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Fuzzy Differential Subordinations Based upon the Mittag-Leffler Type Borel Distribution

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Abstract: In this paper, we investigate several fuzzy differential subordinations that are connected with the Borel distribution series $\mathcal{B}(\lambda,\alpha,\beta)(z)$ of the Mittag-Leffler type, which involves the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$. Using the above-mentioned operator $\mathcal{B}(\lambda,\alpha,\beta)$, we also introduce and study a class $\mathcal{M}^F_{\lambda,\alpha,\beta}(\eta)$ of holomorphic and univalent functions in the open unit disk Δ. The Mittag-Leffler-type functions, which we have used in the present investigation, belong to the significantly wider family of the Fox-Wright function ${}_p\Psi_q(z)$, whose p numerator parameters and q denominator parameters possess a kind of symmetry behavior in the sense that it remains invariant (or unchanged) when the order of the p numerator parameters or when the order of the q denominator parameters is arbitrarily changed. Here, in this article, we have used such special functions in our study of a general Borel-type probability distribution, which may be symmetric or asymmetric. As symmetry is generally present in most works involving fuzzy sets and fuzzy systems, our usages here of fuzzy subordinations and fuzzy membership functions potentially possess local or non-local symmetry features.

Keywords: holomorphic functions; analytic functions; univalent functions; fuzzy differential subordination; fuzzy best dominant; Mittag-Leffler functions; Fox-Wright function; Mittag-Leffler type Borel distribution

MSC: Primary 30C45; Secondary 30A10, 30C20, 33E12

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1. Introduction and Motivation

Our main objective in this article is to investigate several potentially useful results that are based upon second-order fuzzy differential subordinations and their applications in Geometric Function Theory of Complex Analysis and that are intimately connected with the Mittag-Leffler-type Borel distribution series.

The motivation for this investigation was derived from a number of recent works that made use of Borel and other types of probability distributions in the study of such members of the family of holomorphic functions, such as univalent starlike functions and univalent convex functions, which are defined and normalized in the open unit disk in the complex z-plane.

We choose here to mention the works of El-Deeb et al. [1], Murugusundaramoorthy and El-Deeb [2], Srivastava et al. [3], and Wanas and Khuttar [4], in which use was made of the Borel distribution series, involving many different special functions and orthogonal polynomials, in their study of subclasses of normalized holomorphic functions in the open unit disk.

On the other hand, various applications of the concept of fuzzy sets and fuzzy systems in conjunction with the principle of differential subordination between analytic functions can be found in the works of El-Deeb et al. (see [5,6]), Lupaş et al. (see [7–9]), Oros and Oros (see [10–12]), and Wanas [13]. Some other recent publications that are worth mentioning here include those by Eş [14], Laengle et al. [15], Lupaş [16], Oros [17], and Venter [18]. In particular, the recently-published article by Laengle et al. [15] includes an interesting and useful bibliometric and bibliograhic account of notable developments on Fuzzy Sets and Fuzzy Systems over the past 40 years.

The organization of this paper is as follows. In Section 2, we present the definitions and preliminaries that provide the foundation of our paper. Section 3 includes several lemmas that are needed in proving our main results in Section 4. Some corollaries and consequences of our main results are also deduced in Section 4. In Section 5, we present a number of remarks and observations based upon our work. Finally, in the concluding section (Section 6), some potential directions for related further research are presented.

2. Definitions and Preliminaries

Let $\mathbb{C}$ and $\mathbb{N}$ denote the set of complex numbers and the set of positive integers, respectively. For $\Omega \subset \mathbb{C}$, we denote by $\mathcal{H}(\Omega)$ the class of holomorphic functions in Ω.

For $d \in \mathbb{N}$, we denote by $\mathcal{A}_d$ the class of functions defined by

$$\mathcal{A}_d := \left\{ f : f \in \mathcal{H}(\Delta) \quad \text{and} \quad f(z) = z + \sum_{j=d+1}^{\infty} a_j z^j \quad (z \in \Delta;\ d \in \mathbb{N}) \right\},$$

where Δ is the open unit disk given by

$$\Delta := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In particular, we write $\mathcal{A} := \mathcal{A}_1$.

Finally, we let

$$\mathcal{H}[\gamma, d] := \left\{ f : f \in \mathcal{H}(\Delta) \quad \text{and} \quad f(z) = \gamma + \sum_{j=d+1}^{\infty} a_j z^j \quad (z \in \Delta;\ \gamma \in \mathbb{C};\ d \in \mathbb{N}) \right\}$$

and we denote by $\mathcal{S}$, $\mathcal{S}^*$, and $\mathcal{C}$ the classes of functions in $\mathcal{A}$, which are, respectively, univalent, starlike, and convex in Δ, so that, by definition, we have

$$\mathcal{S}^* := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \Delta) \right\}$$

and

$$\mathcal{C} := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \Delta) \right\}.$$

For our present investigation, we need the following definitions.

