Article

Exact Likelihood Inference for a Competing Risks Model with Generalized Type II Progressive Hybrid Censored Exponential Data

Subin Cho [1] **and Kyeongjun Lee** [2,*]

1   Department of Statistics, Daegu University, Gyeongsan 38453, Korea; whtnqls3955@naver.com
2   Division of Mathematics and Big Data Science, Daegu University, Gyeongsan 38453, Korea
*   Correspondence: indra_74@naver.com

Abstract: In many situations of survival and reliability test, the withdrawal of units from the test is pre-planned in order to to free up testing facilities for other tests, or to save cost and time. It is known that several risk factors (RiFs) compete for the immediate failure cause of items. In this paper, we derive an inference for a competing risks model (CompRiM) with a generalized type II progressive hybrid censoring scheme (GeTy2PrHCS). We derive the conditional moment generating functions (CondMgfs), distributions and confidence interval (ConfI) of the scale parameters of exponential distribution (ExDist) under GeTy2PrHCS with CompRiM. A real data set is analysed to illustrate the validity of the method developed here. From the data, it can be seen that the conditional PDFs of MLEs is almost symmetrical.

Keywords: competing risks model; confidence interval; moment generating function; generalized type II progressive hybrid censoring

1. Introduction

In many situations of survival and reliability test, the tester might not obtain complete information on failure times for all items. The withdrawal of units from the test is pre-planned in order to to free up testing facilities for other tests, or to save cost and time. Type I censoring scheme (Ty1CS) and type II censoring scheme (Ty2CS) cannot be used if the teste wants to eliminate the live items at a point other than the end point of the test. Therefore, progressive type II censoring scheme (PrTy2CS) has become popular censoring scheme in a survival and reliability analysis problem (Refs. [1–5]). Though the PrTy2CS assure a number of observed failures, it has the drawback that it might take a long time to terminate the test and to observe a pre-fixed number of failures. In this motivation, Ref. [6] suggest a GeTy2PrHCS in which the test is assured to end at a pre-assigned time. These are designed to fix the drawbacks inherent in the type II progressive hybrid censoring scheme (Ty2PrHCS). The survival and reliability analysis based on the GeTy2PrHCS can save time and costs. GeTy2PrHCS may arise in a situation when the tester has prepaid for the use of the facility.

GeTy2PrHCS can be explained as follows. The m, PrTy2CS $(\Re_1, \Re_2, \cdots, \Re_m; \sum_{i=1}^{m} \Re_i + m = n)$, and times $\mathcal{T}_1$ and $\mathcal{T}_2$ are pre-assigned integers such that $m \leq n$ and $0 < \mathcal{T}_1 < \mathcal{T}_2 < \infty$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the number of failures up to time $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Randomly, $\Re_1$ of the surviving units are removed from the test at the time of 1st failure $(X_{1:m:n})$. Randomly, the $\Re_2$ of the surviving units are removed at the time of 2nd failure $(X_{2:m:n})$. If m-th failure $(X_{m:m:n})$ occurs before the $\mathcal{T}_1$, we continue to observe failures $(\Re_m = \Re_{m+1} = \cdots = \Re_{d_1} = 0$; without any further withdrawals) up to time $\mathcal{T}_1$ (Case (a)). If $\mathcal{T}_1 < X_{m:m:n} < \mathcal{T}_2$, end the test at m-th failure (Case (b)). If $X_{m:m:n} > \mathcal{T}_2$, end the test at time $\mathcal{T}_2$ (Case (c)). This GeTy2PrHCS modifies the Ty2PrHCS by assuring that the test will be finished by $\mathcal{T}_2$. For the GeTy2PrHCS, there are three possible scenarios (Figure 1).
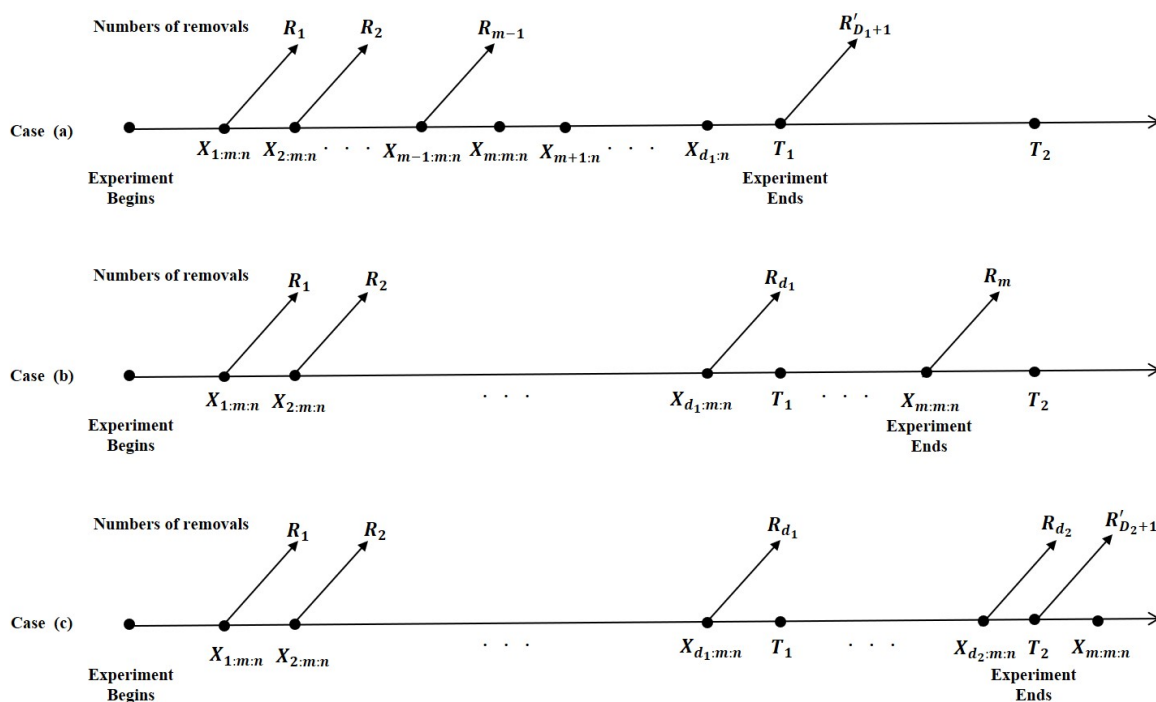
Figure 1. GeTy2PrHCS. **Case (a)**: $\{X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}, X_{m+1:n}, \cdots, x_{d_1:n}\}$, if $X_{m:m:n} < \mathcal{T}_1$, $\Re_m = \Re_{m+1} = \cdots = \Re_{d_1} = 0$. **Case (b)**: $\{X_{1:m:n}, X_{2:m:n}, \cdots, X_{d_1:m:n}, \cdots, X_{m:m:n}\}$, if $\mathcal{T}_1 < X_{m:m:n} < \mathcal{T}_2$. **Case (c)**: $\{X_{1:m:n}, X_{2:m:n}, \cdots, X_{d_1:m:n}, \cdots, X_{d_2:m:n}\}$, if $\mathcal{T}_2 < X_{m:m:n}$.

In Case (c), $X_{d_1:m:n} < \mathcal{T}_1 < X_{d_1+1:m:n}$, $X_{d_2:m:n} < \mathcal{T}_2 < X_{d_2+1:m:n}$, and $X_{d_2+1:m:n}, \cdots$, $X_{m:m:n}$ are not observed.

Due to the complication of external environment and internal structure, it is known that the breakdown of a item results by several reasons of failure. These reasons of breakdown are called the competing risks data (CompD) that compete with each other in life cycle and can be encountered in survival and reliability study, and it has been discussed by many authors (Refs. [7–11]). From Ref. [11], it can be seen that the conditional PDFs of MLEs under CompRiM with censored data is almost symmetrical. In CompRiM, it is assumed that the among RiFs are statistically independent. A CompD and indicator denoting the RiF of failure consists of an observed failure time. In Section 2, we will prove the CondMgf, distributions and ConfI of the scale parameters of ExDist under GeTy2PrHCS with CompRiM. We will present a simulation results to investigate the biases, root mean squared error (rMSE), coverage percentages (CovP) and confidence lengths (ConfL) of the MLEs of parameters of ExDist under GeTy2PrHCS with CompRiM in Section 3. An illustrative example is presented. Finally, in Section 4, the conclusion and summary are presented. The meanings of abbreviations and symbols are listed in Abbreviations Section.

2. Model Description and Conditional Inference for MLEs

2.1. Model Description and MLEs

We suppose that n randomly selected items with CompD for an ExDist data were put on a survival and reliability test. We suppose that the $X_1, X_2, \cdots, X_n$ are iid with an ExDist. Here, $X_i = \min\{X_{i1}, X_{i2}, \cdots, X_{ik}\}$, X_{ik} denotes the life-time of the i-th item under the k-th RiF with cumulative distribution function (CDF) such as $G_k(x) = 1 - \exp(-\lambda_k^{-1}x)$. Recently, researchers are interested with specific factor in the presence of other RiFs. Therefore, in this paper, we suppose that there are two RiFs for the failures. Then, it is to obtain the CDF of life-time as

$$\mathscr{F}(x; \boldsymbol{\lambda}) = 1 - \exp\left[-\left(\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\right)^{-1}x\right], \quad x > 0, \lambda_1 > 0, \lambda_2 > 0,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$.

Let $\boldsymbol{X} = \{x_1, x_2, \cdots, x_n\}$ denote the data of n items, and $\boldsymbol{\Delta} = \{\delta_1, \delta_2, \cdots, \delta_n\}$ denote the indicator of RiF. Here $\delta_i = 1$ denotes the i-th failure caused by 1st RiF. On the other hands, $\delta_i = 0$ denotes that other RiF is responsible for the i-th failure. Each life-time is composed of life-time and the reason for failure under the CompRiM $(\boldsymbol{X}, \boldsymbol{\Delta})$. Therefore, the joint PDF (jPDF) of failure life-time and RiF is

$$f_{\boldsymbol{X},\boldsymbol{\Delta}}(x,k) = \lambda_k^{-1} \exp\left[-\left(\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \right)^{-1} x \right], \quad k = 1, 2.$$

From GeTy2PrHCS, we have the following data;

Case (a) $\{(x_{1:m:n}, \delta_{1:m:n}), (x_{2:m:n}, \delta_{2:m:n}), \cdots, (x_{m:m:n}, \delta_{m:m:n}), (x_{m+1:n}, \delta_{m+1:n}) \cdots, (x_{d_1:n}, \delta_{d_1:n})\}$.

