2,\ldots,n$. Then, the power average operator can be indicated by the following aggregation function*

$$PA(X) = \sum_{i=1}^{n} \frac{1 + T(x_i)}{\sum_{i=1}^{n}(1 + T(x_i))} x_i \quad (4)$$

where $T(x_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(x_i, x_j)(i = 1,2,\ldots,n)$, $\mathrm{Sup}(x_i, x_j)$ represents the degree of support between x_i and x_j, and meets the following conditions

(1) $\mathrm{Sup}(x_i, x_j) \in [0,1]$

(2) $\mathrm{Sup}(x_i, x_j) = \mathrm{Sup}(x_i, x_j)$

(3) *If $d(x_i, x_j) \leq d(x_l, x_k)$, then $\mathrm{Sup}(x_i, x_j) \geq \mathrm{Sup}(x_l, x_k)$.*

Definition 5. (Ref. [23]) *Let $p, q \geq 0$, and $X = (x_1, x_2, \ldots, x_n)$ be a group of nonnegative real numbers, $x_i \in [0,1], i = 1,2,\ldots,n$. Then, the Bonferroni mean operator can be indicated by the following equation*

$$B^{p,q}(X) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} x_i^p x_j^q\right)^{\frac{1}{p+q}} \quad (5)$$

When $n = 2$ and $p = q$, the Bonferroni mean and the geometric mean are equal.

Definition 6. (Ref. [15]) *Let $p, q \geq 0$, and $X = (x_1, x_2, \ldots, x_n)$ be a group of nonnegative real numbers, $x_i \in [0,1], i = 1,2,\ldots,n$. Then, the power Bonferroni mean operator can be indicated by the following equation*

$$\begin{aligned} &PBM^{p,q}(x_1, x_2, \ldots, x_n) \\ &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i\neq j}^{n} \left(\left(\frac{n(1+T(x_i))}{\sum_{t=1}^{n}(1+T(x_t))} x_i\right)^p \otimes \left(\frac{n(1+T(x_j))}{\sum_{t=1}^{n}(1+T(x_t))} x_j\right)^q\right)\right)^{\frac{1}{p+q}} \end{aligned} \quad (6)$$

3. Generalized Fuzzy Soft Power Bonferroni Mean Operator

Considering that the PA operator can determine attribute weights according to the support relationship between attributes, thereby reducing the influence of biased decision makers' abnormal preference values on the decision results, while the BM operator can consider the degree of correlation between different attributes; therefore, according to the characteristics of PA operator and BM operator, this chapter combines the two operators and extends it to the generalized fuzzy soft set environment, and proposes a PBM operator based on generalized fuzzy variables.

Definition 7. *Let $p, q > 0$, $F_\gamma(e_j)(j = 1,2,\ldots,n)$ be a collection of generalized fuzzy variables.*

$$\begin{aligned} &GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) \\ &= \left(\frac{1}{n(n-1)}\right)^{\frac{1}{p+q}} \\ &\cdot \left(\bigoplus_{i,j=1; i\neq j}^{n} \left(\left(\frac{n(1+T(F_\gamma(e_i)))}{\sum_{t=1}^{n}(1+T(F_\gamma(e_t)))} F_\gamma(e_i)\right)^p \otimes \left(\frac{n(1+T(F_\gamma(e_j)))}{\sum_{t=1}^{n}(1+T(F_\gamma(e_t)))} F_\gamma(e_j)\right)^q\right)\right)^{\frac{1}{p+q}} \end{aligned} \quad (7)$$

We call $GFSPBM^{p,q}$ the generalized fuzzy soft power Bonferroni mean operators, where $\mathrm{T}(F_\gamma(e_i)) = \sum_{j=1,j\neq i}^{n} \mathrm{Sup}\left(F_\gamma(e_i), F_\gamma\left(e_j\right)\right)(i = 1, 2, \ldots, n)$, $\mathrm{Sup}\left(F_\gamma(e_i), F_\gamma\left(e_j\right)\right)$ *represents the degree of support between the generalized fuzzy variables $F_\gamma(e_i)$ and $F_\gamma\left(e_j\right)$, and meets the conditions in Definition 2.6.*

Theorem 1. *Let $p, q > 0$, $F_\gamma\left(e_j\right)(j = 1, 2, \ldots, n)$ be a collection of generalized fuzzy variables, then the aggregate value obtained by the GFSPBM operator is still a generalized fuzzy variable.*

Proof. This theorem is easy to prove, and we omit this process. □

Note 1. If one define $\widetilde{\omega}_k$ as follows

$$\widetilde{\omega}_k = \frac{\left(1 + T\left(F\left(e_j\right)\right)\right)}{\sum_{k=1}^{n}(1 + T(F(e_k)))} \tag{8}$$

then $\widetilde{\omega}_k \geq 0$, $\sum_{k=1}^{n} \widetilde{\omega}_k = 1$, and Equation (7) can be transformed into Equation (9):

$$\begin{aligned} &GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) \\ &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} \left((n\widetilde{\omega}_i F_\gamma(e_i))^p \otimes \left(n\widetilde{\omega}_j F_\gamma\left(e_j\right)\right)^q \right) \right)^{\frac{1}{p+q}} \end{aligned} \tag{9}$$

The GFSPBM operator has excellent properties such as idempotence and commutativity.