Definition 1 (see [19–21])**.** *Given two functions f_1 and f_2, which are analytic in Δ, we say that f_1 is subordinate to f_2, denoted by*

$$f_1 \prec f_2 \qquad \textit{or} \qquad f_1(z) \prec f_2(z) \quad (z \in \Delta),$$

provided that a Schwarz function w exisits, which is analytic in Δ and saisfies the condition given by

$$\mathrm{w}(0) = 0 \qquad \textit{and} \qquad |\mathrm{w}(z)| < 1 \quad (\forall\, z \in \Delta),$$

such that

$$f_1(z) = f_2\big(\mathrm{w}(z)\big).$$

Moreover, in the case where f_2 is univalent in Δ, then the following equivalence holds true:

$$f_1(z) \prec f_2(z) \iff f_1(0) = f_2(0) \quad \textit{and} \quad f_1(\Delta) \subset f_2(\Delta).$$

Remark 1. *The widely-applied principle of differential subordination between analytic functions happens to provide an interesting and useful generalization of various inequalities involving complex variables. In fact, the monograph on this subject by Miller and Mocanu* [20] *(see also* [19] *for recent developments on differential subordinations and differential superordinations) is a good source to learn about the theories and applications of differential subordinations and differential superordinations.*

The following definitions and propositions present the notion of *fuzzy differential subordination.*

Definition 2 (see [22])**.** *Assume that the set $\mathcal{X}$ is non-empty, that is, $\mathcal{X} \neq \emptyset$. Then an application $\mathcal{F}_{\mathcal{X}} : \mathcal{X} \to [0,1]$ is called a fuzzy subset of the non-empty set $\mathcal{X}$. More precisely, a pair $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$, where $\mathcal{F}_{\mathcal{B}} : \mathcal{X} \to [0,1]$ and*

$$\mathcal{B} = \{x : x \in \mathcal{X} \quad \textit{and} \quad 0 < \mathcal{F}_{\mathcal{B}}(x) \leqq 1\}, \tag{1}$$

is said to be a fuzzy subset of $\mathcal{X}$. The set $\mathcal{B}$ is referred to as the support of the fuzzy set $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$, written as $\operatorname{supp}(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$, and the set function set function $\mathcal{F}_{\mathcal{B}}$ is called the membership function of the fuzzy set $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$.

Remark 2. *Symmetry type properties and symmetry type features are known to be generally present in most works dealing with fuzzy sets and fuzzy systems. Our usages here of fuzzy subordinations and fuzzy membership functions potentially possess local or non-local symmetric or asymmetric features.*

We now make use of moduli of complex-valued functions in order to introduce and apply the concept of membership functions on the set $\mathbb{C}$ of complex numbers given by

$$z = x + \mathrm{i}y \;\; (x, y \in \mathbb{R}) \qquad \text{and} \qquad |z| = \sqrt{x^2 + y^2} \geqq 0 \;\; (z \in \mathbb{C}).$$

Definition 3 (see [9], p. 120)**.** *Let $F : \mathbb{C} \to \mathbb{R}_+$ be a function such that*

$$F_{\mathbb{C}}(z) = |F(z)| \quad (z \in \mathbb{C}).$$

Denote by

$$F_{\mathbb{C}}(\mathbb{C}) = \{z : z \in \mathbb{C} \quad \textit{and} \quad 0 < |F(z)| \leqq 1\} =: \operatorname{supp}(\mathbb{C}, F_{\mathbb{C}})$$

the fuzzy subset of the set $\mathbb{C}$ of complex numbers. We call the following subset:

$$F_{\mathbb{C}}(\mathbb{C}) = \{z : z \in \mathbb{C} \quad \textit{and} \quad 0 < |F(z)| \leqq 1\} = \Delta_F(0,1)$$

the fuzzy unit disk. It is observed that $(\mathbb{C}, F_{\mathbb{C}})$ is the same as its fuzzy unit disk $\Delta_F(0,1)$.

Proposition 1 (see [9,10])**.** *Each of the following assertions holds true*:
(i) *If $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) = (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, then $\mathcal{B} = \mathcal{U}$, where*

$$\mathcal{B} = \operatorname{supp}(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) \qquad \textit{and} \qquad \mathcal{U} = \operatorname{supp}(\mathcal{U}, \mathcal{F}_{\mathcal{U}}).$$

(ii) *If $(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) \subseteq (\mathcal{U}, \mathcal{F}_{\mathcal{U}})$, then $\mathcal{B} \subseteq \mathcal{U}$, where*

$$\mathcal{B} = \operatorname{supp}(\mathcal{B}, \mathcal{F}_{\mathcal{B}}) \qquad \textit{and} \qquad \mathcal{U} = \operatorname{supp}(\mathcal{U}, \mathcal{F}_{\mathcal{U}}).$$

For $f, g \in H(\Omega)$, we now use the following notations:

$$f(\Omega) = \left\{ f(z) : 0 < |\mathcal{F}_{f(\Omega)} f(z)| \leqq 1 \ \ (z \in \Omega) \right\} = \operatorname{supp}\big(f(\Omega), \mathcal{F}_{f(\Omega)}\big) \tag{2}$$

and

$$g(\Omega) = \left\{ g(z) : \ 0 < |\mathcal{F}_{g(\Omega)} g(z)| \leqq 1 \quad (z \in \Omega) \right\} = \operatorname{supp}\big(g(\Omega), \mathcal{F}_{g(\Omega)}\big). \tag{3}$$

Definition 4 (see [10]). *For a given fixed point $z_0 \in \Omega$, let $f, g \in \mathcal{H}(\Omega)$. Then we say that f is fuzzy subordinate to g, written as $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, provided that*
(i) $f(z_0) = g(z_0)$
and
(ii) $\big|\mathcal{F}_{f(\Omega)} f(z)\big| \leqq \big|\mathcal{F}_{g(\Omega)} g(z)\big| \quad (z \in \Omega)$.