Case (b) $\{(x_{1:m:n}, \delta_{1:m:n}), (x_{2:m:n}, \delta_{2:m:n}), \cdots, (x_{d_1:m:n}, \delta_{d_1:m:n}), \cdots, (x_{m:m:n}, \delta_{m:m:n})\}$.

Case (c) $\{(x_{1:m:n}, \delta_{1:m:n}), (x_{2:m:n}, \delta_{2:m:n}), \cdots, (x_{d_1:m:n}, \delta_{d_1:m:n}), \cdots, (x_{d_2:m:n}, \delta_{d_2:m:n})\}$.

Based on the GeTy2PrHCS, the likelihood function (see, Ref. [6]) is

$$\mathcal{L}(\boldsymbol{\lambda}|x) = \begin{cases} \varsigma'_{\mathscr{D}_1} \prod_{j=1}^{\mathscr{D}_1} f_{\boldsymbol{X},\boldsymbol{\Delta}}\left(x_{j:m:n}, 1\right)^{\delta_j} f_{\boldsymbol{X},\boldsymbol{\Delta}}\left(x_{j:m:n}, 2\right)^{1-\delta_j} \left[1 - \mathscr{F}\left(x_{j:m:n}\right)\right]^{\Re_j} \left[1 - \mathscr{F}\left(\mathscr{T}_1\right)\right]^{\Re'_{\mathscr{D}_1+1}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{D}_1 = m, m+1, \cdots, n, \\[2mm] \varsigma'_m \prod_{j=1}^{m} f_{\boldsymbol{X},\boldsymbol{\Delta}}\left(x_{j:m:n}, 1\right)^{\delta_j} f_{\boldsymbol{X},\boldsymbol{\Delta}}\left(x_{j:m:n}, 2\right)^{1-\delta_j} \left[1 - \mathscr{F}\left(x_{j:m:n}\right)\right]^{\Re_j}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{D}_1 = 0, 1, \cdots, m-1, \ \mathscr{D}_2 = m, \\[2mm] \varsigma'_{\mathscr{D}_2} \prod_{j=1}^{\mathscr{D}_2} f_{\boldsymbol{X},\boldsymbol{\Delta}}\left(x_{j:m:n}, 1\right)^{\delta_j} f_{\boldsymbol{X},\boldsymbol{\Delta}}\left(x_{j:m:n}, 2\right)^{1-\delta_j} \left[1 - \mathscr{F}\left(x_{j:m:n}\right)\right]^{\Re_j} \left[1 - \mathscr{F}\left(\mathscr{T}_2\right)\right]^{\Re'_{\mathscr{D}_2+1}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathscr{D}_2 = 1, 2, \cdots, m-1, \end{cases} \tag{1}$$

where $\Re'_{\mathscr{D}_1+1} = n - \sum_{i=1}^{m-1} \Re_i - \mathscr{D}_1$, $\Re'_{\mathscr{D}_2+1} = n - \sum_{i=1}^{\mathscr{D}_2} \Re_i - \mathscr{D}_2$ and $\varsigma'_{\mathscr{D}} = \prod_{j=1}^{\mathscr{D}} \sum_{k=j}^{m} (\Re_k + 1)$.

Using (1), then, we can obtain the MLEs of λ_k ($k = 1, 2$) as

$$\hat{\lambda}_k = \begin{cases} \frac{1}{n_k} \left[\sum_{j=1}^{\mathscr{D}_1} \left(1 + \Re_j\right) x_{j:m:n} + \mathscr{T}_1 \Re'_{\mathscr{D}_1+1} \right], & \mathscr{D}_1 = m, m+1, \cdots, n, \\[2mm] \frac{1}{n_k} \left[\sum_{j=1}^{m} \left(1 + \Re_j\right) x_{j:m:n} \right], & \mathscr{D}_1 = 0, 1, \cdots, m-1, \ \mathscr{D}_2 = m, \\[2mm] \frac{1}{n_k} \left[\sum_{j=1}^{\mathscr{D}_2} \left(1 + \Re_j\right) x_{j:m:n} + \mathscr{T}_2 \Re'_{\mathscr{D}_2+1} \right], & \mathscr{D}_2 = 1, \cdots, m-1. \end{cases} \tag{2}$$

Here, we denote the total failure number of units due to the RiF k by n_k, $k = 1, 2$, then it is easy to obtain $n_1 = \sum_{i=1}^{u} \delta_i$ and $n_2 = \sum_{i=1}^{u} (1 - \delta_i) = u - n_1$, where $u = d_1$ for Case (a), $u = m$ for Case (b) and $u = d_2$ for Case (c).

From (2), the $\hat{\lambda}_1$ and $\hat{\lambda}_2$ do not exist when $n_1 = 0$ and $n_2 = 0$, respectively. In order to estimate λ_1 and λ_2, we have to observe at least one failure caused by each RiF. That is,

$$\zeta^{(u)} = \{n_1 \geq 1, n_2 \geq 1, n_1 + n_2 = u\}.$$

2.2. Conditional Inference for MLEs

To find the exact conditional inference for $\hat{\lambda}_1$ and $\hat{\lambda}_2$, we first derive the CondMgf of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, respectively. In order to obtain CondMgf of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, we need the following Lemma in Ref. [12].

Lemma 1. *Let $\eta_j > 0$ ($j = 1, 2, \cdots, m$), and let X denote the absolutely continuous RV with $f(x)$ (PDF) and $\mathscr{F}(x)$ (CDF). For $m \geq 1$, then, we have*

$$\int_{\mathscr{T}}^{x_{m+1}} \cdots \int_{\mathscr{T}}^{x_3} \int_{\mathscr{T}}^{x_2} \prod_{j=1}^{m} f(x_j) \{1 - \mathscr{F}(x_j)\}^{\eta_j - 1} dx_1 dx_2 \cdots dx_m$$

$$= \sum_{i=0}^{m} \varsigma_{i,m}(\boldsymbol{\eta_m}) \{1 - \mathscr{F}(x_{m+1})\}^{\kappa_{i,m}(\eta_m)} \{1 - \mathscr{F}(\mathscr{T})\}^{\sum_{j=1}^{m-i} \eta_j},$$

where $\boldsymbol{\eta_m} = (\eta_1, \eta_2, \cdots, \eta_m);$ $\varsigma_{i,m}(\boldsymbol{\eta_m}) = \dfrac{(-1)^i}{\left\{\prod_{j=1}^{i} \sum_{k=m-i+1}^{m-i+j} \eta_k\right\}\left\{\prod_{j=1}^{m-i} \sum_{k=j}^{m-i} \eta_k\right\}},$
$\kappa_{i,m}(\boldsymbol{\eta_m}) = \sum_{j=m-i+1}^{m} \eta_j$ *with the usual conventions that* $\prod_{j=1}^{0} a_j \equiv 1$ *and* $\sum_{j=i}^{0} a_j \equiv 0$.

Using Lemma 1, we have the CondMgf of $\hat{\lambda}_1$, given $\zeta^{(u)}$, as follow Theorem.

Theorem 1. *The CondMgf of $\hat{\lambda}_1$, given $\zeta^{(u)}$, is*

$$M_{\hat{\lambda}_1}(t)$$
$$= E\left(e^{t\hat{\lambda}_1} | \zeta^{(u)}\right)$$
$$= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1 = d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_1}$$
$$\times \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1, \cdots, \Re_{d_1}+1) q_1^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\left[\Re'_{\mathscr{D}_1+1} + \kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]}$$
$$+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-m}$$
$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1, \cdots, \Re_{d_1}+1) \varsigma_{i_2,m-d_1}(\Re_{d_1+1}, \cdots, \Re_m+1) q_1^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}$$
$$\times q_2^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=m-i_2+1}^{m}(\Re_j+1)} + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2 = d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_2}$$
$$\times \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1, \cdots, \Re_{d_2}+1) q_2^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\Re^*_{d_2-i+1}},$$

where $\Re^*_j = \sum_{i=j}^{m}(\Re_i+1)$, $q_1 = \exp\left[-\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\mathscr{T}_1\right]$ *and* $q_2 = \exp\left[-\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\mathscr{T}_2\right]$.

Theorem 2. *The CondMgf of $\hat{\lambda}_2$, given $\zeta^{(u)}$, is*

$$M_{\hat{\lambda}_1}(t)$$
$$= E\left(e^{t\hat{\lambda}_1} | \zeta^{(u)}\right)$$
$$= \sum_{d_1=1}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1 = d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \left(1 - \frac{t}{d_1-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_1}$$
$$\times \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1, \cdots, \Re_{d_1}+1) q_2^{\left(1 - \frac{t}{d_1-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\left[\Re'_{\mathscr{D}_1+1} + \kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]}$$
$$+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m} \left(1 - \frac{t}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-m}$$
$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1, \cdots, \Re_{d_1}+1) \varsigma_{i_2,m-d_1}(\Re_{d_1+1}, \cdots, \Re_m+1) q_1^{\left(1 - \frac{t}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}$$
$$\times q_2^{\left(1 - \frac{t}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=m-i_2+1}^{m}(\Re_j+1)} + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2 = d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \left(1 - \frac{t}{d_2-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_2}$$
$$\times \sum_{j=0}^{d_1} \varsigma_{j,d_2}(\Re_1+1, \cdots, \Re_{d_2}+1) q_2^{\left(1 - \frac{t}{d_2-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\Re^*_{d_2-i+1}}.$$

Using Theorems 1 and 2, then, we readily obtain the first and second moments of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, respectively.

