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Modeling 3D–1D Junction via Very-Weak Formulation

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Abstract: We study the potential flow of an ideal fluid through a domain that consists of a reservoir and a pipe connected to it. The ratio of the pipe's thickness and its length is considered as a small parameter. Using the rigorous asymptotic analysis with respect to that small parameter, we derive an effective model governing the the junction between a 1D and a 3D fluid domain. The obtained boundary-value problem has a measure boundary condition with Dirac mass concentrated in the junction point and is understood in the very-weak sense.

Keywords: asymptotic analysis; very-weak solution; laplace equation; mathematical modeling; potential fluid flow

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1. Introduction

Fluid flows in pipes are important because they appear in many applications. For their description, we typically use one-dimensional approximations. In the case of single pipe, that matter has been extensively studied by various authors (see, e.g., [1–11] and the references therein). A variant of the two-scale convergence for thin domain is developed in [4] for that kind of problems. When there is a structure consisting of several pipes, the main problem is how to derive the effective junction condition. Junctions of elastic structures have been extensively studied for more than 30 years (see, e.g., [12,13]). The study of similar problems in fluid mechanics started a bit later. However, in the last 25 years, several papers can be found, and we mention some of them. The problem of a structure consisting of several pipes is addressed in [7] (see also [1,14,15]) using the classical approach of matched asymptotic expansions or in [16] using the two-scale convergence approach. A particular method of partial domain decomposition was proposed for such problems by Panasenko et al. (see, e.g., [1,17]).

Junction of thin domains of different kind, such as pipe and fracture or a thin domain and a thick 3D domain, is more difficult, from the technical point of view, as the problem of the definition of traces appears. If there is a junction of pipe and 3D reservoir, we can derive the 1D model for the pipe via asymptotic analysis, as the ratio between the pipe's thickness and length tends to zero. However, the junction becomes one point and, in the classical weak formulation setting, the trace of a function defined in 3D reservoir cannot be appropriately defined in one point. Furthermore, the boundary value becomes a measure and not a function. Thus, the usual Sobolev space setting and the weak formulation is not appropriate. Therefore, in this paper, we propose using the very-weak formulation (see, e.g., [18,19]), which appears to be natural tool for our multi-dimensional asymptotic analysis.

That kind of problems, with 3D–1D junction domains, was rigorously studied by Kozlov et al. [20], using the asymptotic expansions and their justification. In fact, Section 2 is devoted to the boundary value problem for the Laplace equation in 3D domain with several thin outlets. Complete asymptotic expansion is derived and the reminder is estimated in H^1 norm. The fact that zero-order approximation contains delta mass on the boundary and is not in H^1 is patched by adding the cut-off function, taking out the "bad part", in the expansion. One difference, compared to the problem treated by Kozlov et al. [20], is that in their original problem the boundary condition is mixed. It is Neumann, except on the end of the thin cylinders, where the condition is Dirichlet (while we have Neumann condition all over). That leads to a different effective model. Another difference is that we use completely a different technique, based on the weak convergence and the very-weak formulation. In comparison, since they derived the complete asymptotics, the information they obtained on the asymptotic behavior of the solution is richer. On the other hand, our method is much simpler and more intuitive. That makes our approach more suitable for further application to more complex problems, such as the Navier–Stokes system.

Thus, the main novelty of the paper is the method. We use of the very-weak formulation of the problem, which allows a direct application of the weak and the two-scale convergence for thin domains and the straightforward rigorous derivation of the singular effective problem. Due to the difference in dimension, the effective model has singular measure boundary data. Unlike the standard weak formulation, the very-weak formulation is designed for treatment of such problems with data lacking regularity.

For the sake of simplicity, we consider the incompressible, potential flow of an ideal fluid. We assume that the fluid is injected in the pipe P_ε with thickness $\varepsilon \ll 1$, by strong injection g_ε. The pipe is connected to the reservoir Ω, so that the fluid enters Ω. Due to the incompressibility it must go out somewhere. We assume that it exits the reservoir on the other side through some part of its boundary.