Proposition 2 (see [10]). *Assume that $z_0 \in \Omega$ is a fixed point and the functions $f, g \in \mathcal{H}(\Omega)$. If $f(z) \prec_{\mathcal{F}} g(z)$ $(z \in \Omega)$, then*
(i) $f(z_0) = g(z_0)$
and
(ii) $f(\Omega) \subseteq g(\Omega)$ *and* $|\mathcal{F}_{f(\Omega)} f(z)| \leqq |\mathcal{F}_{g(\Omega)} g(z)|$ $(z \in \Omega)$,
where $f(\Omega)$ and $g(\Omega)$ are defined by (2) and (3), respectively.

Definition 5 (see [11]). *Assume that $\Phi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ and $h \in \mathcal{S}$, with*

$$\Phi(\alpha, 0, 0; 0) = h(0) = \alpha.$$

Let the function p be analytic in Δ, with $p(0) = \alpha$ and satisfy the following second-order fuzzy differential subordination:

$$\left|\mathcal{F}_{\Phi(\mathbb{C}^3 \times \Delta)} \Phi\Big(p(z), zp'(z), z^2 p''(z); z\Big)\right| \leqq |\mathcal{F}_{h(\Delta)} h(z)|,$$

that is,

$$\Phi\Big(p(z), zp'(z), z^2 p''(z); z\Big) \prec_{\mathcal{F}} h(z) \quad (z \in \Delta). \tag{4}$$

Then p is said to be a fuzzy solution of the fuzzy differential subordination. Moreover, if

$$|\mathcal{F}_{p(\Delta)} p(z)| \leqq \Big|\mathcal{F}_{q(\Delta)} q(z)\Big|,$$

that is, if

$$p(z) \prec_{\mathcal{F}} q(z) \qquad (z \in \Delta)$$

for all functions p that satisfy (4), then we say that the univalent function q is a fuzzy dominant of the fuzzy solutions for the fuzzy differential subordination.

A fuzzy dominant $\widetilde{q}$ satisfying the following condition:

$$|\mathcal{F}_{\widetilde{q}(\Delta)} \widetilde{q}(z)| \leqq |\mathcal{F}_{q(\Delta)} q(z)|,$$

that is,

$$\widetilde{q}(z) \prec_{\mathcal{F}} q(z) \qquad (z \in \Delta)$$

for all fuzzy dominants q of (4), is called the fuzzy best dominant of (4).

Remark 3. *In the literature on probability theory, a nice relationship between Poisson processes and Borel distributions can be found. In addition, Borel distributions are also closely related to the Galton-Watson branching processes. Details can be found in, for example, [23,24].*

Various families of linear or convolution operators are known to play important roles in the Geometric Function Theory of Complex Analysis and its related fields. One can indeed express derivative and integral operators as convolutions of some families of analytic functions. This kind of formalism makes further mathematical investigation much easier and also aids in the better understanding of the geometric properties of the operators involved.

We now introduce the familiar Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ are defined, respectively, by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \quad \text{and} \quad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{5}$$

$$\big(z, \alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0\big),$$

which were first considered by Magnus Gustaf (Gösta) Mittag-Leffler (1846–1927) in 1903 and Anders Wiman (1865–1959) in 1905 (see, for details, [25–27]).

The Mittag-Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ are known to contain, as their special cases, a number of elementary functions, such as the exponential, trigonometric, and hyperbolic functions. In fact, these Mittag-Leffler functions happen to be the most commonly-used special cases of the Fox-Wright function ${}_p\Psi_q(z)$ with p numerator parameters and q denominator parameters, which is defined by the following series (see, for example, Ref. [28] p. 67, Equation (1.12.68) and Ref. [29] p. 21, Equation 1.2(38); see also Ref. [30]):

$${}_p\Psi_q\left[\begin{array}{c} (\alpha_1, A_1), \cdots, (\alpha_p, A_p); \\ \\ (\beta_1, B_1), \cdots, (\beta_q, B_q); \end{array} z\right] := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}. \tag{6}$$

Indeed, by comparing the definitions in (5) and (6), it can be seen that

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{array}{c} (1,1); \\ \\ (\beta, \alpha); \end{array} z\right]. \tag{7}$$

Remark 4. *In the vast and widely-scattered literature on mathematical, physical and engineering sciences, one can find infinitely many usages of the celebrated Gauss hypergeometric function* ${}_2F_1$, *the Kummer (or confluent) hypergeometric function* ${}_1F_1$, *the Clausen hypergeometric function* ${}_3F_2$, *and various other mathematical functions of the hypergeometric type, all of which are contained in the generalized hypergeometric function* ${}_pF_q$, *involving* p *numerator parameters* $a_1, \cdots, a_p$ *and* q *denominator parameters* $b_1, \cdots, b_q$, *as special cases (see, for details, [29,31]). The Fox-Wright function* ${}_p\Psi_q(z)$ *defined by (6) does, in fact, provide a further generalization of the generalized hypergeometric function* ${}_pF_q(z)$, *involving* p *numerator parameters* $a_1, \cdots, a_p$ *and* q *denominator parameters* $b_1, \cdots, b_q$, *given by*

$$\begin{aligned} {}_pF_q\left[\begin{array}{c} \alpha_1, , \cdots, \alpha_p; \\ \\ \beta_1, \cdots, \beta_q; \end{array} z\right] &:= \frac{\prod_{j=1}^{q} \Gamma(\beta_j)}{\prod_{j=1}^{p} \Gamma(\alpha_j)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + n)}{\prod_{j=1}^{q} \Gamma(\beta_j + n)} \frac{z^n}{n!} \\ &= {}_p\Psi_q\left[\begin{array}{c} (\alpha_1, 1), \cdots, (\alpha_p, 1); \\ \\ (\beta_1, 1), \cdots, (\beta_q, 1); \end{array} z\right] \end{aligned}$$

The relatively more familiar Bessel-Wright function $J_\nu^\mu(z)$ is also a very specialized case of the Fox-Wright function ${}_p\Psi_q(z)$ defined by (6).