Theorem 2. (Idempotence). *Let $p, q > 0$, $F_\gamma\left(e_j\right)(j = 1, 2, \ldots, n)$ be a set of generalized fuzzy variables. If $F_\gamma\left(e_j\right)(j = 1, 2, \ldots, n)$ are all equal, i.e., $F_\gamma\left(e_j\right) = F_\gamma\left(e_j\right)(j = 1, 2, \ldots, n)$, then*

$$GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) = F_\gamma(e). \tag{10}$$

Proof. Since $F_\gamma\left(e_j\right)(j = 1, 2, \ldots, n)$ is equal, their weights are also equal. Let $\widetilde{\omega}_1 = \widetilde{\omega}_2 = \cdots = \widetilde{\omega}_n = \widetilde{\omega}$, then

$$\begin{aligned} &GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) \\ &= GFSPBM^{p,q}(F_\gamma(e), F_\gamma(e), \ldots, F_\gamma(e)) \\ &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} \left((n\widetilde{\omega}_i F_\gamma(e))^p \otimes \left(n\widetilde{\omega}_j F_\gamma(e)\right)^q \right) \right)^{\frac{1}{p+q}} \\ &= n\widetilde{\omega} F_\gamma(e) \\ &= F_\gamma(e) \end{aligned}$$

□

Theorem 3. (Commutativity). *Let $p, q > 0$, $F_\gamma\left(e_j\right)(j = 1, 2, \ldots, n)$ be a collection of generalized fuzzy variables, in the same time $\left(F'_\gamma(e_1), \ldots, F'_\gamma(e_n)\right)$ are any permutation of $(F_\gamma(e_1), \ldots, F_\gamma(e_n))$, then*

$$\begin{aligned} &GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) \\ &= GFSPBM^{p,q}\left(F'_\gamma(e_1), F'_\gamma(e_2), \ldots, F'_\gamma(e_n)\right) \end{aligned} \tag{11}$$

Proof. Since $\left(F'_\gamma(e_1), \ldots, F'_\gamma(e_n)\right)$ is any permutation of $(F_\gamma(e_1), \ldots, F_\gamma(e_n))$, then

$$
\begin{aligned}
&GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \dots, F_\gamma(e_n)) \\
&= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} \left((n\widetilde{\omega}_i F_\gamma(e_i))^p \otimes \left(n\widetilde{\omega}_j F_\gamma(e_j) \right)^q \right) \right)^{\frac{1}{p+q}} \\
&= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1;i\neq j}^{n} \left(\left(n\widetilde{\omega}_i F'_\gamma(e_i) \right)^p \otimes \left(n\widetilde{\omega}_j F'_\gamma(e_j) \right)^q \right) \right)^{\frac{1}{p+q}} \\
&= GFSPBM^{p,q}\left(F'_\gamma(e_1), F'_\gamma(e_2), \dots, F'_\gamma(e_n) \right)
\end{aligned}
$$

□

The $GFSPBM^{p,q}(F_\gamma(e_1), \dots, F_\gamma(e_n))$ operator only considers the correlation between the weight vector based on the power operator and the generalized fuzzy variable to be integrated, and does not consider the importance of the data; that is, the integrated operator is gathering defined when variables are equally important. However, in many actual decision-making processes, the importance of different attributes may be different, so their weights are not equal. To consider the importance of the index weight, the generalized fuzzy soft weighted power Bonferroni mean(GFSWPBM) operator will be defined next.

Definition 8. *Let $p, q > 0$, $F_\gamma(e_j)(j = 1, 2, \dots, n)$ be a set of generalized fuzzy variables. Suppose $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T (0 \leq \omega_j \leq 1, \sum_{j=1}^{n} \omega_j = 1)$ is the weight vector of attribute $e_j(j = 1, 2, \dots, n)$. Then, the GFSWPBM operator can be defined as follows*

$$
\begin{aligned}
&GFSWPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \dots, F_\gamma(e_n)) \\
&= \left(\frac{1}{n(n-1)} \right)^{\frac{1}{p+q}} \\
&\cdot \left(\bigoplus_{i,j=1;i\neq j}^{n} \left(\left(\frac{n\omega_i(1 + T(F_\gamma(e_i)))}{\sum_{t=1}^{n} \omega_t(1 + T(F_\gamma(e_t)))} F_\gamma(e_i) \right)^p \otimes \left(\frac{n\omega_j\left(1 + T\left(F_\gamma\left(e_j\right)\right)\right)}{\sum_{t=1}^{n} \omega_t(1 + T(F_\gamma(e_t)))} F_\gamma\left(e_j\right) \right)^q \right) \right)^{\frac{1}{p+q}}
\end{aligned} \tag{12}
$$

where $T(F_\gamma(e_i)) = \sum_{j=1, j\neq i}^{n} \omega_j \operatorname{Sup}\left(F_\gamma(e_i), F_\gamma\left(e_j\right)\right)(i = 1, 2, \dots, n)$, $\operatorname{Sup}\left(F_\gamma(e_i), F_\gamma\left(e_j\right)\right)$ represents the degree of support between the generalized fuzzy variables $F_\gamma(e_i)$ and $F_\gamma\left(e_j\right)$, and satisfies the conditions in Definition 2.4.