Corollary 1. *The $E_{\lambda_1}(\hat{\lambda}_1)$ and $E_{\lambda_1}(\hat{\lambda}_1^2)$ are given by*

$$E_{\lambda_1}(\hat{\lambda}_1) = M'_{\hat{\lambda}_1}(0)$$

$$= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)$$

$$\times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \left\{ \frac{d_1}{i} \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]\right\}$$

$$+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}$$

$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}$$

$$\times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \left\{ \frac{m}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1) + \frac{\mathscr{T}_2}{i}\sum_{j=m-i_2+1}^{m}(\Re_j+1)\right\}$$

$$+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}}$$

$$\times \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}} \left\{ \frac{d_2}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_2}{i}\Re^*_{d_2-i+1}\right\},$$

and

$$E_{\lambda_1}(\hat{\lambda}_1^2) = M''_{\hat{\lambda}_1}(0)$$

$$= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)$$

$$\times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \left[\frac{d_1}{i}\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left\{\frac{d_1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]\right\}^2\right]$$

$$+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}$$

$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}$$

$$\times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \left[\frac{m}{i}\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left\{\frac{m}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1) + \frac{\mathscr{T}_2}{i}\sum_{j=m-i_2+1}^{m}(\Re_j+1)\right\}^2\right]$$

$$+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}}$$

$$\times \left[\frac{d_2}{i}\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left\{\frac{d_2}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_2}{i}\Re^*_{d_2-i+1}\right\}^2\right].$$

Corollary 2. *The* $E_{\lambda_2}(\hat{\lambda}_2)$ *and* $E_{\lambda_2}(\hat{\lambda}_2^2)$ *are given by*

$$E_{\lambda_2}(\hat{\lambda}_2) = M'_{\hat{\lambda}_2}(0)$$

$$= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)$$

$$\times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \left\{ \frac{d_1}{d_1-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{d_1-i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]\right\}$$

$$+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}$$

$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}$$

$$\times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \left\{ \frac{m}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{m-i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1) + \frac{\mathscr{T}_2}{m-i}\sum_{j=m-i_2+1}^{m}(\Re_j+1)\right\}$$

$$+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}}$$

$$\times \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}} \left\{ \frac{d_2}{d_2-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_2}{d_2-i}\Re^*_{d_2-i+1}\right\},$$

and

$$E_{\lambda_2}(\hat{\lambda}_2^2) = M''_{\hat{\lambda}_2}(0)$$

$$= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)$$

$$\times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)}$$

$$\times \left[\frac{d_1}{d_1-i}\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left\{ \frac{d_1}{d_1-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{d_1-i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]\right\}^2 \right]$$

$$+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}$$

$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)} q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)}$$

$$\times \left[\frac{m}{m-i}\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left\{ \frac{m}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_1}{m-i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1) + \frac{\mathscr{T}_2}{m-i}\sum_{j=m-i_2+1}^{m}(\Re_j+1)\right\}^2 \right]$$

$$+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}}$$

$$\times \left[\frac{d_2}{d_2-i}\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^2 + \left\{ \frac{d_2}{d_2-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} + \frac{\mathscr{T}_2}{i}\Re^*_{d_2-i+1}\right\}^2 \right].$$

In order to obtain CondPDF of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, respectively, we need the following Lemma in Ref. [13].

Lemma 2. *If RV* $Y \sim \gamma(a,b)$, *let* $X = Y + \delta$, *then the Mgf or RV X is*

$$M_X(t) = e^{t\delta}(1-bt)^{-a}, \ t < 1/b,$$

where $\gamma(x-\delta;a,b)$ *is a gamma distribution with shift* (δ), *shape* $(a > 0)$ *and rate* $(b > 0)$ *parameters.*

Using Lemma 2, we have the CondPDF of $\hat{\lambda}_1$, given as $\zeta^{(u)}$, as in the following theorem.

Theorem 3. *The CondPDF of $\hat{\lambda}_1$, given $\zeta^{(u)}$, is*

$$
\begin{aligned}
&f_{\hat{\lambda}_1}(x) \\
&= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1) \\
&\quad \times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \gamma\left(x - \frac{\mathscr{T}_1}{i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]; d_1, \frac{1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right) \\
&\quad + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m} \\
&\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)} \\
&\quad \times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \gamma\left(x - \frac{\mathscr{T}_1}{i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1) - \frac{\mathscr{T}_2}{i}\sum_{j=m-i_2+1}^{m}(\Re_j+1); m, \frac{1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right) \\
&\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}} \\
&\quad \times \gamma\left(x - \frac{\mathscr{T}_2}{i}\Re^*_{d_2-i+1}; d_2, \frac{1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right),
\end{aligned}
$$

Theorem 4. *The CondPDF of $\hat{\lambda}_2$, given $\zeta^{(u)}$, is*

$$
\begin{aligned}
&f_{\hat{\lambda}_2}(x) \\
&= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1) \\
&\quad \times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \gamma\left(x - \frac{\mathscr{T}_1}{d_1-i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]; d_1, \frac{1}{d_1-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right) \\
&\quad + \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m} \\
&\quad \times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)} \\
&\quad \times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \gamma\left(x - \frac{\mathscr{T}_1}{m-i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1) - \frac{\mathscr{T}_2}{m-i}\sum_{j=m-i_2+1}^{m}(\Re_j+1); m, \frac{1}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right) \\
&\quad + \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}} \\
&\quad \times \gamma\left(x - \frac{\mathscr{T}_2}{d_2-i}\Re^*_{d_2-i+1}; d_2, \frac{1}{d_2-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right).
\end{aligned}
$$

Corollary 3. *The tail probabilities of $\hat{\lambda}_1$ and $\hat{\lambda}_2$, given $\zeta^{(u)}$, are*

$$
P_{\lambda_1}(\hat{\lambda}_1 > w)
$$

$$
= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)
$$

$$
\times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \Gamma\left(d_1, \frac{1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left\langle d_1-\frac{\mathscr{T}_1}{i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]\right\rangle\right)
$$

$$
+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}
$$

$$
\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}
$$

$$
\times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \Gamma\left(m, \frac{1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left\langle m-\frac{\mathscr{T}_1}{i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)-\frac{\mathscr{T}_2}{i}\sum_{j=m-i_2+1}^{m}(\Re_j+1)\right\rangle\right)
$$

$$
+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}}
$$

$$
\times \Gamma\left(d_2, \frac{1}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left\langle d_2-\frac{\mathscr{T}_2}{i}\Re^*_{d_2-i+1}\right\rangle\right),
$$

and

$$
P_{\lambda_2}(\hat{\lambda}_2 > w)
$$

$$
= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathscr{D}_1=d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}} \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)
$$

$$
\times q_1^{\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)} \Gamma\left(d_1, \frac{1}{d_1-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left\langle d_1-\frac{\mathscr{T}_1}{d_1-i}\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]\right\rangle\right)
$$

$$
+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathscr{D}_1=d_1,\mathscr{D}_2=m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}
$$

$$
\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1},\cdots,\Re_m+1)q_1^{\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}
$$

$$
\times q_2^{\sum_{j=m-i_2+1}^{m}(\Re_j+1)} \Gamma\left(m, \frac{1}{m-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left\langle m-\frac{\mathscr{T}_1}{m-i}\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)-\frac{\mathscr{T}_2}{m-i}\sum_{j=m-i_2+1}^{m}(\Re_j+1)\right\rangle\right)
$$

$$
+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathscr{D}_2=d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_2}} \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1+1,\cdots,\Re_{d_2}+1)q_2^{\Re^*_{d_2-i+1}}
$$

$$
\times \Gamma\left(d_2, \frac{1}{d_2-i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left\langle d_2-\frac{\mathscr{T}_2}{d_2-i}\Re^*_{d_2-i+1}\right\rangle\right),
$$

respectively, where w is the arbitrary constant, $<x>=\max\{x,0\}$ and $\Gamma(a,b)=\int_b^\infty(1/(a-1)!)x^{a-1}e^{-x}dx$.

The proofs of Theorems are given in Appendices A and B. Based on Corollary 3, we derive $100(1-\alpha)\%$ ConfI of $\hat{\lambda}_k$. Here, when the other parameter is fixed, we suppose that $P(\hat{\lambda}_k > w)$ is an increasing function of λ_k. Then, we can derive the ConfP for λ_k, denoted by $(\lambda_k^{(L)}, \lambda_k^{(U)})$, satisfying the following equation with $\hat{\lambda}_k^{(obs)}$ being the observed value of $\hat{\lambda}_k$:

$$
P_{\lambda_k^{(L)}}(\hat{\lambda}_k^{(L)} > \hat{\lambda}_k^{(obs)}) = \frac{\alpha}{2}, \quad P_{\lambda_k^{(U)}}(\hat{\lambda}_k^{(U)} > \hat{\lambda}_k^{(obs)}) = 1 - \frac{\alpha}{2}, \quad k = 1, 2.
$$

3. Data Analysis and Simulation Results

3.1. Data Analysis

In order to analyze the illustrative example data, we use the data in Ref. [14]. The full data are presented in Table 1. This data was analyzed by Refs. [11,15,16]. From the data (Table 1), $\delta_i = 1$ denotes the failure of the i-th unit caused by 9th RiF, and $\delta_i = 0$ denotes the failure of the i-th unit caused by other RiFs. Here, we suppose that this data follow the ExDist based on the PrTy2CS (i.e., $n = 36$, $m = 24$, $\Re_1 = 6$, $\Re_{24} = 6$ and $\Re_i = 0$ for $i = 2, \ldots, 23$). Then, PrTy2CS data are presented in Table 2.

Table 1. Full data for illustrative example.

x_i	11	35	49	170	329	381	708	958	1062
δ_i	0	0	0	0	0	0	0	0	0
x_i	1167	1594	1925	1990	2223	2327	2400	2451	2471
δ_i	1	0	1	1	1	0	1	0	1
x_i	2551	2565	2831	2568	2694	2702	2761	3034	3059
δ_i	1	0	1	1	0	0	0	1	0
x_i	3112	3214	3478	3504	4329	6367	6976	7846	13403
δ_i	1	1	1	1	1	0	1	1	0

Table 2. PrTy2CS data for illustrative example.

x_i	11	35	49	170	329	381	708	958
δ_i	0	0	0	0	0	0	0	0
$\Re_i$	6	0	0	0	0	0	0	0
x_i	1062	1167	1594	1925	1990	2223	2327	2400
δ_i	0	1	0	1	1	1	0	1
$\Re_i$	0	0	0	0	0	0	0	0
x_i	2451	2471	2551	2565	2568	2694	2702	2761
δ_i	0	1	1	0	1	1	0	0
$\Re_i$	0	0	0	0	0	0	0	6

We set three GeTy2PrHCS (Case (a): $\mathcal{T}_1 = 3000$ and $\mathcal{T}_2 = 4000$, Case (b): $\mathcal{T}_1 = 2000$ and $\mathcal{T}_2 = 3000$, and Case (c): $\mathcal{T}_1 = 2000$ and $\mathcal{T}_1 = 2500$). Table 3 presents the each 95% ConfIs for $\hat{\lambda}_1$ and $\hat{\lambda}_2$, and we have contained the standard error (StE) and MSE calculated from Corollary 1–3. Each PDF of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ based on the illustrative example data is shown in Figure 2.

Table 3. Inference of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ for illustrative example.