In view of Remark 4, it is clear that almost all of the special functions of hypergeometric class as well as most (if not all) of the Mittag-Leffler-type functions, including those that we have used in our present investigation, belong to the much wider family of the Fox-Wright function ${}_p\Psi_q(z)$, whose p numerator parameters and q denominator parameters possess some kind of symmetry behavior in the sense that the Fox-Wright function ${}_p\Psi_q(z)$ remains invariant (or unchanged) when the order of the numerator parameters or when the order of the denominator parameters is arbitrarily changed.

It is easy to rewrite the second definition in (5) as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=d}^{\infty} \frac{z^{k-d}}{\Gamma\big(\alpha(k-d)+\beta\big)} \qquad (z,\alpha,\beta \in \mathbb{C};\ \Re(\alpha)>0;\ d \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}). \tag{8}$$

In the solutions of many real-world problems, which are modeled by means of fractional-order differential, integral and integro-differential equations, Mittag-Leffler-type functions are known to arise naturally. Some important examples include fractional-order generalizations of the kinetic equation, random walks, Lévy flights, super-diffusive transport and the study of complex systems. Potentially useful properties of the Mittag-Leffler-type functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ can be found in, for example, [28,32–38].

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ does not belong to the normalized analytic function class $\mathcal{A}$. We, therefore, normalize the Mittag-Leffler function $E_{\alpha,\beta}(z)$ as follows:

$$\mathfrak{E}_{\alpha,\beta}(z) := z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{j=d+1}^{\infty} \frac{\Gamma(\beta)}{\Gamma\big(\alpha(j-d)+\beta\big)} z^{j-d+1} \qquad (d \in \mathbb{N}), \tag{9}$$

$\alpha \in \mathbb{C}$ $\big(\Re(\alpha) > 0\big)$ and $\beta, z \in \mathbb{C}$. For convenience, hereafter we only consider the case when the parameters α and β are real-valued and for $z \in \Delta$.

Named after the French mathematician, Félix Édouard Justin Émile Borel (1871–1956), the widely- and extensively-studied *Borel distribution* is a discrete probability distribution that arises in such contexts as (for example) branching processes and queueing theory. We recall that a discrete random variable x defines a Borel distribution if it takes on the values $1, 2, 3, \ldots,$ together with the probabilities given by

$$\frac{e^{-\lambda}}{1!},\ \frac{2\lambda e^{-2\lambda}}{2!},\ \frac{9\lambda^2 e^{-3\lambda}}{3!},\ \cdots,$$

respectively, λ being the parameter of the Borel distribution.

Recently, Wanas and Khuttar [4] applied the Borel distribution (BD) in their study of certain convexity and other geometric properties of analytic functions. Its probability mass function, given by

$$\text{Prob}\{x=\rho\} = \frac{(\rho\lambda)^{\rho-1}\, e^{-\lambda\rho}}{\rho!} \qquad (\rho \in \mathbb{N}),$$

was studied by Wanas and Khuttar [4]. Following the work of Wanas and Khuttar [4], we introduce the following series $\mathcal{M}(\lambda; z)$ in which the coefficients involve probabilities of the Borel distribution (BD):

$$\mathcal{M}(\lambda; z) = z + \sum_{j=d+1}^{\infty} \frac{[\lambda(j-d)]^{j-d-1}\, e^{-\lambda(j-d)}}{(j-d)!} z^{j-d+1} \qquad (0 < \lambda \leqq 1;\ d \in \mathbb{N}). \tag{10}$$

We now recall the following Mittag-Leffler type Borel distribution that was studied by Murugusundaramoorthy and El-Deeb [2] (see also [1,3]):

$$\mathcal{P}(\lambda,\alpha,\beta;\rho)=\frac{(\lambda\rho)^{\rho-1}}{\Gamma(\alpha\rho+\beta)\ E_{\alpha,\beta}(\lambda\rho)}\qquad(\rho\in\mathbb{N}),$$

where the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined in (5). Thus, by using (9) and (10), and by means of the Hadamard product (or convolution), we now define the Mittag-Leffler-type Borel distribution series as follows:

$$\mathcal{B}(\lambda,\alpha,\beta)(z)=z+\sum_{j=d+1}^{\infty}\frac{\Gamma\big(\lambda(j-d)+1\big)[\lambda(j-d)]^{j-d-1}\ e^{-\lambda(j-d)}}{(j-d)!\ E_{\alpha,\beta}\big(\lambda(j-d)\big)\Gamma\big(\alpha(j-d)+\beta\big)}\ z^{j}\qquad(0<\lambda\leqq 1).$$