Using the rigorous asymptotic analysis, as the thickness of the pipe tends to zero, we obtain the effective junction condition in the form of a Dirac mass concentrated in the junction point. Such problem cannot have a weak solution, but it is uniquely solvable in the very-weak sense.

1.1. The Geometry

Let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain, such that the point $O(0,0,0) \in \partial\Omega$. We assume that there there is a flat part of the boundary around O, i.e., there exists some $\delta > 0$ such that

$$\Gamma_\delta = \{(0, x_2, x_3) \in \mathbf{R}^3 \,;\, x_2^2 + x_3^2 < \delta^2\} \subset \partial\Omega \ .$$

For $\omega \subset \mathbf{R}^2$ smooth and convex domain contained in the unit ball $B(0,1)$, and a small parameter $\varepsilon \ll 1$ such that $\varepsilon < \delta$, we define the small set

$$\omega_\varepsilon = \varepsilon\, \omega$$

and the thin pipe

$$P_\varepsilon =]\,0, L\,[\, \times \omega_\varepsilon \ .$$

The fluid domain is now defined as

$$\Omega_\varepsilon = \Omega \cup P_\varepsilon \ .$$

We denote

$$\begin{aligned} &\Sigma_\varepsilon =]\,0, L\,[\, \times \partial\omega_\varepsilon && - \text{the pipe's wall} \\ &\gamma_\varepsilon = \{\, L\,\} \times \omega_\varepsilon && - \text{the pipe's entrance} \\ &\Gamma = \partial\Omega_\varepsilon \backslash (\gamma_\varepsilon \cup \Sigma_\varepsilon) && - \text{the reservoir's boundary.} \end{aligned}$$

Please see Figure 1.

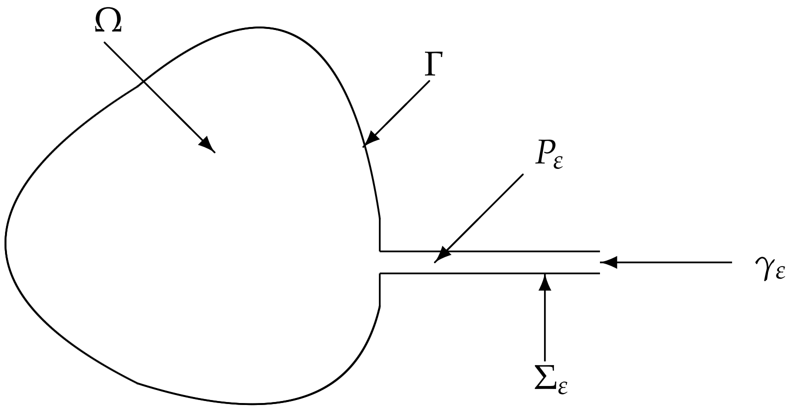

Figure 1. The reservoir Ω and the pipe P_ε.

1.2. The Equations

We denote by θ^ε the velocity potential and study the Neumann problem for the Laplace equation. For the boundary condition, we first define the entering velocity as

$$g_\varepsilon(x_2, x_3) = \frac{1}{\varepsilon^2}\, g\Big(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) \ , \tag{1}$$

where $g \in L^2(\omega)$. We denote the total flux through the entrance of the pipe by

$$\begin{aligned} \tau \ &= \int_{\omega_\varepsilon} g_\varepsilon(x_2, x_3)\ dx_2\, dx_3 = \\ &\int_\omega g(y_2, y_3)\ dy_2\, dy_3. \end{aligned} \tag{2}$$

Next, we choose the function $h \in L^2(\Gamma)$ such that

$$\int_\Gamma h\ dS = -\tau \ . \tag{3}$$

Now, our problem reads

$$\Delta\theta^\varepsilon = 0 \quad \text{in} \ \ \Omega_\varepsilon \tag{4}$$
$$\frac{\partial\theta^\varepsilon}{\partial\mathbf{n}} = g_\varepsilon \quad \text{on} \ \ \gamma_\varepsilon \tag{5}$$
$$\frac{\partial\theta^\varepsilon}{\partial\mathbf{n}} = 0 \quad \text{on} \ \ \Sigma_\varepsilon \tag{6}$$
$$\frac{\partial\theta^\varepsilon}{\partial\mathbf{n}} = h \quad \text{on} \ \ \Gamma \ . \tag{7}$$

It is well posed due to (2) and (3) .