Theorem 4. *Let $p, q > 0$, $F_\gamma\left(e_j\right)(j = 1, 2, \dots, n)$ be a collection of generalized fuzzy variables, then the aggregate value obtained by the GFSWPBM operator is still a generalized fuzzy variable.*

Proof. This theorem is easy to prove, and we omit this process. □

Similar to the GFSPBM operator, GFSWPBM also has permutation invariance.

Several special cases about GFSWPBM operators will be discussed below. It can be found that many existing operators are special cases of GFSWPBM operators mentioned in this article.

Note 2. If $\frac{(1+T(F_\gamma(e_i)))}{\sum_{k=1}^{n}\omega_t(1+T(F_\gamma(e_i)))} = 1(i = 1, 2, \ldots, n)$, then

$$
\begin{aligned}
&GFSWPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) \\
&= \left(\frac{1}{n(n-1)}\right)^{\frac{1}{p+q}} \\
&\cdot \left(\bigoplus_{i,j=1;i\neq j}^{n}\left(\left(\frac{n\omega_i(1+T(F_\gamma(e_i)))}{\sum\limits_{t=1}^{n}\omega_t(1+T(F_\gamma(e_t)))}F_\gamma(e_i)\right)^{p} \otimes \left(\frac{n\omega_j(1+T(F_\gamma(e_j)))}{\sum\limits_{t=1}^{n}\omega_t(1+T(F_\gamma(e_t)))}F_\gamma(e_j)\right)^{q}\right)\right)^{\frac{1}{p+q}} \\
&= \left(\frac{1}{n(n-1)}\bigoplus_{i,j=1;i\neq j}^{n}\left((n\omega_i F_\gamma(e_i))^{p} \otimes (n\omega_j F_\gamma(e_j))^{q}\right)\right)^{\frac{1}{p+q}} \\
&= GFSSWBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n))
\end{aligned}
$$

Note 3. If $\omega = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^{T}$, then

$$
\begin{aligned}
&GFSWPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n)) \\
&= \left(\frac{1}{n(n-1)}\right)^{\frac{1}{p+q}} \\
&\cdot \left(\bigoplus_{i,j=1;i\neq j}^{n}\left(\left(\frac{n\omega_i(1+T(F_\gamma(e_i)))}{\sum\limits_{t=1}^{n}\omega_t(1+T(F_\gamma(e_t)))}F_\gamma(e_i)\right)^{p} \otimes \left(\frac{n\omega_j(1+T(F_\gamma(e_j)))}{\sum\limits_{t=1}^{n}\omega_t(1+T(F_\gamma(e_t)))}F_\gamma(e_j)\right)^{q}\right)\right)^{\frac{1}{p+q}} \\
&= \left(\frac{1}{n(n-1)}\right)^{\frac{1}{p+q}} \\
&\cdot \left(\bigoplus_{i,j=1;i\neq j}^{n}\left(\left(\frac{n(1+T(F_\gamma(e_i)))}{\sum\limits_{t=1}^{n}(1+T(F_\gamma(e_t)))}F_\gamma(e_i)\right)^{p} \otimes \left(\frac{n(1+T(F_\gamma(e_j)))}{\sum\limits_{t=1}^{n}(1+T(F_\gamma(e_t)))}F_\gamma(e_j)\right)^{q}\right)\right)^{\frac{1}{p+q}} \\
&= GFSPBM^{p,q}(F_\gamma(e_1), F_\gamma(e_2), \ldots, F_\gamma(e_n))
\end{aligned}
$$

At this time, the GFSWPBM operator degenerates into the GFSPBM operator. Obviously, the GFSPBM operator is a special case of the GFSWPBM operator.

4. Solving Multi-Attribute Decision-Making Problem with GFSWPBM Operator

4.1. Similarity Measure between GFSSs

To correct the shortcomings of similarity measures defined in the existing literature, Chen [5] defined a new GFSS similarity in 2020, as shown in Definition 4.3.

Definition 9. (Ref. [5]) *Let $U = \{x_1, x_2, \cdots, x_m\}$ be the universal collection of elements, and $E = \{e_1, e_2, \cdots, e_n\}$ be the universal collection of parameters. Suppose F_γ and G_δ are two GFSS over the parameterized universe (U, E), $F_\gamma = \{(F(e_j), \gamma(e_j)), j = 1, 2, \cdots, n\}$ and $G_\delta = \{(G(e_j), \delta(e_j)), j = 1, 2, \cdots, n\}$. The similarity measure between the GFSS F_γ and G_δ is is given by the following equation:*

$$
S\left(\widetilde{F}_\gamma, \widetilde{G}_\delta\right) = \frac{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\left[1-\left|\widetilde{F}_{ij}-\widetilde{G}_{ij}\right|\right]\right)\cdot\left[1-|\gamma(e_j)-\delta(e_j)|\right]}{mn} \tag{13}
$$

where $\widetilde{F}_{ij} = \gamma_{F(e_j)}(x_i)$ and $\widetilde{G}_{ij} = \delta_{\tilde{G}(e_j)}(x_i)$.