Moreover, by making use of the Hadamard product (or convolution), for

$$f(z)=z+\sum_{j=d+1}^{\infty}a_j\ z^j\qquad(z\in\Delta),$$

we define

$$\begin{aligned}\mathcal{B}(\lambda,\alpha,\beta)f(z)&:=\mathcal{B}(\lambda,\alpha,\beta)(z)*f(z)\\&=z+\sum_{j=d+1}^{\infty}\frac{\Gamma\big(\lambda(j-d)+1\big)[\lambda(j-d)]^{j-d-1}\ e^{-\lambda(j-d)}}{(j-d)!\ E_{\alpha,\beta}\big(\lambda(j-d)\big)\ \Gamma\big(\alpha(j-d)+\beta\big)}\ a_jz^j\\&=z+\sum_{j=d+1}^{\infty}\phi_ja_jz^j\end{aligned}\tag{11}$$

$$\big(\alpha\in\mathbb{C};\ \Re(\alpha)>0;\ \beta\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ 0<\lambda\leqq 1\big),$$

where $\mathbb{Z}_0^-$ denotes the set of *non-positive* integers and

$$\phi_j=\frac{\Gamma\big(\lambda(j-d)+1\big)[\lambda(j-d)]^{j-d-1}\ e^{-\lambda(j-d)}}{(j-d)!\ E_{\alpha,\beta}\big(\lambda(j-d)\big)\Gamma\big(\alpha(j-d)+\beta\big)}.\tag{12}$$

We have used, in this paper, such special functions as the Mittag-Leffler type functions in our study of a general Borel type probability distribution, which may be conditioned to be symmetric or asymmetric.

3. A Set of Lemmas

Each of the following lemmas will be needed in proving our main results.

Lemma 1 (see [20])**.** *Let $\psi\in\mathcal{A}$ and suppose that*

$$\mathcal{G}(z)=\frac{1}{z}\int_0^z\psi(t)\ \mathrm{d}t\qquad(z\in\Delta).$$

If

$$\Re\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)>-\frac{1}{2}\qquad(z\in\Delta),$$

then $\mathcal{G}\in\mathcal{C}$.

Lemma 2 (see [12] Theorem 2.6)**.** *Let ψ be a convex function with $\psi(0)=\gamma$ and $\nu\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ with $\Re(\nu)\geqq 0$. If $p\in\mathcal{H}[\gamma,d]$ with $p(0)=\gamma$, $\Phi:\mathbb{C}^2\times\Delta\to\mathbb{C}$, the function $\Phi\big(p(z),zp'(z);z\big)=p(z)+\frac{1}{\nu}zp'(z)$ is analytic in Δ, and*

$$\left|\mathcal{F}_{\Phi(\mathbb{C}^2\times\Delta)}\left(p(z)+\frac{1}{\nu}\ zp'(z)\right)\right|\leqq\left|\mathcal{F}_{h(\Delta)}h(z)\right|\implies p(z)+\frac{1}{\nu}\ zp'(z)\prec_{\mathcal{F}}h(z)\qquad(z\in\Delta),$$

then

$$\mathcal{F}_{p(\Delta)}p(z) \leqq \mathcal{F}_{q(\Delta)}q(z) \leqq \mathcal{F}_{h(\Delta)}h(z) \implies p(z) \prec_{\mathcal{F}} q(z) \qquad (z \in \Delta),$$

where

$$q(z) = \frac{\nu}{d\, z^{\frac{\nu}{d}}} \int_0^z \psi(t)\, t^{\frac{\nu}{d}-1}\, \mathrm{d}t (z \in \Delta).$$

The function q is convex in Δ and it is the fuzzy best dominant.

Lemma 3 (see [12] Theorem 2.7)**.** *Let the function g be convex in Δ and suppose that*

$$\psi(z) = g(z) + d\, \gamma z g'(z) \qquad (z \in \Delta;\ d \in \mathbb{N};\ \gamma > 0).$$

If the function p given by

$$p(z) = g(0) + p_d\, z^d + p_{d+1}\, z^{d+1} + \cdots$$

belongs to the class $\mathcal{H}(\Delta)$ and

$$\left|\mathcal{F}_{p(\Delta)}\big(p(z) + \gamma z p'(z)\big)\right| \leqq \left|\mathcal{F}_{\psi(\Delta)}\ \psi(z)\right| \implies p(z) + \gamma z p'(z) \prec_{\mathcal{F}} \psi(z) \qquad (z \in \Delta),$$

then

$$\left|\mathcal{F}_{p(\Delta)}\big(p(z)\big)\right| \leqq \left|\mathcal{F}_{g(\Delta)}g(z)\right| \implies p(z) \prec_{\mathcal{F}} g(z) \qquad (z \in \Delta).$$

This result is sharp, that is, the equality holds true for a suitably specified function.

For the general theory of fuzzy differential subordination and its applications, we refer the reader to the recent works [5–9,13].

In Section 4 below, we obtain several fuzzy differential subordinations that are associated with the operator $\mathcal{B}(\lambda, \alpha, \beta)$ by using the method of fuzzy differential subordination.

4. Main Results and Their Consequences

Throughout this paper, we assume that $\eta \in [0,1)$, $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$, $\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $z \in \Delta$. By using the operator $\mathcal{B}(\lambda, \alpha, \beta)$, we define a new class of normalized analytic functions $\mathcal{M}^F_{\lambda,\alpha,\beta}(\eta)$ for which we derive several fuzzy differential subordinations.