2. A Priori Estimates

We start with $H^1(\Omega_\varepsilon)$ estimate.

Proposition 1. *Let θ^ε be the solution of the problem (4)–(7). Then, there exists $C > 0$, independent on ε, such that*

$$|\theta^\varepsilon|_{H^1(\Omega_\varepsilon)} \le C(1 + \varepsilon^{-1}) \tag{8}$$

Proof. We test (4) with θ^ε. It gives

$$\int_{\Omega_\varepsilon} |\nabla\theta^\varepsilon|^2 = \int_\Gamma h\,\theta^\varepsilon + \int_{\gamma_\varepsilon} g_\varepsilon\,\theta^\varepsilon \ .$$

Using the trace theorem, we obviously have

$$\left| \int_\Gamma h\, \theta^\varepsilon \right| \le |h|_{L^2(\Gamma)}\, |\theta^\varepsilon|_{L^2(\Gamma)} \le C|\theta^\varepsilon|_{H^1(\Omega_\varepsilon)} \ .$$

By direct integration, it is easy to prove that

$$|\theta^\varepsilon|_{L^2(\gamma_\varepsilon)} \le C|\theta^\varepsilon|_{H^1(P_\varepsilon)} \ ,$$

with $C > 0$, independent from ε. Thus,

$$\begin{aligned} &\left| \int_{\gamma_\varepsilon} g_\varepsilon\, \theta^\varepsilon \right| \le |g_\varepsilon|_{L^2(\gamma_\varepsilon)}\, |\theta^\varepsilon|_{L^2(\gamma_\varepsilon)} \le \\ &\le C|g_\varepsilon|_{L^2(\gamma_\varepsilon)}|\theta^\varepsilon|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{-1}|\theta^\varepsilon|_{H^1(\Omega_\varepsilon)} \ . \end{aligned}$$

□

2.1. $L^2(\Omega)$ Estimate

We proceed with sharp (and essential) $L^2(\Omega)$ estimate.

Proposition 2. *Let θ^ε be the solution of the problem (4)–(7). Then, there exists $C > 0$, independent on ε, such that*

$$\left| \theta^\varepsilon - \frac{1}{|\Omega|} \int_\Omega \theta^\varepsilon \right|_{L^2(\Omega)} \le C \tag{9}$$

Proof. We assume, at the beginning, that $\int_\Omega \theta^\varepsilon = 0$.
We need an auxiliary problem:

$$\Delta\phi = \theta^\varepsilon \quad \text{in} \ \ \Omega \tag{10}$$

$$\frac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{on} \ \ \partial\Omega \quad . \tag{11}$$

Since the right hand-side is an $H^1(\Omega) \cap L^2_0(\Omega)$ function, the solution $\phi \in H^3(\Omega)$ (standard elliptic regularity; see, e.g., [21]). Thus, $\phi \in C^{1,\sigma}(\Omega)$ for some $\sigma > 0$. Furthermore, there exists some $C > 0$, independent on ε, such that

$$\begin{aligned} &|\phi|_{H^2(\Omega)} \le C|\theta^\varepsilon|_{L^2(\Omega)} \\ &|\phi|_{L^\infty(\Omega)} \le C|\theta^\varepsilon|_{L^2(\Omega)} \\ &|\phi|_{H^3(\Omega)} \le C|\theta^\varepsilon|_{H^1(\Omega)} \\ &|\nabla\phi|_{L^\infty(\Omega)} \le C|\theta^\varepsilon|_{H^1(\Omega)} \le \frac{C}{\varepsilon} \ . \end{aligned}$$