4.2. Bidirectional Projection

Definition 10. (Ref. [33]) *Let alternative $A_i(i=1,2,\cdots,m)$ be denoted as $A_i=(a_{i1},a_{i2},\cdots,a_{in})$, where a_{ij} is a fuzzy number, which indicates the degree to which alternative A_i conforms to attribute e_j. Then, the modulus length of the vector corresponding to alternative A_i is*

$$|A_i|=\sqrt{\sum_{j=1}^{n}|a_{ij}|^2} \tag{14}$$

Definition 11. (Ref. [33]) *Suppose $DM=\left(a_{ij}\right)_{m\times n}$ is the decision matrix, and $A^+=\left(a_1^+,a_2^+,\cdots,a_n^+\right)$ and $A^-=\left(a_1^-,a_2^-,\cdots,a_n^-\right)$ are the vectors formed by positive ideal alternatives and negative ideal alternatives, where $r_j^+=\max_{1\le i\le n}\{r_{ij}\}$, $r_j^-=\min_{1\le i\le n}\{r_{ij}\}$, $j=1,2,\cdots,n$.*

Definition 12. (Ref. [33]) *For two alternatives $A_i=(a_{i1},a_{i2},\cdots,a_{in})$ and $A_j=\left(a_{j1},a_{j2},\cdots,a_{jn}\right)$, $i,j=1,2,\cdots,m$, we defined*

$$A_iA_j=\left(a_{j1}-a_{i1},a_{j2}-a_{i2},\cdots,a_{jn}-a_{in}\right) \tag{15}$$

It is the vector formed by alternative A_i and alternative A_j.

Definition 13. (Ref. [33]) *Let the alternatives be $A_i=(a_{i1},a_{i2},\cdots,a_{in})$. $A^+=\left(a_1^+,a_2^+,\cdots,a_n^+\right)$ and $A^-=\left(a_1^-,a_2^-,\cdots,a_n^-\right)$ are positive ideal alternatives and negative ideal alternatives, respectively. Then, the vectors formed by positive ideal alternatives and negative ideal alternatives, negative ideal alternatives and alternatives are*

$$A^-A^+=\left\{a_1^+-a_1^-,a_2^+-a_2^-,\cdots,a_n^+-a_n^-\right\} \tag{16}$$

$$A^-A_i=\left\{a_{i1}-a_1^-,a_{i2}-a_2^-,\cdots,a_{in}-a_n^-\right\} \tag{17}$$

The corresponding vector modulus lengths are

$$\left|A^-A^+\right|=\sqrt{\sum_{j=1}^{n}\left|a_j^+-a_j^-\right|^2} \tag{18}$$

$$\left|A^-A_i\right|=\sqrt{\sum_{j=1}^{n}\left|a_{ij}-a_j^-\right|^2} \tag{19}$$

Then we defined

$$\cos\left(A^-A_i,A^-A^+\right)=\frac{\sum_{j=1}^{n}\left(a_j^+-a_j^-\right)\left(a_{ij}-a_j^-\right)}{|A^-A_i||A^-A^+|} \tag{20}$$

it is the cosine of the angle between A^-A^+ and A^-A_i.

Note 4. Ref. [33] The bidirectional projection has the following properties

(1) Symmetry: $\cos(A^-A_i,A^-A^+)=\cos(A^-A^+,A^-A_i)$.
(2) Boundedness: $0\le\cos(A^-A_i,A^-A^+)\le 1$. $\cos(A^-A_i,A^-A^+)=1$, if and only if the A^-A_i and A^-A^+ directions are the same.

Definition 14. (Ref. [33]) *Let the alternatives, alternatives with positive ideals, and alternatives with negative ideals be A_i, A^+, and A^-, respectively, we defined*

$$\Pr j_{A^-A^+}(A^-A_i) = |A^-A_i|\cos(A^-A_i, A^-A^+) = \frac{\sum_{j=1}^{n}\left(a_j^+ - a_j^-\right)\left(a_{ij} - a_j^-\right)}{|A^-A^+|} \tag{21}$$

$$\Pr j_{A_iA^+}(A^-A^+) = |A^-A^+|\cos(A^-A^+, A_iA^+) = \frac{\sum_{j=1}^{n}\left(a_j^+ - a_j^-\right)\left(a_j^+ - a_{ij}\right)}{|A_iA^+|} \tag{22}$$

These are, respectively, called the projection of the vector formed by the negative ideal alternative and the alternative on the vector formed by the positive ideal alternative and the negative ideal alternative, and the vector formed by the positive ideal alternative and the negative ideal alternative on the alternative and the projection on the vector formed by the positive ideal alternative.