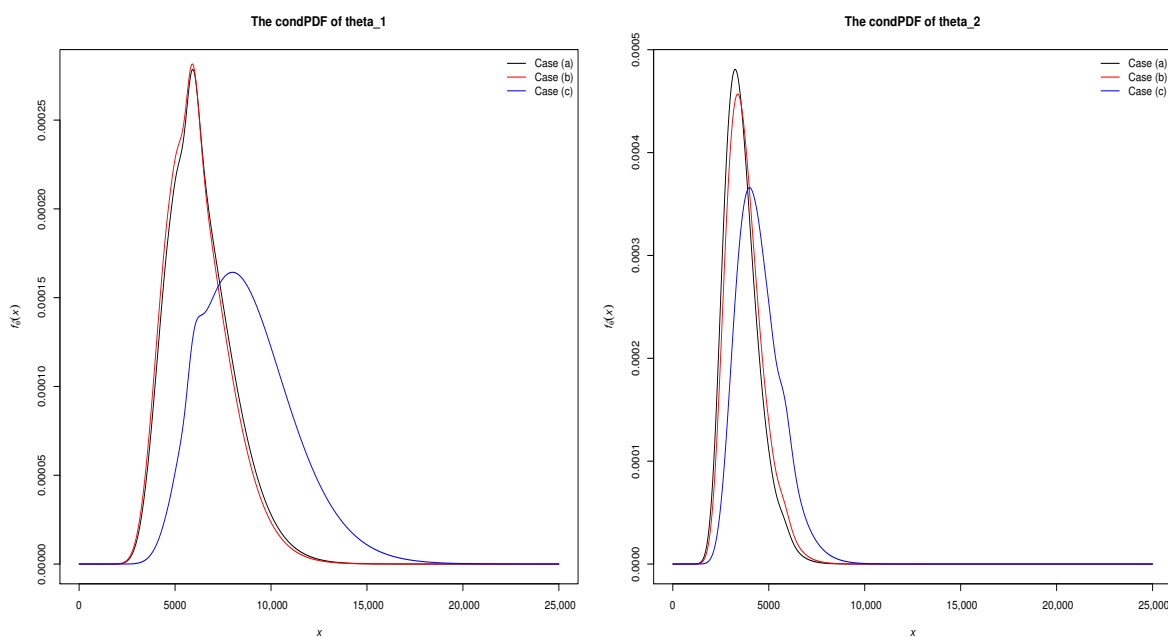| $\mathcal{T}_1$	$\mathcal{T}_2$	n_1	n_2	$\hat{\lambda}_1$	SE($\hat{\lambda}_1$)	95% ConfI for $\hat{\lambda}_1$
				$\hat{\lambda}_2$	SE($\hat{\lambda}_2$)	95% ConfI for $\hat{\lambda}_2$
3000	4000	9	16	6221.000	2073.667	(3236.839, 11,956.370)
				3499.312	874.828	(2143.771, 5711.985)
2000	3000	9	15	6080.444	2026.815	(3163.707, 11,686.230)
				3648.267	941.978	(2199.395, 6051.596)
2000	2500	6	12	8719.500	3559.721	(3917.271, 19,408.840)
				4359.750	1258.551	(2475.919, 7676.917)

Figure 2. The CondPDFs of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ for example.

3.2. Simulation Results

In this Section, we consider various GeTy2PrHCS. First of all, we have used four PrTy2CS as; Sch (a): $\Re_m = n - m$ and $\Re_1 = \Re_2 = \cdots = \Re_{m-1} = 0$, Sch (b): $\Re_1 = n - m$ and $\Re_2 = \Re_3 = \cdots = \Re_m = 0$, Sch (c): $\Re_1 = \Re_m = (n - m)/2$ and $\Re_2 = \Re_3 = \cdots = \Re_{m-1} = 0$, Sch (d): $\Re_{m/2} = n - m$ and $\Re_1 = \Re_2 = \cdots = \Re_{m/2-1} = \Re_{m/2+1} = \cdots = \Re_m = 0$.

First of all, for four PrTy2CS, we generate PrTy2CS data. We generate new RV $\boldsymbol{U} = (u_1, u_2, \cdots, u_n)$. Now if $u_i < \theta_1/(\theta_1 + \theta_2)$, then assign $\delta_i = 1$, otherwise $\delta_i = 0$. Then, the corresponding GeTy2PrHCS CompD is $\{(x_{1:m:n}, \delta_1), (x_{2:m:n}, \delta_2), \cdots, (x_{d_1:n}, \delta_{d_1})\}$ for Case (a) $(x_{m:m:n} < \mathscr{T}_1)$, $\{(x_{1:m:n}, \delta_1), (x_{2:m:n}, \delta_2), \cdots, (x_{m:m:n}, \delta_m)\}$ for Case (b) $(\mathscr{T}_1 < x_{m:m:n} < \mathscr{T}_2)$, and $\{(x_{1:m:n}, \delta_1), (x_{2:m:n}, \delta_2), \cdots, (x_{d_2:m:n}, \delta_{d_2})\}$ for Case (c) $(\mathscr{T}_2 < x_{m:m:n})$. Without loss of generality, we take $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$ in each case. We replicate the process 1000 times in each GeTy2PrHCS. We calculate the rMSEs, biases, ConfL and CovP of the estimator. The simulation results are presented in Figure 3 (Table 4) and Figure 4 (Table 5).

From Tables 4 and 5, the MSEs and ConfL increase as n decreases. For fixed n, the MSEs and ConfL increase generally as the number of m decreases. For fixed n and m, the rMSEs and ConfL increases generally as the pre-fixed time $\mathscr{T}_2$ decreases. From Figures 3 and 4, it can be seen that the concentration becomes thicker. Therefore, we can make it easy to grasp the simulation results. The estimaor for Sch (a) has smaller rMSE and ConfL than the corresponding estimaor for the other three Sch (a, b, c). We can observe that the ConfI works well for all GeTy2PrHCS.

Frome Table 4, it can be seen that $\hat{\lambda}_1$ is more precise compared to the $\hat{\lambda}_2$ in terms of the rMSE and bias. It is because, when λ_2 is bigger than λ_1, we may observe smaller failure number due to the 2nd RiF than those due to the 1st RiF. Therefore, $\hat{\lambda}_1$ is more efficient than $\hat{\lambda}_2$.
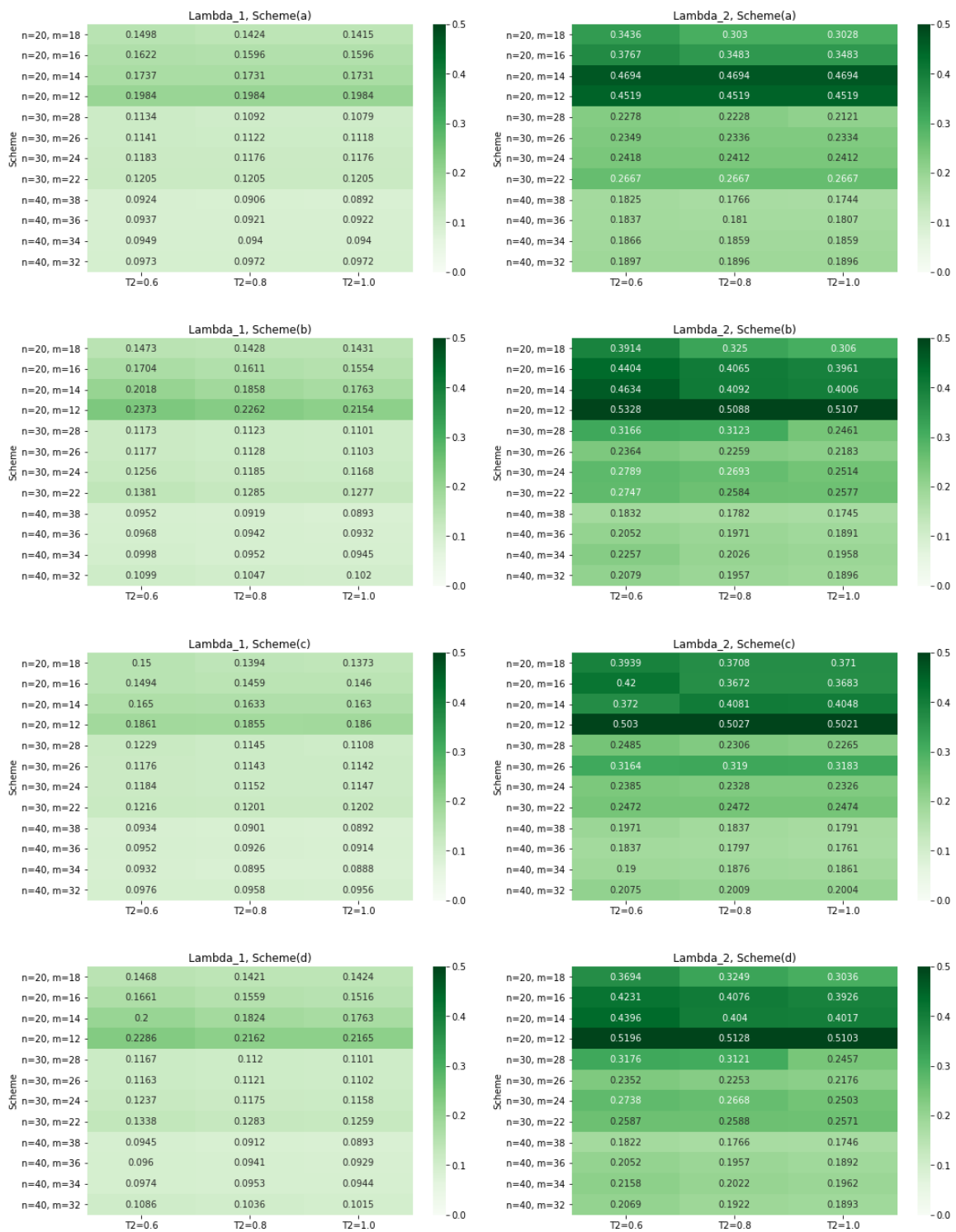
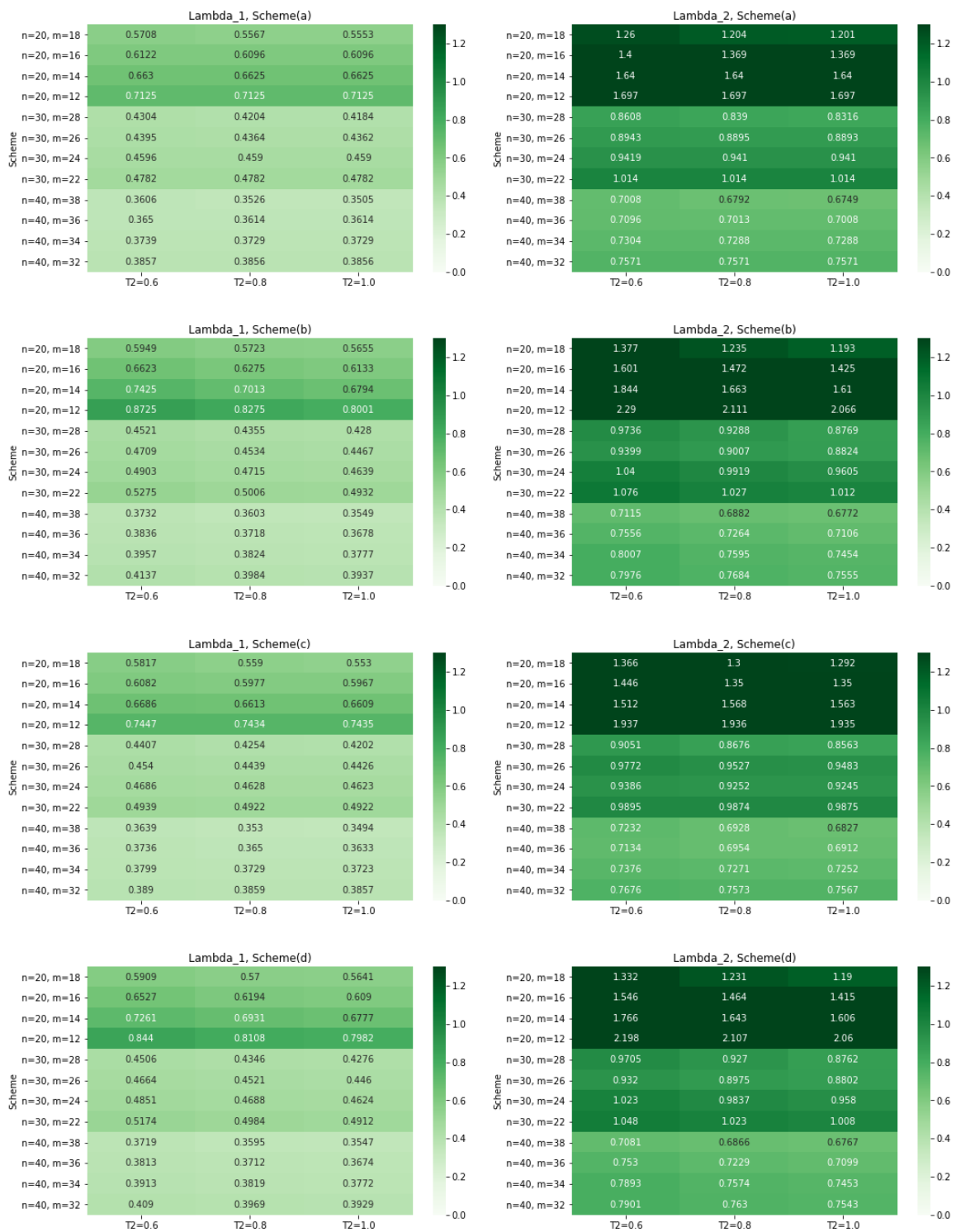**Figure 3.** Relative rMSEs for $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