Definition 6. *Let $0 \leqq \eta < 1$. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{M}^F_{\lambda,\alpha,\beta}(\eta)$, if it satisfies the following inequality*:

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)f\right)'(\Delta)}\big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)'\right| > \eta \qquad (z \in \Delta).$$

Theorem 1. *Let the function k be in the normalized convex function class $\mathcal{C}$ on Δ and suppose that*

$$h(z) = k(z) + \frac{1}{\lambda+2}\, zk'(z).$$

If $f \in \mathcal{M}^F_{\lambda,\alpha,\beta}(\eta)$ and

$$G(z) = \mathcal{J}^{\lambda} f(z) = \frac{\lambda+2}{z^{\lambda+1}} \int_0^z t^{\lambda}\, f(t)\, \mathrm{d}t, \tag{13}$$

then the following fuzzy differential subordination:

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)f\right)'(\Delta)}\big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)'\right| \leqq \left|F_{h(\Delta)}h(z)\right| \implies \big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)' \prec_{\mathcal{F}} h(z) \tag{14}$$

implies that

$$\left|F_{(\mathcal{B}(\lambda,\alpha,\beta)G)'(\Delta)}\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)'\right| \leqq \left|F_{k(\Delta)}k(z)\right| \implies \big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)' \prec_{\mathcal{F}} k(z).$$

This result is sharp, that is, the equality holds true for a suitably specified function.

Proof. Since

$$z^{\lambda+1}G(z) = (\lambda+2)\int_0^z t^{\lambda} f(t)\, \mathrm{d}t,$$

by differentiating both sides with respect to z, we obtain

$$(\lambda+1)G(z) + zG'(z) = (\lambda+2)f(z),$$

so that

$$(\lambda+1)\mathcal{B}(\lambda,\alpha,\beta)G(z) + z\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)' = (\lambda+2)\mathcal{B}(\lambda,\alpha,\beta)f(z), \tag{15}$$

which, by differentiating with respect to z, yields

$$\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)' + \frac{1}{\lambda+2}\, z\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)'' = \big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)'. \tag{16}$$

By using (16), the fuzzy differential subordination (14) can be written as follows:

$$\begin{aligned}&\left|F_{(\mathcal{B}(\lambda,\alpha,\beta)f)'(\Delta)}\left(\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)' + \frac{1}{\lambda+2}\, z\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)''\right)\right| \\ &\qquad \leqq \left|F_{h(\Delta)}\Big(k(z) + \frac{1}{\lambda+2}\, zk'(z)\Big)\right|.\end{aligned} \tag{17}$$

We now set

$$q(z) = \big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)' \tag{18}$$

such that $q \in \mathcal{H}[1,n]$. Thus, by substituting from (18) into (17), we have

$$\left|F_{(\mathcal{B}(\lambda,\alpha,\beta)f)'(\Delta)}\Big(q(z) + \frac{1}{\lambda+2}\, zq'(z)\Big)\right| \leqq \left|F_{h(\Delta)}\Big(k(z) + \frac{1}{\lambda+2}\, zk'(z)\Big)\right|. \tag{19}$$

By applying Lemma (3), we find that

$$F_{q(\Delta)}q(z) \leqq F_{k(\Delta)}k(z),$$

that is, that

$$\left|F_{(\mathcal{B}(\lambda,\alpha,\beta)G(z))'(\Delta)}\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)'\right| \leqq \left|F_{k(\Delta)}k(z)\right|.$$

Therefore, we obtain

$$\big(\mathcal{B}(\lambda,\alpha,\beta)G(z)\big)' \prec_{\mathcal{F}} k(z)$$

and k is the fuzzy best dominant. This completes our proof of Theorem 1. □

Theorem 2. *Assume that*

$$h(z) = \frac{1+(2\eta-1)z}{1+z}, \quad \eta \in [0,1) \quad \textit{and} \quad \lambda > 0.$$

Let the operator $\mathcal{I}^{\lambda}$ be given by (13). Then,

$$\mathcal{I}^{\lambda}\left[\mathcal{M}^{F}_{\lambda,\alpha,\beta}(\eta)\right] \subset \mathcal{M}^{F}_{\lambda,\alpha,\beta}(\eta^{*}), \tag{20}$$

where

$$\eta^* := 2\eta - 1 + (\lambda+2)(2-2\eta)\int_0^1 \frac{t^{\lambda+2}}{t+1}\,\mathrm{d}t. \tag{21}$$

Proof. Since the function h belongs to the normalized convex function class $\mathcal{C}$ in Δ, by using the same technique as in the proof of Theorem 1, we find from the hypothesis of Theorem 2 that

$$\left|F_{q(\Delta)}\left(q(z) + \frac{1}{\lambda+2}\,zq'(z)\right)\right| \leqq \left|F_{h(\Delta)}h(z)\right|,$$

where $q(z)$ is defined in (18). Thus, by using Lemma 2, we obtain

$$\left|F_{q(\Delta)}q(z)\right| \leqq \left|F_{k(\Delta)}k(z)\right| \leqq \left|F_{h(\Delta)}h(z)\right|,$$

which implies that

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)G\right)'(\Delta)}\left(\mathcal{B}(\lambda,\alpha,\beta)G(z)\right)'\right| \leqq \left|F_{k(\Delta)}k(z)\right| \leqq \left|F_{h(\Delta)}h(z)\right|,$$

where the function $k(z)$, given by

$$\begin{aligned} k(z) &= \frac{\lambda+2}{z^{\lambda+2}}\int_0^z t^{\lambda+1}\,\frac{1+(2\eta-1)t}{1+t}\,\mathrm{d}t \\ &= (2\eta-1) + \frac{(\lambda+2)(2-2\eta)}{z^{\lambda+2}}\int_0^z \frac{t^{\lambda+1}}{1+t}\,\mathrm{d}t, \end{aligned}$$

belongs to $\mathcal{C}$ on Δ and $k(\Delta)$ is symmetric with respect to the real axis. Consequently, we have