Note 5. Ref. [33] The bigger $Prj_{A^-A^+}(A^-A_i)$ is, the closer alternative A_i is to the ideal alternative A^+. The smaller $Prj_{A^-A^+}(A^-A_i)$ is, the farther away alternative A_i is from the ideal alternative A^+. The bigger $Prj_{A_iA^+}(A^-A^+)$ is, the closer alternative A_i is to negative ideal alternative A^-. The smaller $Prj_{A_iA^+}(A^-A^+)$ is, the farther away alternative A_i is from negative ideal alternative A^-.

Definition 15. (Ref. [33]) *To obtain the optimal alternative, the closeness $C(A_i)$ is construed as follows*

$$C(A_i) = \frac{\Pr j_{A^-A^+}(A^-A_i)}{\Pr j_{A^-A^+}(A^-A_i) + \Pr j_{A_iA^+}(A^-A^+)} \tag{23}$$

The above formula actually refers to the closeness formula of TOPSIS and other methods. Obviously, the larger the $C(A_i)$, the better the alternative A_i. The opposite is true. The above decision-making methods are defined when the importance of every attribute is the same. In the practical decision-making process, the importance of different attributes may be different, so they have different weights. Next, define the weighted bidirectional projection method of GFSS.

Definition 16. (Ref. [33]) *Suppose the attribute weight is $w = (w_1, w_2, \cdots, w_n)$; then, the vector formed by the i-th alternative and the negative ideal alternative is a weighted projection on the vector formed by the positive ideal alternative and the negative ideal alternative, and the positive ideal alternative and the negative ideal alternative are formed The weighted projections of the vector formed by the alternative and the positive ideal alternative are, respectively,*

$$\begin{aligned}&\Pr j_{A^-A^+}(A^-A_i)_w \\ &= |A^-A_i|_w \cos(A^-A_i, A^-A^+)_w = \frac{\sum_{j=1}^{n} w_j^2\left(a_j^+ - a_j^-\right)\left(a_{ij} - a_j^-\right)}{|A^-A^+|_w}\end{aligned} \tag{24}$$

$$\begin{aligned}&\Pr j_{A_iA^+}(A^-A^+)_w \\ &= |A^-A^+|_w \cos(A^-A^+, A_iA^+)_w = \frac{\sum_{j=1}^{n} w_j^2\left(a_j^+ - a_j^-\right)\left(a_j^+ - a_{ij}\right)}{|A_iA^+|_w}\end{aligned} \tag{25}$$

where

$$|A^-A^+|_w = \sqrt{\sum_{j=1}^{n}\left(w_j\left|a_j^+ - a_j^-\right|\right)^2} \tag{26}$$

$$\left|A^{-}A_i\right|_w = \sqrt{\sum_{j=1}^{n}\left(w_j\left|a_{ij} - a_j^{-}\right|\right)^2} \tag{27}$$

$$\cos\left(A^{-}A_i, A^{-}A^{+}\right)_w = \frac{\sum_{j=1}^{n} w_j^2\left(a_j^{+} - a_j^{-}\right)\left(a_{ij} - a_j^{-}\right)}{|A^{-}A_i|_w |A^{-}A^{+}|_w} \tag{28}$$

$$\cos\left(A^{-}A^{+}, A_iA^{+}\right)_w = \frac{\sum_{j=1}^{n} w_j^2\left(a_j^{+} - a_j^{-}\right)\left(a_j^{+} - a_{ij}\right)}{|A_iA^{+}|_w |A^{-}A^{+}|_w} \tag{29}$$

Refer to Equation (23) to define the closeness formula considering the attribute weight:

$$C(A_i)_w = \frac{\Pr j_{A^{-}A^{+}}(A^{-}A_i)_w}{\Pr j_{A^{-}A^{+}}(A^{-}A_i)_w + \Pr j_{A_iA^{+}}(A^{-}A^{+})_w} \tag{30}$$

4.3. Algorithm

In this section, we will introduce a multi-attribute decision-making method based on the GFSS environment that considers the cognition of the decision maker. The algorithm steps are as follows.

Step 1: Suppose experts use generalized fuzzy soft sets to express their opinions, which contain decision information about attribute alternatives. Therefore, DM $l(l = 1, 2, \cdots, L)$ expresses the judgment for attribute $e_j(j = 1, 2, \cdots, n)$, which can be indicated as $\widetilde{F}^l\left(e_j\right)$.

$$\widetilde{F}^l\left(e_j\right) = \left(\left\{\frac{A_1}{\widetilde{F}^l\left(e_j\right)(A_1)}, \frac{A_2}{\widetilde{F}^l\left(e_j\right)(A_2)}, \cdots, \frac{A_m}{\widetilde{F}^l\left(e_j\right)(A_m)}\right\}, \gamma^l\left(e_j\right)\right) \tag{31}$$

Step 2: Calculate the similarity measure between DMs, get the weight of DMs, and use Equation (13) to find the similarity coefficient $S\left(\widetilde{F}_\gamma^l, \widetilde{F}_\gamma^k\right)$ between DMs. In this way, a consensus matrix of preferences of all DMs is obtained.