Figure 4. Relative ConfL for $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

Table 4. Relative rMSEs and biases for $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

n	$\mathscr{T}_1$	m	Scheme	$\mathscr{T}_2 = 0.6$		$\mathscr{T}_2 = 0.8$		$\mathscr{T}_2 = 1.0$	
				$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
20	0.3	18	(a)	0.1498(0.0233) [†]	0.3436(0.0678)	0.1424(0.0198)	0.3030(0.0606)	0.1415(0.0195)	0.3028(0.0601)
			(b)	0.1473(0.0224)	0.3914(0.0792)	0.1428(0.0203)	0.3250(0.0616)	0.1431(0.0205)	0.3060(0.0560)
			(c)	0.1500(0.0240)	0.3939(0.0891)	0.1394(0.0196)	0.3708(0.0797)	0.1373(0.0182)	0.3710(0.0790)
			(d)	0.1468(0.0217)	0.3694(0.0733)	0.1421(0.0198)	0.3249(0.0610)	0.1424(0.0201)	0.3036(0.0555)
		16	(a)	0.1622(0.0276)	0.3767(0.0819)	0.1596(0.0268)	0.3483(0.0786)	0.1596(0.0268)	0.3483(0.0786)
			(b)	0.1704(0.0313)	0.4404(0.0958)	0.1611(0.0256)	0.4065(0.0813)	0.1554(0.0232)	0.3961(0.0762)
			(c)	0.1494(0.0211)	0.4200(0.0794)	0.1459(0.0189)	0.3672(0.0659)	0.1460(0.0187)	0.3683(0.0661)
			(d)	0.1661(0.0300)	0.4231(0.0891)	0.1559(0.0234)	0.4076(0.0799)	0.1516(0.0219)	0.3926(0.0743)
		14	(a)	0.1737(0.0346)	0.4694(0.1370)	0.1731(0.0344)	0.4694(0.1037)	0.1731(0.0344)	0.4694(0.1037)
			(b)	0.2018(0.0393)	0.4634(0.1251)	0.1858(0.0342)	0.4092(0.1059)	0.1763(0.0302)	0.4006(0.1004)
			(c)	0.1650(0.0279)	0.3720(0.0814)	0.1633(0.0258)	0.4081(0.0906)	0.1630(0.0257)	0.4048(0.0897)
			(d)	0.2000(0.0365)	0.4396(0.1154)	0.1824(0.0329)	0.4040(0.1038)	0.1763(0.0301)	0.4017(0.1003)
		12	(a)	0.1984(0.0437)	0.4519(0.1122)	0.1984(0.0437)	0.4519(0.1122)	0.1984(0.0437)	0.4519(0.1122)
			(b)	0.2373(0.0486)	0.5328(0.1392)	0.2262(0.0441)	0.5088(0.1223)	0.2154(0.0401)	0.5107(0.1183)
			(c)	0.1861(0.0338)	0.5030(0.1113)	0.1855(0.0336)	0.5027(0.1114)	0.1860(0.0336)	0.5021(0.1111)
			(d)	0.2286(0.0443)	0.5196(0.1302)	0.2162(0.0415)	0.5128(0.1228)	0.2165(0.0401)	0.5103(0.1180)
30	0.3	28	(a)	0.1134(0.0126)	0.2278(0.0358)	0.1092(0.0103)	0.2228(0.0315)	0.1079(0.0097)	0.2121(0.0301)
			(b)	0.1173(0.0182)	0.3166(0.0582)	0.1123(0.0161)	0.3123(0.0517)	0.1101(0.0146)	0.2461(0.0463)
			(c)	0.1229(0.0144)	0.2485(0.0483)	0.1145(0.0116)	0.2306(0.0430)	0.1108(0.0103)	0.2265(0.0404)
			(d)	0.1167(0.0182)	0.3176(0.0581)	0.1120(0.0158)	0.3121(0.0515)	0.1101(0.0145)	0.2457(0.0462)
		26	(a)	0.1141(0.0111)	0.2349(0.0390)	0.1122(0.0101)	0.2336(0.0381)	0.1118(0.0100)	0.2334(0.0380)
			(b)	0.1177(0.0177)	0.2364(0.0398)	0.1128(0.0150)	0.2259(0.0358)	0.1103(0.0141)	0.2183(0.0333)
			(c)	0.1176(0.0172)	0.3164(0.0565)	0.1143(0.0152)	0.3190(0.0517)	0.1142(0.0149)	0.3183(0.0505)
			(d)	0.1163(0.0166)	0.2352(0.0387)	0.1121(0.0148)	0.2253(0.0353)	0.1102(0.0140)	0.2176(0.0328)
		24	(a)	0.1183(0.0130)	0.2418(0.0433)	0.1176(0.0127)	0.2412(0.0430)	0.1176(0.0127)	0.2412(0.0430)
			(b)	0.1256(0.0165)	0.2789(0.0582)	0.1185(0.0140)	0.2693(0.0518)	0.1168(0.0129)	0.2514(0.0465)
			(c)	0.1184(0.0151)	0.2385(0.0382)	0.1152(0.0137)	0.2328(0.0356)	0.1147(0.0135)	0.2326(0.0353)
			(d)	0.1237(0.0160)	0.2738(0.0555)	0.1175(0.0134)	0.2668(0.0502)	0.1158(0.0126)	0.2503(0.0462)
		22	(a)	0.1205(0.0138)	0.2667(0.0547)	0.1205(0.0138)	0.2667(0.0547)	0.1205(0.0138)	0.2667(0.0547)
			(b)	0.1381(0.0208)	0.2747(0.0479)	0.1285(0.0148)	0.2584(0.0423)	0.1277(0.0139)	0.2577(0.0411)
			(c)	0.1216(0.0183)	0.2472(0.0396)	0.1201(0.0178)	0.2472(0.0394)	0.1202(0.0179)	0.2474(0.0394)
			(d)	0.1338(0.0186)	0.2587(0.0432)	0.1283(0.0147)	0.2588(0.0424)	0.1259(0.0133)	0.2571(0.0403)
40	10.0	38	(a)	0.0924(0.0084)	0.1825(0.0237)	0.0906(0.0072)	0.1766(0.0191)	0.0892(0.0065)	0.1744(0.0179)
			(b)	0.0952(0.0102)	0.1832(0.0183)	0.0919(0.0081)	0.1782(0.0161)	0.0893(0.0070)	0.1745(0.0145)
			(c)	0.0934(0.0076)	0.1971(0.0300)	0.0901(0.0059)	0.1837(0.0237)	0.0892(0.0049)	0.1791(0.0207)
			(d)	0.0945(0.0099)	0.1822(0.0173)	0.0912(0.0077)	0.1766(0.0156)	0.0893(0.0070)	0.1746(0.0143)
		36	(a)	0.0937(0.0079)	0.1837(0.0230)	0.0921(0.0068)	0.1810(0.0208)	0.0922(0.0069)	0.1807(0.0206)
			(b)	0.0968(0.0101)	0.2052(0.0293)	0.0942(0.0093)	0.1971(0.0253)	0.0932(0.0096)	0.1891(0.0222)
			(c)	0.0952(0.0097)	0.1837(0.0180)	0.0926(0.0081)	0.1797(0.0149)	0.0914(0.0078)	0.1761(0.0142)
			(d)	0.0960(0.0098)	0.2052(0.0297)	0.0941(0.0093)	0.1957(0.0242)	0.0929(0.0095)	0.1892(0.0221)
		34	(a)	0.0949(0.0076)	0.1866(0.0239)	0.0940(0.0072)	0.1859(0.0235)	0.0940(0.0072)	0.1859(0.0235)
			(b)	0.0998(0.0100)	0.2257(0.0370)	0.0952(0.0083)	0.2026(0.0293)	0.0945(0.0081)	0.1958(0.0271)
			(c)	0.0932(0.0095)	0.1900(0.0239)	0.0895(0.0070)	0.1876(0.2019)	0.0888(0.0067)	0.1861(0.0214)
			(d)	0.0974(0.0088)	0.2158(0.0350)	0.0953(0.0086)	0.2022(0.0293)	0.0944(0.0081)	0.1962(0.0274)
		32	(a)	0.0973(0.0081)	0.1897(0.0257)	0.0972(0.0080)	0.1896(0.0258)	0.0972(0.0080)	0.1896(0.0258)
			(b)	0.1099(0.0130)	0.2079(0.0236)	0.1047(0.0104)	0.1957(0.0212)	0.1020(0.0104)	0.1896(0.0197)
			(c)	0.0976(0.0080)	0.2075(0.0261)	0.0958(0.0072)	0.2009(0.0231)	0.0956(0.0071)	0.2004(0.0229)
			(d)	0.1086(0.0121)	0.2069(0.0232)	0.1036(0.0103)	0.1922(0.0199)	0.1015(0.0102)	0.1893(0.0197)

[†] rMSE(Bias).