Testing (10) with θ^ε gives

$$\int_\Omega |\theta^\varepsilon|^2 = \int_\Gamma h\, \phi + \int_{\gamma_\varepsilon(0)} \phi \frac{\partial \theta^\varepsilon}{\partial x_1} \ .$$

The first integral is easily estimated as above, and

$$\left| \int_\Gamma h\, \phi \right| \le C|\phi|_{H^1(\Omega)} \le C|\theta^\varepsilon|_{L^2(\Omega)} \ . \tag{12}$$

For the second one, we proceed as follows:

$$\begin{aligned}&\int_{\gamma_\varepsilon(0)} \phi(0,x')\frac{\partial\theta^\varepsilon}{\partial x_1} = \\ &\int_{P_\varepsilon} \nabla_{x'}\phi(0,x')\nabla_{x'}\theta^\varepsilon + \int_{\gamma_\varepsilon} g_\varepsilon\phi(0,x') \le \\ &\le C\varepsilon|\nabla_{x'}\phi|_{L^\infty(\Omega)}|\nabla_{x'}\theta^\varepsilon|_{L^2\infty(P_\varepsilon)} \\ &+|g_\varepsilon|_{L^1(\gamma_\varepsilon)}|\phi|_{L^\infty(\Omega)} \le C(1+|\theta^\varepsilon|_{L^2(\Omega)}) \ .\end{aligned} \tag{13}$$

so that

$$|\theta^\varepsilon|_{L^2(\Omega)} \le C \quad , \tag{14}$$

of course under the condition that $\int_\Omega \theta^\varepsilon = 0$. Thus, in general,

$$\left|\theta^\varepsilon - \frac{1}{|\Omega|}\int_\Omega \theta^\varepsilon\right|_{L^2(\Omega)} \le C \quad . \tag{15}$$

□

2.2. $L^2(P_\varepsilon)$ Estimate

The estimate in $L^2(P_\varepsilon)$ is less complicated and follows from the Ponicaré inequality:

Lemma 1. *There exists a constant $C > 0$, independent from ε, such that $\forall\ \psi \in H^1(P_\varepsilon)$*

$$\left|\psi - \frac{1}{|P_\varepsilon|}\int_{P_\varepsilon} \psi\right|_{L^2(P_\varepsilon)} \le C\varepsilon|\nabla\psi|_{L^2(P_\varepsilon)} \ . \tag{16}$$

Proof. The classical Ponicaré inequality on ω yields that $\forall\ \Psi = \Psi(y_2, y_3) \in H^1(\omega)$

$$\left|\Psi - \frac{1}{|\omega|}\int_\omega \Psi\right|_{L^2(\omega)} \le C\left|\nabla_{y'}\Psi\right|_{L^2(\omega)} \ , \tag{17}$$

with $C > 0$ depending only on ω. Let $\psi \in H^1(P_\varepsilon)$. Then,

$$\Phi(y_2, y_3) = \psi(x_1, \varepsilon y_2, \varepsilon y_3) \in H^1(\omega)$$

for all $0 < x_1 < L$. A simple change of variables and (17) imply the claim. □

3. The Limit

The very-weak formulation of the problems (4) and (7) reads:
Find $\theta^\varepsilon \in L^2_0(\Omega_\varepsilon) = \{v \in L^2(\Omega_\varepsilon)\ ;\ \int_{\Omega_\varepsilon} v = 0\}$ such that

$$\int_{\Omega_\varepsilon} \theta^\varepsilon\ \Delta\phi + \int_\Gamma h\ \phi + \int_{\gamma_\varepsilon} g_\varepsilon\phi = 0 \ , \tag{18}$$

for any $\phi \in H^2(\Omega_\varepsilon)$ such that $\frac{\partial\phi}{\partial \mathbf{n}} = 0$ on $\partial\Omega_\varepsilon$.
It is easy to see that it has a unique solution (see, e.g., [18]).
Our main result is the following convergence theorem:

Theorem 1. *Let θ^ε be the solution to the problems (4) and (7). Then, its restriction on Ω satisfies*

$$\theta^\varepsilon \rightharpoonup \theta \ \ \textit{weakly in} \ \ L^2(\Omega) \ , \tag{19}$$

where $\theta \in L^2_0(\Omega)$ is the unique very-weak solution to the problem

$$\Delta\theta = 0 \quad in \ \ \Omega \tag{20}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = h + \tau\,\delta \quad on \ \ \Gamma \ \ , \tag{21}$$

δ is the Dirac measure defined by

$$\langle\, \delta \mid \psi \,\rangle = \psi(0) \ \ \textit{for any} \ \ \psi \in C(\overline{\Omega}) \ \ ,$$

and $\tau \in \mathbf{R}$ is the boundary flux defined by (2).

Proof. Next, we take the function $\psi \in C^2(\overline{\Omega})$ such that $\frac{\partial\psi}{\partial\mathbf{n}} = 0$ on $\partial\Omega$. Then,

$$\int_\Omega \theta^\varepsilon \Delta\psi = \int_\Gamma h\,\psi + \int_{\gamma_\varepsilon(0)} \psi(0,x')\ \frac{\partial\theta^\varepsilon}{\partial x}(0,x')dx' \ \ .$$

The $L^2(\Omega)$ bound for θ^ε implies the existence of a subsequence (denoted by the same symbol) and function $\theta \in L^2(\Omega)$, such that (19) holds. Then, for the first integral, we have

$$\int_\Omega \theta^\varepsilon \Delta\psi \to \int_\Omega \theta\ \Delta\psi\ .$$

For the last integral, we obtain

$$\begin{aligned}
&\int_{\gamma_\varepsilon(0)} \psi(0,x')\ \frac{\partial\theta^\varepsilon}{\partial x}(0,x')dx' = \\
&= \int_{\gamma_\varepsilon(0)} \left[\psi(0,x') - \psi(0)\right]\ \frac{\partial\theta^\varepsilon}{\partial x}(0,x')dx' + \\
&+\psi(0)\int_{\gamma_\varepsilon(0)} \frac{\partial\theta^\varepsilon}{\partial x}(0,x')dx' = \\
&= \int_{\gamma_\varepsilon(0)} \left[\psi(0,x') - \psi(0)\right]\ \frac{\partial\theta^\varepsilon}{\partial x}(0,x')dx' + \\
&+\tau\,\psi(0)\ .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\gamma_\varepsilon(0)} \left[\psi(0,x') - \psi(0)\right]\ \frac{\partial\theta^\varepsilon}{\partial x}(0,x')dx' \le \\
&\le \left|\psi(0,x') - \psi(0)\right|_{H^{\frac{1}{2}}(\gamma_\varepsilon(0))} \times \\
&\times \left|\frac{\partial\theta^\varepsilon}{\partial x}(0,x')\right|_{H^{-\frac{1}{2}}(\gamma_\varepsilon(0))} \le C\varepsilon\ .
\end{aligned}$$

We use here two estimates:

$$
\begin{aligned}
1.)\quad & \int_{\gamma_\varepsilon(0)} \frac{\partial \theta^\varepsilon}{\partial x}\, w = \\
& = -\int_\Omega \Delta\theta^\varepsilon + \int_\Omega \nabla\theta^\varepsilon \nabla w = \\
& = \int_\Omega \nabla\theta^\varepsilon \nabla w \le |\nabla\theta^\varepsilon|_{L^2(\Omega)} |\nabla w|_{L^2(\Omega)} \le \\
& \le \frac{C}{\varepsilon} |\nabla w|_{L^2(\Omega)} \le \frac{C}{\varepsilon} |w|_{H^{1/2}(\gamma_\varepsilon(0))} \ , \\
& \forall \ \ w \in H^1(\Omega)\ , \\
2.)\quad & \left|\psi(0,x') - \psi(0)\right|_{H^{\frac{1}{2}}(\gamma_\varepsilon(0))} \le \\
& \le |\gamma_\varepsilon(0)|^{\frac{1}{2}} \varepsilon \left| D^2\psi \right|_{L^\infty(\Omega)} \le C\varepsilon^2\ .
\end{aligned}
$$