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m1} & S_{m2} & \cdots & S_{mn} \end{bmatrix} \tag{32}$$

Next, we define the weight coefficient ω_l of DM l as follows

$$\omega_l = \frac{\sum_{j=1}^{n} S_{lj}}{\sum_{i=1}^{m}\sum_{j=1}^{n} S_{ij}} \tag{33}$$

Step 3: Utilize the GFSWPBM operator, to aggregate all of the individual GFSSs $\widetilde{F}_\gamma^l\left(e_j\right)(l = 1, 2, \cdots, L)$ into a comprehensive GFSS $\widetilde{F}_\gamma\left(e_j\right)$, then derive the comprehensive overall assessed value for attribute $e_j(j = 1, 2, \cdots, n)$ of alternative $A_i(i = 1, 2, \cdots, m)$.

$$GFSWPBM^{p,q}\left(\omega_1\widetilde{F}_\gamma^1\left(e_j\right), \omega_2\widetilde{F}_\gamma^2\left(e_j\right), \ldots, \omega_L\widetilde{F}_\gamma^L\left(e_j\right)\right)$$
$$= \left(\frac{1}{L(L-1)}\right)^{\frac{1}{p+q}}$$
$$\cdot\left(\bigoplus_{i,j=1;i\neq j}^{L}\left(\left(\frac{L\omega_i\left(1 + T\left(\widetilde{F}_\gamma^k\left(e_j\right)\right)\right)}{\sum_{t=1}^{L}\omega_t\left(1 + T\left(\widetilde{F}_\gamma^t\left(e_j\right)\right)\right)}\widetilde{F}_\gamma^k\left(e_j\right)\right)^p \otimes \left(\frac{L\omega_j\left(1 + T\left(\widetilde{F}_\gamma^l\left(e_j\right)\right)\right)}{\sum_{t=1}^{L}\omega_t\left(1 + T\left(\widetilde{F}_\gamma^t\left(e_j\right)\right)\right)}\widetilde{F}_\gamma^l\left(e_j\right)\right)^q\right)\right)^{\frac{1}{p+q}}$$

Step 4: Determine the positive ideal alternative A^+ and negative ideal alternative A^- in the decision matrix GFSS $\widetilde{F}_\gamma(e_j)$. Take the adjustment factor γ_j as the attribute weight vector, $w = (w_1, w_2, \cdots, w_n)$, according to the Equation (30) to find $C(A_i)_w (i = 1, 2, \cdots, m)$ and rank the alternatives.

5. Illustrative Example

5.1. Case

We take the group decision-making problem proposed by Wang and Lee and Zhang [34,35] as an example. To improve work efficiency, the administrators of the university computer center need to consider a choice of computer software. There are four alternatives $U = \{A_1, A_2, A_3, A_4\}$ remaining on the alternative list. The expert evaluation team includes three DMs $D = \{d_1, d_2, d_3\}$. The DMs evaluate the four alternatives using a group of attributes: hardware/software is cheap (e_1), benefits the organization (e_2), easy migration from current system (e_3), and Outsourcing software developers have high reliability (e_4). Next, we use the decision-making method proposed in this paper to evaluate these four alternatives.

Step 1: Three evaluation experts DMs evaluated four alternatives $A_i (i = 1, 2, 3, 4)$ by using the information provided by the generalized fuzzy soft set. Table 1 shows the results of alternative A_i evaluated by each DM under criteria $e_j (j = 1, 2, 3, 4)$.

Step 2: We calculate the similarity between DM and use it to calculate the weight of each DM.

Table 1. Tabular representation of GFSS of each DM.

DM	U	e_1	e_2	e_3	e_4
DM1	A_1	0.7	0.4	0.5	0.6
	A_2	0.6	0.4	0.8	0.7
	A_3	0.8	0.2	0.6	0.8
	A_4	0.9	0.9	0.7	0.8
	γ	0.33	0.62	0.31	0.23
DM2	A_1	0.8	0.9	0.3	0.6
	A_2	0.3	0.6	0.7	0.4
	A_3	0.7	0.4	0.9	0.5
	A_4	0.9	0.5	0.5	0.8
	γ	0.36	0.48	0.44	0.38
DM3	A_1	0.5	0.4	0.7	0.7
	A_2	0.6	0.8	0.5	0.5
	A_3	0.8	0.5	0.9	0.9
	A_4	0.9	0.7	0.6	0.8
	γ	0.36	0.33	0.41	0.34

By Equation (13), We calculate the similarity measure between DM1 and DM2.

$$
\begin{aligned}
S_{12} &= S(DM1, DM2) \\
&= \frac{\sum_{j=1}^{n} \left(\sum_{i=1}^{m} \left[1 - \left|DM1_{ij} - DM2_{ij}\right|\right] \right) \cdot \left[1 - \left|\gamma(e_j) - \delta(e_j)\right|\right]}{mn} \\
&= \frac{(0.9 + 0.7 + 0.9 + 1) \times 0.97 + (0.5 + 0.8 + 0.8 + 0.6) \times 0.86}{16} \\
&+ \frac{(0.8 + 0.9 + 0.7 + 0.8) \times 0.87 + (1 + 0.7 + 0.7 + 1) \times 0.85}{16} \\
&= 0.7119
\end{aligned}
$$

In the same way, we can calculate the similarity measure between DM1 and DM3 and the similarity measure between DM2 and DM3: $S_{13} = 0.7426$, $S_{23} = 0.7697$.