Table 5. Relative ConfL and CovP for $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

| n | $\mathscr{T}_1$ | m | Scheme | $\mathscr{T}_2 = 0.6$ | | $\mathscr{T}_2 = 0.8$ | | $\mathscr{T}_2 = 1.0$ | |
				$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
20	0.3	18	(a)	0.5708(940.1) [†]	10.2596(960.1)	0.5567(940.1)	10.2036(960.0)	0.5553(940.0)	10.2010(950.9)
			(b)	0.5949(940.7)	10.3765(940.7)	0.5723(940.8)	10.2353(950.1)	0.5655(940.5)	10.1928(950.3)
			(c)	0.5817(940.9)	10.3657(950.3)	0.5590(950.2)	10.3001(950.6)	0.5530(950.2)	10.2922(950.2)
			(d)	0.5909(940.7)	10.3323(950.0)	0.5700(940.8)	10.2307(950.1)	0.5641(940.5)	10.1896(950.3)
		16	(a)	0.6122(940.0)	10.3998(950.5)	0.6096(940.1)	10.3694(950.5)	0.6096(940.1)	10.3694(950.5)
			(b)	0.6623(940.8)	10.6005(950.7)	0.6275(940.5)	10.4725(950.4)	0.6133(940.5)	10.4249(950.5)
			(c)	0.6082(950.1)	10.4458(940.6)	0.5977(950.1)	10.3503(950.2)	0.5967(950.2)	10.3503(950.2)
			(d)	0.6527(940.7)	10.5460(950.4)	0.6194(940.7)	10.4641(950.5)	0.6090(940.6)	10.4145(950.6)
		14	(a)	0.6630(940.8)	10.6396(950.7)	0.6625(940.8)	10.6396(950.7)	0.6625(940.8)	10.6396(950.7)
			(b)	0.7425(950.0)	10.8444(950.4)	0.7013(950.0)	10.6625(950.2)	0.6794(950.3)	10.6101(950.1)
			(c)	0.6686(940.5)	10.5124(950.1)	0.6613(930.7)	10.5683(940.9)	0.6609(930.8)	10.5634(950.0)
			(d)	0.7261(940.9)	10.7664(950.2)	0.6931(940.7)	10.6427(950.0)	0.6777(950.3)	10.6064(950.1)
		12	(a)	0.7125(940.5)	10.6972(960.1)	0.7125(940.5)	10.6972(960.1)	0.7125(940.5)	10.6972(960.1)
			(b)	0.8725(940.8)	20.2903(940.6)	0.8275(950.0)	20.1114(940.6)	0.8001(940.8)	20.0659(940.4)
			(c)	0.7447(940.5)	10.9367(960.0)	0.7434(940.6)	10.9358(960.0)	0.7435(940.6)	10.9347(960.0)
			(d)	0.8440(940.8)	20.1976(940.7)	0.8108(940.9)	20.1065(940.6)	0.7982(940.8)	20.0598(940.6)
30	0.3	28	(a)	0.4304(950.5)	0.8608(950.7)	0.4204(950.7)	0.8390(950.6)	0.4184(950.6)	0.8316(950.6)
			(b)	0.4521(940.6)	0.9736(950.7)	0.4355(940.2)	0.9288(950.2)	0.4280(940.6)	0.8769(950.2)
			(c)	0.4407(940.0)	0.9051(950.0)	0.4254(940.1)	0.8676(940.5)	0.4202(940.0)	0.8563(940.7)
			(d)	0.4506(940.4)	0.9705(950.4)	0.4346(940.3)	0.9270(950.4)	0.4276(940.7)	0.8762(950.3)
		26	(a)	0.4395(950.5)	0.8943(950.2)	0.4364(950.4)	0.8895(950.2)	0.4362(950.4)	0.8893(950.3)
			(b)	0.4709(950.6)	0.9399(950.9)	0.4534(950.4)	0.9007(950.6)	0.4467(950.4)	0.8824(950.7)
			(c)	0.4540(940.5)	0.9772(960.1)	0.4439(940.2)	0.9527(950.6)	0.4426(940.2)	0.9483(950.7)
			(d)	0.4664(950.5)	0.9320(950.8)	0.4521(950.5)	0.8975(950.8)	0.4460(950.3)	0.8802(950.7)
		24	(a)	0.4596(950.7)	0.9419(940.7)	0.4590(950.8)	0.9410(940.8)	0.4590(950.8)	0.9410(940.8)
			(b)	0.4903(940.1)	10.0400(940.7)	0.4715(940.7)	0.9919(940.7)	0.4639(940.7)	0.9605(940.9)
			(c)	0.4686(940.7)	0.9386(940.7)	0.4628(940.7)	0.9252(940.6)	0.4623(940.7)	0.9245(940.5)
			(d)	0.4851(940.2)	10.0230(940.8)	0.4688(940.6)	0.9837(940.8)	0.4624(940.6)	0.9580(940.9)
		22	(a)	0.4782(950.7)	10.0145(950.1)	0.4782(950.7)	10.0145(950.1)	0.4782(950.7)	10.0145(950.1)
			(b)	0.5275(940.0)	10.0762(940.1)	0.5006(930.8)	10.0271(940.8)	0.4932(930.8)	10.0124(940.4)
			(c)	0.4939(960.2)	0.9895(950.9)	0.4922(960.2)	0.9874(950.8)	0.4922(960.2)	0.9875(950.8)
			(d)	0.5174(930.9)	10.0478(940.6)	0.4984(930.8)	10.0232(940.5)	0.4912(930.9)	10.0082(940.5)
40	10.0	38	(a)	0.3606(940.9)	0.7008(940.2)	0.3526(950.0)	0.6792(940.0)	0.3505(940.9)	0.6749(940.0)
			(b)	0.3732(950.0)	0.7115(930.8)	0.3603(940.9)	0.6882(940.2)	0.3549(940.9)	0.6772(940.3)
			(c)	0.3639(960.0)	0.7232(930.8)	0.3530(940.7)	0.6928(940.6)	0.3494(940.7)	0.6827(940.5)
			(d)	0.3719(940.6)	0.7081(930.9)	0.3595(940.6)	0.6866(940.1)	0.3547(940.8)	0.6767(940.2)
		36	(a)	0.3650(940.1)	0.7096(930.6)	0.3614(940.0)	0.7013(930.5)	0.3614(930.9)	0.7008(930.4)
			(b)	0.3836(940.8)	0.7556(930.9)	0.3718(940.5)	0.7264(930.6)	0.3678(940.7)	0.7106(930.7)
			(c)	0.3736(940.8)	0.7134(930.8)	0.3650(950.1)	0.6954(930.7)	0.3633(940.8)	0.6912(930.8)
			(d)	0.3813(950.0)	0.7530(930.3)	0.3712(940.6)	0.7229(930.9)	0.3674(940.8)	0.7099(930.8)
		34	(a)	0.3739(940.5)	0.7304(940.4)	0.3729(940.7)	0.7288(940.6)	0.3729(940.7)	0.7288(940.6)
			(b)	0.3957(940.2)	0.8007(940.8)	0.3824(950.3)	0.7595(950.1)	0.3777(940.8)	0.7454(950.0)
			(c)	0.3799(950.6)	0.7376(940.4)	0.3729(960.1)	0.7271(940.5)	0.3723(960.0)	0.7252(940.6)
			(d)	0.3913(940.4)	0.7893(940.6)	0.3819(950.3)	0.7574(950.0)	0.3772(940.9)	0.7453(950.1)
		32	(a)	0.3857(940.7)	0.7571(940.3)	0.3856(940.7)	0.7571(940.3)	0.3856(940.7)	0.7571(940.3)
			(b)	0.4137(940.3)	0.7976(940.4)	0.3984(940.5)	0.7684(940.3)	0.3937(940.3)	0.7555(940.6)
			(c)	0.3890(950.1)	0.7676(930.9)	0.3859(940.9)	0.7573(940.0)	0.3857(950.0)	0.7567(940.0)
			(d)	0.4090(940.2)	0.7901(940.3)	0.3969(940.3)	0.7630(940.5)	0.3929(940.2)	0.7543(940.6)

[†] ConfL (CovP).

4. Conclusions

In many situations of survival and reliability tests, the withdrawal of units from the test is pre-planned in order to to free up testing facilities for other tests, or to save cost and time. Recently, Ref. [6] suggest a GeTy2PrHCS. It is known that more than one RiF may be present at the same time. Therefore, we derive inference for CompRiM with GeTy2PrHCS ExDist data. We derive the CondMgf of the $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of ExDist and the resulting ConfI under GeTy2PrHCS. Consequently, for fixed n (sample size) and m (PrTy2CS sample size), the rMSEs and ConfL increase as the time $\mathscr{T}_2$ decreases. $\hat{\lambda}_1$ is more efficient compared to the $\hat{\lambda}_2$ in terms of the rMSE and bias. Although we focused on the inference for CompRiM with GeTy2PrHCS ExDist data, the suggested GeTy2PrHCS CompRiM can be extended to other distributions. In these cases, the exact condPDF of the MLEs under GeTy2PrHCS CompRiM cannot be explicitly obtained.