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)G\right)'(\Delta)}\left(\mathcal{B}(\lambda,\alpha,\beta)G(z)\right)'\right| \geqq \min_{|z|=1}\left\{\left|F_{k(\Delta)}k(z)\right|\right\} = \left|F_{k(\Delta)}k(1)\right| \tag{22}$$

and

$$\eta^* = k(1) = 2\eta - 1 + (\lambda+2)(2-2\eta)\int_0^1 \frac{t^{\lambda+2}}{t+1}\,\mathrm{d}t.$$

This evidentally completes the proof of Theorem 2. □

Theorem 3. *Let the function k belong to the normalized convex function class $\mathcal{C}$ in Δ, $k(0) = 1$, and*

$$h(z) = k(z) + zk'(z) \qquad (z \in \Delta).$$

If $f \in \mathcal{A}$ and satisfies the following fuzzy differential subordination:

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)f\right)'(\Delta)}\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)'\right| \leqq \left|F_{h(\Delta)}h(z)\right| \implies \left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)' \prec_{\mathcal{F}} h(z), \tag{23}$$

then

$$\left|F_{\mathcal{B}(\lambda,\alpha,\beta)f(\Delta)}\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}\right| \leqq \left|F_{k(\Delta)}k(z)\right| \implies \frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} \prec_{\mathcal{F}} k(z). \tag{24}$$

The result is sharp, that is, the assertion holds true for a suitably specified function.

Proof. For

$$q(z) = \frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} = \frac{z + \sum\limits_{j=d+1}^{\infty} \frac{\Gamma\left(\lambda(j-d)+1\right)[\lambda(j-d)]^{j-d-1}\, e^{-\lambda(j-d)}}{(j-d)!\, E_{\alpha,\beta}(\lambda(j-d))\Gamma\left(\alpha(j-d)+\beta\right)}\, a_j z^j}{z}$$
$$= 1 + \sum_{j=d+1}^{\infty} \frac{\Gamma\left(\lambda(j-d)+1\right)[\lambda(j-d)]^{j-d-1}\, e^{-\lambda(j-d)}}{(j-d)!\, E_{\alpha,\beta}(\lambda(j-d))\Gamma\left(\alpha(j-d)+\beta\right)}\, a_j z^{j-1},$$

we find that

$$q(z) + zq'(z) = \left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)'.$$

We, thus, see that the following inequality:

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)f\right)'(\Delta)}\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)'\right| \leqq \left|F_{h(\Delta)}h(z)\right|$$

implies that

$$\left|F_{q(\Delta)}\left(q(z)+zq'(z)\right)\right| \leqq \left|F_{h(\Delta)}h(z)\right| = \left|F_{k(\Delta)}\left(k(z)+zk'(z)\right)\right|.$$

Now, by applying Lemma 3, we have

$$\left|F_{q(\Delta)}q(z)\right| \leqq \left|F_{k(\Delta)}k(z)\right| \implies \left|F_{\mathcal{B}(\lambda,\alpha,\beta)f(\Delta)}\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}\right| \leqq \left|F_{k(\Delta)}k(z)\right|,$$

which implies that

$$\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} \prec_{\mathcal{F}} k(z).$$

The result is easily seen to be sharp, that is, the result holds true for a suitably specified function. The proof of Theorem 3 is, thus, completed. □

Theorem 4. *Let $h \in \mathcal{H}(\Delta)$, with $h(0) = 1$, such that*

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2} \qquad (z \in \Delta).$$

If $f \in \mathcal{A}$ and the following fuzzy differential subordination holds true:

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)f\right)'(\Delta)}\left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)'\right| \leqq \left|F_{h(\Delta)}h(z)\right|$$
$$\implies \left(\mathcal{B}(\lambda,\alpha,\beta)f(z)\right)' \prec_{\mathcal{F}} h(z), \tag{25}$$

then

$$\left|F_{\mathcal{B}(\lambda,\alpha,\beta)f(\Delta)}\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}\right| \leqq \left|F_{k(\Delta)}k(z)\right|$$
$$\implies \frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} \prec_{\mathcal{F}} k(z), \tag{26}$$

where the function $k(z)$, given by

$$k(z) = \frac{1}{z}\int_0^z h(t)\,\mathrm{d}t, \tag{27}$$

is convex and it is the fuzzy best dominant.

Proof. Let

$$q(z) = \frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} = 1 + \sum_{j=d+1}^{\infty} \frac{\Gamma\big(\lambda(j-d)+1\big)[\lambda(j-d)]^{j-d-1}\, e^{-\lambda(j-d)}}{(j-d)!\, E_{\alpha,\beta}(\lambda(j-d))\Gamma\big(\alpha(j-d)+\beta\big)}\, a_j z^{j-1}, \tag{28}$$

where $q \in \mathcal{H}[1,1]$. Suppose also that $h \in \mathcal{H}(\Delta)$, with $h(0) = 1$ such that

$$\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2} \qquad (z \in \Delta).$$

From Lemma 1, we have

$$k(z) = \frac{1}{z}\int_0^z h(t)\,\mathrm{d}t,$$

which belongs to the class $\mathcal{C}$ and satisfies the fuzzy differential subordination (25). Since

$$k(z) + zk'(z) = h(z),$$

it is the fuzzy best dominant.