Finally,

$$\int_{\gamma_\varepsilon(0)} \psi(0,x')\, \frac{\partial \theta^\varepsilon}{\partial x}(0,x')dx' \to \tau\ \psi(0) \quad . \tag{22}$$

We conclude from the above that the limit θ satisfies

$$\int_\Omega \theta\ \Delta\psi = \int_\Gamma h\,\psi + \tau\ \psi(0) \quad , \tag{23}$$

which is exactly the very-weak formulation of the problems (20) and (21). It remains to prove that it has a unique solution (implying that the whole sequence θ^ε converges, and not only a subsequence) in $L^2_0(\Omega)$. However, (22) is the very weak formulation of the linear boundary value problem for the Laplace equation with non-smooth data, and it is well known that it has the unique very weak solution in $L^2_0(\Omega) = \{v \in L^2(\Omega)\ ;\ \int_\Omega v = 0\}$ (see, e.g., [18]). □

4. Example

To illustrate the obtained model, we solve the effective problems (20) and (21) with measure boundary data for rectangular domain $\Omega = \langle -1,1\rangle \times \langle 0,1\rangle$. The problem reads

$$
\left\{
\begin{array}{l}
\Delta\theta = 0 \ \text{ for } \ -1 < x,y < 1\ ,\ 0 < z < 1 \\
\frac{\partial\theta}{\partial x}(-1,y,z) = \frac{\partial\theta}{\partial x}(1,y,z) = \frac{\partial\theta}{\partial x}(x,-1,z) = \frac{\partial\theta}{\partial x}(x,1,z) = 0 \ \ ,\ 0 < z < 1 \\
\frac{\partial\theta}{\partial y}(x,y,1) = 0 \ \ ,\ \ -1 < x < 1\ ,\ -1 < y < 1 \\
-\frac{\partial\theta}{\partial y}(x,,y,0) = h(x,y) + \tau\,\delta \ \ ,\ \ -1 < x < 1 \ \ -1 < y < 1
\end{array}
\right. \quad . \tag{24}
$$

We assume, for simplicity, that h is a smooth even function defined on $[-1,1]^2$. Of course,

$$\tau = -\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} h(s,t)\,ds\,dt \quad .$$

Due to the simple geometry, we can solve the problem (in the very-weak sense) using the Fourier method. We look for the solution of the form

$$
\begin{aligned}
\theta(x,y) \quad &= \sum_{k,j=0}^{\infty} A_{kj} \cos(k\pi x)\, \cos(j\pi y) \left[\sinh(\pi\, z\, \sqrt{k^2+j^2}\,) - \right. \\
&\left. - \tanh(\pi\,\sqrt{k^2+j^2})\, \sinh(\pi\, z\,\sqrt{k^2+j^2}\,)\right] \ ,
\end{aligned} \tag{25}
$$

where the coefficients A_{kj} are picked such that

$$A_{kj} = \frac{4}{\pi\,\sqrt{k^2+j^2}\ \tanh(\pi\,\sqrt{k^2+j^2}\,)} \left[\int_0^1 \int_0^1 h(t)\ \cos(k\pi t)\ \cos(j\pi s) dt\, ds + \tau \right] \quad . \tag{26}$$

If, in particular, we choose h as a constant

$$h = H_0 = const. \ ,$$

then

$$A_{kj} = -\frac{H_0}{\pi\ \sqrt{k^2+j^2}\ \tanh(\pi\ \sqrt{k^2+j^2}\,)}\ .$$

5. Conclusions

We rigorously derive a model for describing the potential flow of an ideal fluid through a reservoir with several pipes connected to it. The flow through the pipes is, usually, described by mono-dimensional models, while the flow through the reservoir is described by a three-dimensional model. The effective junction condition between the pipe and the reservoir is described by a Dirac delta measure concentrated in a junction point. The obtained problem has unique solution, but only in the very-weak sense, since the boundary value is not a function but a measure. The future goal is to do the same for the viscous flow using the concept of the very-weak solution for the Navier–Stokes system developed in [19].

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