Author Contributions: Conceptualization, K.L. and S.C.; Software, K.L.; Supervision, K.L.; Writing—original draft preparation, K.L.; Writing—review and editing, K.L. and S.C.; Visualization, S.C.; Funding acquisition, K.L. Both authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

RiFs	Risk factors
CompRiM	Competing risks model
GeTy2PrHCS	generalized type II progressive hybrid censoring
CondMgf	Conditional moment generating function
ExDist	Exponential distribution
ConfI	Confidence interval
Ty1CS	Type I censoring scheme
Ty2CS	Type II censoring scheme
PrTy2CS	Progressive type II censoring scheme
Ty2PrHCS	Type II progressive hybrid censoring scheme
CompD	Competing risk data
rMSE	Root mean squared error
ConfL	Confidence length
CovP	Coverage percentage
CDF	Cumulative distribution function
jPDF	Joint probability density function
jDist	Joint distribution
OS	Order statistics
condPDF	Conditional probability density function
StE	Standard error
$X_{i:m:n}$	i-th failure time under progressive censoring scheme
$(\Re_1, ... \Re_m)$	Progressive censoring scheme
$\{\delta_1, \delta_2, \cdots, \delta_n\}$	The indicator of risk factor cause corresponding to the data
$\mathscr{D}_1$	The number of observed failures up to time $\mathscr{T}_1$
$\mathscr{D}_2$	The number of observed failures up to time $\mathscr{T}_2$

Appendix A. Proof of Theorem 1

Conditional on $\zeta^{(u)}$, the condMgf of $\hat{\lambda}_1$ is given by

$$
\begin{aligned}
M_{\hat{\lambda}_1}(t) =& E\left(e^{t\hat{\lambda}_1}|\zeta^{(u)}\right) \\
=& \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \zeta^{(d_1)}\right) P(\mathscr{D}_1 = d_1) \\
&+ \sum_{d_1=0}^{m-1} E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, \zeta^{(m)}\right) P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m) \\
&+ \sum_{d_2=1}^{m-1} E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_2 = d_2, \zeta^{(d_2)}\right) P(\mathscr{D}_2 = d_2).
\end{aligned}
\tag{A1}
$$

For convenience, let us denote the subset of indicator of failure causes as $\mathcal{Q}_u^*$, where

$$
\mathcal{Q}_u^* = \{\boldsymbol{\Delta} = (\delta_1, \cdots, \delta_u) : \delta_i = 0 \text{ or } 1; i = 1, \cdots, u\}.
$$

(1) Case I ($\mathscr{D}_1 = m, m+1, \cdots, n$): Conditional on $\mathscr{D}_1 = d_1, n_1 = i$, the joint distribution (jDist) of order statistics (OS) $x_{1:m:n} < \cdots < x_{m:m:n} < x_{m+1:n} < \cdots < x_{d_1:n} < \mathscr{T}_1$ has the form

$$
\begin{aligned}
& f(x_{1:m:n}, \cdots, x_{m:m:n}, x_{m+1:n}, \cdots, x_{d_1:n}|\mathscr{D}_1 = d_1, n_1 = i) \\
&= \frac{1}{P(\mathscr{D}_1 = d_1, n_1 = i)} f(x_{1:m:n}, \cdots, x_{m:m:n}, x_{m+1:n}, \cdots, x_{d_1:n}; \mathscr{D}_1 = d_1, \sum_{j=1}^{d_1} z_j = i) \\
&= \frac{1}{P(\mathscr{D}_1 = d_1, n_1 = i)} \sum_{Z_{d_1} \in \mathcal{Q}_{d_1}^*, \sum_{j=1}^{d_1} \delta_j = i} \cdots \sum f((x_{1:m:n}, \delta_1), \cdots, (x_{m:m:n}, \delta_m) \cdots, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x_{d_1+1:n}, \delta_{d_1}); \mathscr{D}_1 = d_1) \\
&= \frac{1}{P(\mathscr{D}_1 = d_1, n_1 = i)} \sum_{Z_{d_1} \in \mathcal{Q}_{d_1}^*, \sum_{j=1}^{d_1} \delta_j = i} \cdots \sum \varsigma_{d_1}' \prod_{j=1}^{d_1} f_{\mathbf{X}, \boldsymbol{\Delta}}(x_{j:m:n}, 1)^{\delta_j} f_{X, \Delta}(x_{j:m:n}, 2)^{1-\delta_j} \\
&\quad \times \left[1 - \mathscr{F}(x_{j:m:n})\right]^{\Re_j} \left[1 - \mathscr{F}(\mathscr{T}_1)\right]^{\Re'_{\mathscr{D}_1+1}} \\
&= \frac{\varsigma_{d_1}'}{P(\mathscr{D}_1 = d_1, n_1 = i)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i} \lambda_2^i}{(\lambda_1 + \lambda_2)^{d_1}} \\
&\quad \times \exp\left[-\left(\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\right)^{-1} \sum_{j=1}^{d_1} (1 + \Re_j) x_{j:m:n} + \mathscr{T}_1 \Re'_{\mathscr{D}_1+1}\right].
\end{aligned}
$$

Upon the conditional PDF (CondPDF) obtained above, we can readily have

$$
\begin{aligned}
& E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \zeta^{(d_1)}\right) \\
&= \sum_{i=1}^{d_1-1} E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, n_1 = i\right) P(n_1 = i|\zeta^{(d_1)}, \mathscr{D}_1 = d_1) \\
&= \sum_{i=1}^{d_1-1} \frac{\varsigma_{d_1}'}{P(\mathscr{D}_1 = d_1, n_1 = i)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i} \lambda_2^i}{(\lambda_1 + \lambda_2)^{d_1}} q_1^{\Re'_{\mathscr{D}_1+1}\left(1 - \frac{t}{i}\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\right)} \\
&\quad \times \int_0^{\mathscr{T}_1} \cdots \int_0^{x_{2:m:n}} \prod_{j=1}^{d_1} f(x_{j:m:n})[1 - F(x_{j:m:n})]^{(1+\Re_j)\left(1 - \frac{t}{i}\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}\right) - 1} dx_{1:m:n} \cdots dx_{d_1:n}.
\end{aligned}
$$

From Lemma 1 with $\eta_j = \left(1 + \Re_j\right)\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)$ and then factor $\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)$ out of all of the η_j's, $E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \zeta^{(d_1)}\right)$ can be easily simplified as

$$\sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\mathscr{D}_1 = d_1, n_1 = i)}\binom{d_1}{i}\frac{\lambda_1^{d_1-i}\lambda_2^i}{(\lambda_1+\lambda_2)^{d_1}}\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_1}$$
$$\times \sum_{j=0}^{d_1}\varsigma_{j,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1)q_1^{\left(1-\frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\left[\Re'_{\mathscr{D}_1+1}+\kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]}. \quad \text{(A2)}$$

(2) Case II $(\mathscr{D}_1 = 0, \cdots, m-1, \mathscr{D}_2 = m)$: Conditional on $\mathscr{D}_1 = 1, \cdots, m-1, \mathscr{D}_2 = m$ and $n_1 = i$, the jDist of OS $x_{1:m:n} < \cdots < x_{d_1:m:n} < \mathscr{T}_1 < \cdots < x_{m:m:n} < \mathscr{T}_2$ has the form

$$f(x_{1:m:n}, \cdots, x_{m:m:n}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)$$
$$= \frac{1}{P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)}f(x_{1:m:n}, \cdots, x_{m:m:n}; \mathscr{D}_1 = d_1, \mathscr{D}_2 = m, \sum_{j=1}^m \delta_j = i)$$
$$= \frac{1}{P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)}\sum_{Z_m \in \mathscr{Q}_m^*, \sum_{j=1}^m \delta_j = i}f((x_{1:m:n}, \delta_1), \cdots,$$
$$(x_{m:m:n}, \delta_m); \mathscr{D}_1 = d_1, \mathscr{D}_2 = m)$$
$$= \frac{\varsigma'_m}{P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)}\binom{m}{i}\frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}\exp\left[-\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-1}\sum_{j=1}^m(1+\Re_j)x_{j:m:n}\right].$$

Then, immediately we have

$$E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, \zeta^{(m)}\right)$$
$$= \sum_{i=1}^{m-1}E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i\right)P(n_1 = i|\zeta^{(m)}, \mathscr{D}_1 = d_1, \mathscr{D}_2 = m)$$
$$= \sum_{i=1}^{m-1}\frac{\varsigma'_m}{P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)}\binom{m}{i}\frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1+\lambda_2)^m}$$
$$\times \int_{\mathscr{T}_1}^{\mathscr{T}_2}\cdots\int_{x_{m-1:m:n}}^{\mathscr{T}_2}\int_0^{\mathscr{T}_1}\cdots\int_0^{x_{2:m:n}}\prod_{j=1}^m f(x_{j:m:n})[1 - F(x_{j:m:n})]^{(1+\Re_j)\left(1-\frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)-1}$$
$$\times dx_{1:m:n}\cdots dx_{d_1:m:n}dx_{m:m:n}\cdots dx_{d_1+1:m:n}.$$

From Lemma 1 with $\eta_j = \left(1 + \Re_j\right)\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)$ and then factor $\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)$ out of all of the η_j's, $E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, \zeta^{(m)}\right)$ can be easily simplified as

$$\sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1 + \lambda_2)^m}$$

$$\times \sum_{i_1=0}^{d_1} \varsigma_{i_1,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1) q_1^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)}$$

$$\times \int_{\mathscr{T}_1}^{\mathscr{T}_2} \cdots \int_{x_{m-1:m:n}}^{\mathscr{T}_2} \prod_{j=d_1+1}^{m} f(x_{j:m:n})[1 - F(x_{j:m:n})]^{(1+\Re_j)\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)-1} dx_{m:m:n} \cdots dx_{d_1+1:m:n}$$

$$= \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\mathscr{D}_1 = d_1, \mathscr{D}_2 = m, n_1 = i)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^i}{(\lambda_1 + \lambda_2)^m} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}\right)^{-m}$$

$$\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1)\varsigma_{i_2,m-d_1}(\Re_{d_1+1} + 1, \cdots, \Re_m + 1)$$

$$\times q_1^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)} q_2^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=m-i_2+1}^{m}(\Re_j+1)}. \tag{A3}$$

Here, Equation (A3) is obtained by the integration process on the basis of identity that

$$\int_{\mathscr{T}_1}^{\mathscr{T}_2} \cdots \int_{x_{m-2}}^{\mathscr{T}_2} \int_{x_{m-1}}^{\mathscr{T}_2} \prod_{j=d_1+1}^{m} f(x_j)\{1 - \mathscr{F}(x_j)\}^{\eta_j-1} dx_m \cdots dx_{d_1+1}$$

$$= \sum_{i=0}^{m-d_1} \varsigma_{i,m-d_1}(\boldsymbol{\eta_{m-d_1}})\{1 - \mathscr{F}(\mathscr{T}_1)\}^{\sum_{j=d_1+1}^{m-i}\eta_j}\{1 - \mathscr{F}(\mathscr{T}_2)\}^{\kappa_{i,m-d_1}}(\boldsymbol{\eta_{m-d_1}}),$$

where $\boldsymbol{\eta_{m-d_1}} = (\eta_{d_1+1}, \cdots, \eta_m)$.