We next observe that

$$q(z) + zq'(z) = \big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)' \qquad (z \in \Delta),$$

so that (25) becomes

$$\left|F_{q(\Delta)}\big(q(z) + zq'(z)\big)\right| \leqq \left|F_{h(\Delta)}h(z)\right|.$$

Thus, by applying Lemma 3, we find that

$$\left|F_{q(\Delta)}q(z)\right| \leqq \left|F_{k(\Delta)}k(z)\right| \implies \left|F_{\mathcal{B}(\lambda,\alpha,\beta)f(\Delta)}\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z}\right| \leqq \left|F_{k(\Delta)}k(z)\right|.$$

Consequently, we obtain

$$\frac{\mathcal{B}(\lambda,\alpha,\beta)f(z)}{z} \prec_{\mathcal{F}} k(z),$$

which completes the proof of Theorem 4. □

Upon setting

$$h(z) = \frac{1+(2\beta-1)z}{1+z} \qquad (z \in \Delta)$$

in Theorem 4, we can deduce the following corollary.

Corollary 1. *Let*

$$h(z) = \frac{1+(2\beta-1)z}{1+z} \qquad (z \in \Delta)$$

be in the normalized convex function class $\mathcal{C}$ in Δ, with $h(0) = 1$ and $0 \leqq \beta < 1$. If the function $f \in \mathbb{A}$ satisfies the following fuzzy differential subordination:

$$\left|F_{\left(\mathcal{B}(\lambda,\alpha,\beta)f\right)'(\Delta)}\big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)'\right| \leqq \left|F_{h(\Delta)}h(z)\right| \implies \big(\mathcal{B}(\lambda,\alpha,\beta)f(z)\big)' \prec_{\mathcal{F}} h(z), \tag{29}$$

then the function $k(z)$, given by

$$k(z) = 2\beta - 1 + \frac{2(1-\beta)}{z}\log(1+z), \tag{30}$$

is convex and is the fuzzy best dominant.

5. Further Remarks and Observations

In our present investigation, we derived several results involving fuzzy differential subordinations that are connected with the Mittag-Leffler-type Borel distribution series given by

$$\mathcal{B}(\lambda,\alpha,\beta)(z) = z + \sum_{j=d+1}^{\infty} \frac{\Gamma\big(\lambda(j-d)+1\big)\ [\lambda(j-d)]^{j-d-1}\ e^{-\lambda(j-d)}}{(j-d)!\ E_{\alpha,\beta}\big(\lambda(j-d)\big)\Gamma\big(\alpha(j-d)+\beta\big)}\ z^j$$

or, equivalently, by

$$\mathcal{B}(\lambda,\alpha,\beta)(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma\big(\lambda(k-1)+1\big)\ [\lambda(k-1)]^{k-2}\ e^{-\lambda(k-1)}}{(k-1)!\ E_{\alpha,\beta}\big(\lambda(k-1)\big)\Gamma\big(\alpha(k-1)+\beta\big)}\ z^{k+d-1},$$

where d is a positive integer and $0 < \lambda \leqq 1$. We successfully applied the above operator $\mathcal{B}(\lambda,\alpha,\beta)$ with a view to introduce and study the class $\mathcal{M}^F_{\lambda,\alpha,\beta}(\eta)$ of holomorphic and univalent functions in the open unit disk Δ. Upon specialization, one of our main results (Theorems 1 to 4) yields an interesting special case, which we have recorded here as a corollary.

Recently, in his survey-cum-expository review article, Srivastava [39] demonstrated how the theories of the basic (or $\mathfrak{q}$-) calculus and the fractional $\mathfrak{q}$-calculus have significantly encouraged and motivated further developments in Geometric Function Theory of Complex Analysis. It is, therefore, worthwhile to reiterate an important observation, which was made in the above-mentioned review-cum-expository review article by Srivastava [39], who pointed out the fact that the basic (or $\mathfrak{q}$-) extensions of the results, which we have presented here, can easily, and almost trivially, be translated into the corresponding results for the so-called $(\mathfrak{p},\mathfrak{q})$-analogues (with $0 < |\mathfrak{q}| < \mathfrak{p} \leqq 1$) by making use of some obvious and straightforward variations of parameters and arguments. This is so, because the additional parameter $\mathfrak{p}$ is *redundant*.

6. Conclusions

In our present investigation of applications of fuzzy differential subordinations in Geometric Function Theory of Complex Analysis, we successfully made use of a general Mittag-Leffler type Borel distribution involving the two-parameter Mittag-Leffler function $M_{\alpha,\beta}(z)$. As we indicated in Remark 3 above, almost all of the higher transcendental functions of the hypergeometric class as well as most (if not all) of the Mittag-Leffler-type functions, including those that we used in this article, belong to the much wider family of the Fox-Wright function ${}_p\Psi_q(z)$.

Consequently, one could possibly generalize the results presented in this paper by analogously using the Borel distribution and other suitable probability distributions with Mittag-Leffler-type functions that are more general than the two-parameter Mittag-Leffler function $M_{\alpha,\beta}(z)$ that we used herein.

Another avenue for further research on this subject is provided by the fact that, in the theory of differential subordinations and differential superordinations, there are differential subordinations and differential superordinations of the third and higher orders as well (see, for details, [20]; see also [19] for recent developments on this subject). In this presentation, we only used and explored the second-order differential subordinations and differential superordinations.

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