Using Equation (33), we get the weights of the DMs. The calculation process of $\omega_1 = 0.3295$ is as follows

$$\begin{aligned}\omega_1 &= \frac{\sum_{j=1}^{3} S_{1j}}{\sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}} \\ &= \frac{1 + 0.7119 + 0.7426}{3 \times 1 + 0.7119 \times 2 + 0.7426 \times 2 + 0.7697 \times 2} \\ &= 0.3295\end{aligned}$$

We use the same calculation method to get the values of ω_2 and ω_3: $\omega_2 = 0.3332$, $\omega_3 = 0.3373$.

Step 3: GFSWPBM operators combine the evaluation information of each DM to calculate the comprehensive decision-making GFSS. To facilitate the calculation, we choose the parameter $p = 1, q = 1$, as shown in Table 2.

Table 2. Tabular representation of the collective decision GFSS $\widetilde{F}_\gamma\left(e_j\right)$.

U	e_1	e_2	e_3	e_4
A_1	0.6599	0.5412	0.4864	0.6327
A_2	0.4910	0.5902	0.6599	0.5259
A_3	0.7662	0.3571	0.7950	0.7242
A_4	0.9000	0.6901	0.5967	0.8000
γ	0.3498	0.4680	0.3851	0.3137

Step 4: Calculate the score $C(A_i)_w (i = 1,2,3,4)$ by Equation (30). The calculation process of $C(A_1)_w$ is as follows:

Take out the positive ideal alternative, the negative ideal alternative and the weight vector:$A^+ = (0.9000, 0.6901, 0.7950, 0.8000)$, $A^- = (0.4910, 0.3571, 0.4864, 0.5259)$, $w = (0.3498, 0.4680, 0.3851, 0.3137)$. Calculate the vector formed by the positive ideal alternative and the negative ideal alternative, and the vector formed by the alternative and the positive ideal alternative.

$$\left|A^- A^+\right|_w = 0.2574$$
$$\left|A^1 A_+\right|_w = 0.1697$$

Calculate the value of $\Pr j_{A^- A^+}(A^- A_1)_w$ and $\Pr j_{A_1 A^+}(A^- A^+)_w$

$$\begin{aligned}&\Pr j_{A^- A^+}\left(A^- A_1\right)_w \\ &= \frac{\sum_{j=1}^{4} w_j^2\left(a_j^+ - a_j^-\right)\left(a_{1j} - a_j^-\right)}{|A^- A^+|_w} \\ &= \frac{0.3498^2 \times 0.4090 \times 0.1689 + 0.4680^2 \times 0.3339 \times 0.1841}{0.2574} \\ &+ \frac{0.3851^2 \times 0.3086 \times 0 + 0.3137^2 \times 0.2741 \times 0.1068}{0.2574} \\ &= 0.0962\end{aligned}$$

$$
\begin{aligned}
&\Pr j_{A_1A^+}\left(A^-A^+\right)_w \\
&= \frac{\sum_{j=1}^{4} w_j^2\left(a_j^+ - a_j^-\right)\left(a_j^+ - a_{1j}\right)}{|A_1A^+|_w} \\
&= \frac{0.3498^2 \times 0.4090 \times 0.2410 + 0.4680^2 \times 0.3339 \times 0.1489}{0.1697} \\
&+ \frac{0.3851^2 \times 0.3086 \times 0.3086 + 0.3137^2 \times 0.2741 \times 0.1673}{0.1697} \\
&= 0.2447
\end{aligned}
$$

Calculate the score of alternative A_1.

$$
\begin{aligned}
C(A_1)_w &= \frac{\Pr j_{A^-A^+}(A^-A_1)_w}{\Pr j_{A^-A^+}(A^-A_1)_w + \Pr j_{A_1A^+}(A^-A^+)_w} \\
&= \frac{0.0962}{0.0962 + 0.2447} \\
&= 0.2822
\end{aligned}
$$

The same calculation method can get the scores of A_2, A_3, and A_4: $C(A_2)_w = 0.2980$, $C(A_3)_w = 0.3914$, and $C(A_4)_w = 0.6515$. According to the score, the alternatives are sorted: $A_4 > A_3 > A_2 > A_1$. The best alternative is A_4.

5.2. Sensitivity Analysis

In decision-making process, the decision makers can choose appropriate parameters p, q according to their own risk preferences. The calculated sorting result may change with the change of parameter selection. For the convenience of calculation, the above analysis is calculated when the parameters are selected as $p = 1, q = 1$.