(3) Case III ($\mathscr{D}_2 = 1, \cdots, m - 1$): Conditional on $\mathscr{D}_2 = 1, \cdots, m - 1, n_1 = i$, the jDist of OS $x_{1:m:n} < \cdots < x_{d_2:m:n} < \mathscr{T}_2$ has the form

$$f(x_{1:m:n}, \cdots, x_{d_2:m:n}|\mathscr{D}_2 = d_2, n_1 = i)$$

$$= \frac{1}{P(\mathscr{D}_2 = d_2, n_1 = i)} f(x_{1:m:n}, \cdots, x_{d_2:m:n}; \mathscr{D}_2 = d_2, \sum_{j=1}^{d_2} \delta_j = i)$$

$$= \frac{1}{P(\mathscr{D}_2 = d_2, n_1 = i)} \sum_{Z_{d_2} \in \mathscr{Q}_{d_2}^*, \sum_{j=1}^{d_2}\delta_j=i} \cdots \sum f((x_{1:m:n}, \delta_1), \cdots, (x_{d_2:m:n}, \delta_{d_2}); \mathscr{D}_2 = d_2)$$

$$= \frac{\varsigma'_{d_2}}{P(\mathscr{D}_2 = d_2, n_1 = i)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1 + \lambda_2)_2^d} \exp\left[-\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-1}\sum_{j=1}^{d_2}(1 + \Re_j)x_{j:m:n} + \mathscr{T}_2 R'_{\mathscr{D}_2+1}\right].$$

Then, we have

$$E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_2 = d_2, \zeta^{(d_2)}\right)$$

$$= \sum_{i=1}^{d_2-1} E\left(e^{t\hat{\lambda}_1}|\mathscr{D}_2 = d_2, n_1 = i\right) P(n_1 = i|\zeta^{(d_2)}, \mathscr{D}_2 = d_2)$$

$$= \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\mathscr{D}_2 = d_2, n_1 = i)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^i}{(\lambda_1 + \lambda_2)_2^d} q_2^{R'_{d_2+1}\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)}$$

$$\times \int_0^{\mathscr{T}_2} \cdots \int_0^{x_{2:m:n}} \prod_{j=1}^{d_2} f(x_{j:m:n})[1 - F(x_{j:m:n})]^{(1+\Re_j)\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)-1} dx_{1:m:n} \cdots dx_{d_2:m:n}.$$

From Lemma 1 with $\eta_j = (1 + \Re_j)\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)$ and then factor $\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)$ out of all of the η_j's, $E\left(e^{t\hat{\lambda}_1}|\mathcal{D}_2 = d_2, \zeta^{(d_2)}\right)$ can be easily simplified as

$$
\sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\mathcal{D}_2 = d_2, n_1 = i)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^{i}}{(\lambda_1 + \lambda_2)_2^d} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_2}
$$
$$
\times \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1 + 1, \cdots, \Re_{d_2} + 1) q_2^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\Re^*_{d_2-i+1}}. \tag{A4}
$$

The theorem then follows readily upon substituting Equations (A2)–(A4) into (A1).

Appendix B. Proof of Theorem 3

From Theorem 1, the CondMgf of $\hat{\lambda}_1$ is given by

$$
M_{\hat{\lambda}_1}(t) = E\left(e^{t\hat{\lambda}_1}|\zeta^{(u)}\right)
$$
$$
= \sum_{d_1=m}^{n-(\Re_1+\cdots+\Re_{m-1})} \sum_{i=1}^{d_1-1} \frac{\varsigma'_{d_1}}{P(\zeta^{(d_1)}|\mathcal{D}_1 = d_1)} \binom{d_1}{i} \frac{\lambda_1^{d_1-i}\lambda_2^{i}}{(\lambda_1 + \lambda_2)^{d_1}} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_1}
$$
$$
\times \sum_{j=0}^{d_1} \varsigma_{j,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1) q_1^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\left[\Re'_{\mathcal{D}_1+1} + \kappa_{j,d_1}(\Re_1+1,\cdots,\Re_{d_1}+1)\right]}
$$
$$
+ \sum_{d_1=0}^{m-1} \sum_{i=1}^{m-1} \frac{\varsigma'_m}{P(\zeta^{(m)}|\mathcal{D}_1 = d_1, \mathcal{D}_2 = m)} \binom{m}{i} \frac{\lambda_1^{m-i}\lambda_2^{i}}{(\lambda_1 + \lambda_2)^m} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-m}
$$
$$
\times \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{m-d_1} \varsigma_{i_1,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1) \varsigma_{i_2,m-d_1}(\Re_{d_1+1}, \cdots, \Re_m + 1)
$$
$$
\times q_1^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=d_1-i_1+1}^{d_1}(\Re_j+1)} q_2^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\sum_{j=m-i_2+1}^{m}(\Re_j+1)}
$$
$$
+ \sum_{d_2=1}^{m-1} \sum_{i=1}^{d_2-1} \frac{\varsigma'_{d_2}}{P(\zeta^{(d_2)}|\mathcal{D}_2 = d_2)} \binom{d_2}{i} \frac{\lambda_1^{d_2-i}\lambda_2^{i}}{(\lambda_1 + \lambda_2)^{d_2}} \left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)^{-d_2}
$$
$$
\times \sum_{j=0}^{d_2} \varsigma_{j,d_2}(\Re_1 + 1, \cdots, \Re_{d_2} + 1) q_2^{\left(1 - \frac{t}{i}\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\right)\Re^*_{d_2-i+1}}.
$$

From Lemma 2, $(1 - t\lambda_1\lambda_2/\{i(\lambda_1 + \lambda_2)\})^{-d_1} \exp(\mathscr{T}_1[\Re'_{\mathcal{D}_1+1} + \kappa_{j,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1)]/i)$ is the Mgf of RV X at t, where X is a gamma RV with shape parameter d_1, rate parameter $\lambda_1\lambda_2/\{i(\lambda_1 + \lambda_2)\}$ and shift parameter $\mathscr{T}_1[\Re'_{\mathcal{D}_1+1} + \kappa_{j,d_1}(\Re_1 + 1, \cdots, \Re_{d_1} + 1)]/i$. Therefore, the theorem readily follows.

References

1. Alshenawy, R.; Al-Alwan, A.; Almetwally, E.M.; Afify, A.Z.; Almongy, H.M. Progressive type-II censoring schemes of extended odd Weibull exponential distribution with applications in medicine and engineering. *Mathematics* **2020**, *8*, 1679. [CrossRef]
2. Jha, M.K.; Dey, S.; Alotaibi, R.M.; Tripathi, Y.M. Reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive Type II censoring. *Qual. Reliab. Eng. Int.* **2020**, *36*, 965–987. [CrossRef]
3. Chen, S.; Gui, W. Statistical analysis of a lifetime distribution with a bathtub-shaped failure rate function under adaptive progressive type-II censoring. *Mathematics* **2020**, *8*, 670. [CrossRef]
4. Helu, A.; Samawi, H.; Rochani, H.; Yin, J.; Vogel, R. Kernel density estimation based on progressive type-II censoring. *J. Korean Stat. Soc.* **2020**, *49*, 475–498. [CrossRef]
5. Lee, H.; Lee, K. Exact likelihood inference for an exponential parameter under generalized adaptive progressive hybrid censoring. *Symmetry* **2020**, *12*, 1149. [CrossRef]
6. Lee, K.; Sun, H.; Cho, Y. Exact likelihood inference of the exponential parameter under generalized Type II progressive hybrid censoring. *J. Korean Stat. Soc.* **2016**, *45*, 123–136. [CrossRef]

7. Mahto, A.K.; Lodhi, C.; Tripathi, Y.M.; Wang, L. Inference for partially observed competing risks model for Kumaraswamy distribution under generalized progressive hybrid censoring. *J. Appl. Stat.* **2021**, 1–29. [CrossRef]

8. Ahmed, E.A.; Ali Alhussain, Z.; Salah, M.M.; Haj Ahmed, H.; Eliwa, M.S. Inference of progressively type-II censored competing risks data from Chen distribution with an application. *J. Appl. Stat.* **2020**, *47*, 2492–2524. [CrossRef]

9. Azizi, F.; Haghighi, F.; Tabibi Gilani, N. Statistical inference for competing risks model under progressive interval censored Weibull data. *Commun. Stat. Simul. Comput.* **2020**, *49*, 1931–1944. [CrossRef]

10. Liu, S.; Gui, W. Estimating the Parameters of the Two-Parameter Rayleigh Distribution Based on Adaptive Type II Progressive Hybrid Censored Data with Competing Risks. *Mathematics* **2020**, *8*, 1783. [CrossRef]

11. Cho, Y.; Lee, K. Exact Inference for an Exponential Parameter under Generalized Adaptive Progressive Hybrid Censored Competing Risks Data. *Symmetry* **2020**, *12*, 2005. [CrossRef]

12. Balakrishnan, N.; Childs, A.; Chandrasekar, B. An efficient computational method for moments of order statistics under progressive censoring. *Stat. Probab. Lett.* **2002**, *60*, 359–365. [CrossRef]

13. Johnson, N.L.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*; Wiley: New York, NY, USA, 1994.

14. Lawless, J. *Statistical Models and Methods for Lifetime Data*; Wiley: Hoboken, NJ, USA, 2011.

15. Mao, S.; Shi, Y.M.; Sun, Y.D. Exact inference for competing risks model with generalized type I hybrid censored exponential data. *J. Stat. Comput. Simul.* **2014**, *84*, 2506–2521. [CrossRef]

16. Cho, Y.; Lee, K. Exact inference for competing risks model with generalized progressive hybrid censored exponential data. *J. Korean Data Anal. Soc.* **2017**, *19*, 565–575. [CrossRef]