To reflect the influence of different parameters on the ranking order, we conducted a sensitivity analysis. From Table 3, we can find that, as the parameters p and q change, the score values of each scheme have changed accordingly. The best solution is always A_4. When the gap between p and q is large enough, the worst solution changes from A_1 to A_2. From the structure of the operator, we can easily see that the calculation results of the operator are symmetrical at about the values of p and q (for example, the calculation result of parameter $p = 1, q = 2$ is the same as the calculation result of parameter $p = 2, q = 1$).

Table 3. Sorting results of GFSWPBM operator under different parameter values.

Parameter Value	Score Value	Ranking Results
$p = q = 0.5$	$C = (0.2817, 0.2961, 0.3923, 0.6531)$	$A_4 > A_3 > A_2 > A_1$
$p = q = 1$	$C = (0.2822, 0.2980, 0.3914, 0.6515)$	$A_4 > A_3 > A_2 > A_1$
$p = q = 2$	$C = (0.2809, 0.3018, 0.3902, 0.6473)$	$A_4 > A_3 > A_2 > A_1$
$p = q = 5$	$C = (0.2700, 0.3154, 0.3894, 0.6332)$	$A_4 > A_3 > A_2 > A_1$
$p = q = 10$	$C = (0.2576, 0.3307, 0.3897, 0.6144)$	$A_4 > A_3 > A_2 > A_1$
$p = 1, q = 2$	$C = (0.2886, 0.3024, 0.3885, 0.6498)$	$A_4 > A_3 > A_2 > A_1$
$p = 1, q = 5$	$C = (0.3501, 0.3295, 0.3678, 0.6523)$	$A_4 > A_3 > A_1 > A_2$
$p = 2, q = 1$	$C = (0.2886, 0.3024, 0.3885, 0.6498)$	$A_4 > A_3 > A_2 > A_1$
$p = 5, q = 1$	$C = (0.3501, 0.3295, 0.3678, 0.6523)$	$A_4 > A_3 > A_1 > A_2$

5.3. Comparative Analysis with Existing Methods

To better show the advantages of the method proposed in this paper, the following further compares and analyzes with the existing methods, and selects the GFSSWBM operator and the FSSWBM operator in the literature [5] (To facilitate the calculation, we choose the parameter $p = 1, q = 1$). The comparison results are shown in Table 4.

Table 4. Comparative analysis results.

Integration Method	Score Value	Ranking Results
GFSSWBM	$S = (-0.0735, -0.0825, -0.0339, 0.1899)$	$A_4 > A_3 > A_1 > A_2$
FSSWBM	$S = (-0.1987, -0.2663, -0.0242, 0.4891)$	$A_4 > A_3 > A_1 > A_2$
GFSWPBM	$C = (0.2822, 0.2980, 0.3914, 0.6515)$	$A_4 > A_3 > A_2 > A_1$

From the above integration results, It can be seen that the optimal alternative by the GFSWPBM operator in this article and the optimal alternative calculated by the GFSSWBM operator and the FSSWBM operator [5] are both A_4. The ranking results calculated by the three operators are roughly similar. It is further discovered that the score value of this method is also different from other methods. The main factor that causes the difference in the sorting results is that the operators of the above models all adopt different information integration methods. Although they are all based on the idea of arithmetic average, the focus of the operators in the assembly process is different. The GFSSWBM operator and the FSSWBM operator do not consider the relationship between data information. The GFSWPBM operator given in this paper combines the advantages of the PA operator and the BM operator. It not only considers the possible relationships between attributes, but also reflects the overall balance between the data, thereby preventing biased decision makers from giving anomalies. The preference value (the value that is too large or too small in the original data) affects the result of the decision, making the decision more fair and objective. At the same time, this article has different opinions on the ranking results of A_1 and A_2, which can provide a new reference angle for judging the quality of alternatives. The main reason for the difference in the score value is that this paper uses the weighted bidirectional projection method to calculate the score value, while the GFSSWBM operator and the FSSWBM operator use the weighted method to calculate the score value. Observing the calculation results, we can see that the weighted bidirectional projection method in this paper has better discrimination.

6. Conclusions

Aiming at the problem of group decision-making, the cognition of decision makers is considered. In this paper, generalized fuzzy soft sets are used to solve the influence of decision makers' cognition on the effectiveness of information provided and introduces two new integrated methods for GFSS, namely generalized fuzzy soft Bonferroni mean(GFSPBM) operator and generalized fuzzy soft weighted power Bonferroni mean(GFSWPBM) operator. The new operator combines the excellent characteristics of the power average operator and the Bonferroni operator. It not only considers the overall balance between existing data and information, but also considers the possible correlation between attributes. Furthermore, some excellent properties and special situations of the new operator are also discussed. On this basis, a multi-attribute decision-making method based on the generalized fuzzy soft weighted power Bonferroni mean (GFSWPBM) operator and detailed steps are given. The weighted bidirectional projection method is introduced to calculate the score value of the alternative, and the calculation is carried out in an example. The practicability of this method is explained, and its advantages are illustrated by comparing with existing methods.

The decision-making methods provided in this article can also be further applied to such fields as supplier selection evaluation, product program selection evaluation, human resources department talent introduction recommendation, etc., with certain theoretical and application value.

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