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Properties of Certain Subclass of Meromorphic Multivalent Functions Associated with q-Difference Operator

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Abstract: A new subclass $\Sigma_{p,q}(\alpha, A, B)$ of meromorphic multivalent functions is defined by means of a q-difference operator. Some properties of the functions in this new subclass, such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radius of starlikeness and convexity, partial sums and closure theorems, are investigated.

Keywords: q-difference operator; Janowski function; meromorphic multivalent function; distortion theorem; partial sum; closure theorem

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1. Introduction

In recent years, q-analysis has attracted the interest of scholars because of its numerous applications in mathematics and physics. Jackson [1,2] was the first to consider the certain application of q-calculus and introduced the q-analog of the derivative and integral. Very recently, several authors published a set of articles [3–13] in which they concentrated upon the classes of q-starlike functions related to the Janowski functions [14] from some different aspects. Further, a recently published survey-cum-expository review paper by Srivastava [15] is very useful for scholars working on these topics. In this review paper, Srivastava [15] gave certain mathematical explanation and addressed applications of the fractional q-derivative operator in Geometric Function Theory. In the same survey-cum-expository review paper [15], the trivial and inconsequential (p, q) variations of various known q-results by adding an obviously redundant parameter p were clearly exposed (see, for details, [15] p. 340).

In this article, motivated essentially by the above works, we shall define a new subclass of meromorphic multivalent functions by using the q-difference operator and Janowski functions and study its geometric properties, such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radius of starlikeness and convexity, partial sums and closure theorems.

Let M_p denote the class of meromorphic multivalent functions of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

which are analytic in the punctured open unit disk $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = D \setminus \{0\}$ with a pole of order p at the origin.

A function $f(z) \in M_p$ is said to be the meromorphic p-valent starlike function of order σ if

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \sigma \qquad (0 \leqq \sigma < p)$$

for all $z \in D^*$. We denote this class by $MS_p^*(\sigma)$.

A function $f(z) \in M_p$ is said to be the meromorphic p-valent convex function of order σ if

$$\mathrm{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \sigma \quad (0 \leqq \sigma < p)$$

for all $z \in D^*$. We denote this class by $MC_p(\sigma)$.

For two functions, $f(z)$ and $g(z)$, which are analytic in D, we can say that $g(z)$ is subordinate to $f(z)$ and denote $g(z) \prec f(z)$ $(z \in D)$, if there exists a Schwarz function $w(z)$, analytic in D with $w(0) = 0$ and $|w(z)| < 1$ $(z \in D)$, such that $g(z) = f(w(z))$ $(z \in D)$. Further, if $f(z)$ is univalent in D, then we have the following equivalence:

$$g(z) \prec f(z) \quad (z \in D) \Longleftrightarrow g(0) = f(0) \quad \text{and} \quad g(D) \subset f(D).$$

A function $\varphi(z)$ is said to be in the class $P[A, B]$, if it is analytic in D with $\varphi(0) = 1$ and

$$\varphi(z) \prec \frac{1+Az}{1+Bz} \quad (-1 \leqq B < A \leqq 1),$$

equivalently, we can write

$$\left|\frac{\varphi(z)-1}{A-B\varphi(z)}\right| < 1.$$

Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1+q+q^2+\cdots+q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Particularly, when $\lambda = 0$, we write $[0]_q = 0$.

Definition 1. *For $q \in (0, 1)$, the q-difference operator D_q of a function $f(z)$ is defined by*

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases}$$

provided that $f'(0)$ exists.

From Definition 1, we observe that

$$\lim_{q\to 1^-} D_q f(z) = \lim_{q\to 1^-} \frac{f(qz)-f(z)}{(q-1)z} = f'(z)$$

for a differentiable function $f(z)$.

For $f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^n \in M_p$, we can see that

$$D_q f(z) = [-p]_q z^{-p-1} + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \quad (z \neq 0),$$

where $[-p]_q = \frac{1-q^{-p}}{1-q} = -q^{-p}[p]_q$ and $[p]_q = \frac{1-q^{-p}}{1-q} = 1+q+q^2+\cdots+q^{p-1}$.

We now define a new subclass $\Sigma_{p,q}(\alpha, A, B)$ of M_p as the following.

Definition 2. *For $q \in (0, 1)$, $\alpha > 1$ and $-1 \leqq B < A \leqq 1$, a function $f(z) \in M_p$ is said to belong to the class $\Sigma_{p,q}(\alpha, A, B)$, if it satisfies*

$$\frac{1}{1-\alpha}\left(\frac{zD_q f(z)}{[-p]_q f(z)} - \alpha\frac{z^2 D_q^2 f(z)}{[-p]_q[-p-1]_q f(z)}\right) \prec \frac{1+Az}{1+Bz},$$

or equivalently

$$\left| \frac{\frac{zD_qf(z)}{[-p]_qf(z)} - \alpha\frac{z^2D_q^2f(z)}{[-p]_q[-p-1]_qf(z)} - (1-\alpha)}{(1-\alpha)A - B\left(\frac{zD_qf(z)}{[-p]_qf(z)} - \alpha\frac{z^2D_q^2f(z)}{[-p]_q[-p-1]_qf(z)}\right)} \right| < 1 \quad (z \in D). \tag{1}$$

2. Main Results

Theorem 1. *Let* $1 < \alpha \leqq 1 - \frac{1}{[-p]_q}$ *and*

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geqq 0) \in M_p.$$

Then $f(z) \in \Sigma_{p,q}(\alpha, A, B)$ *if*

$$\sum_{n=1}^{\infty}\big((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q)\big)a_n \leqq (1-\alpha)(B-A)[-p]_q[-p-1]_q. \tag{2}$$

Proof. Suppose that the inequality (2) holds true. Then we have

$$\begin{aligned}
&\left| \frac{\frac{zD_qf(z)}{[-p]_qf(z)} - \alpha\frac{z^2D_q^2f(z)}{[-p]_q[-p-1]_qf(z)} - (1-\alpha)}{(1-\alpha)A - B\left(\frac{zD_qf(z)}{[-p]_qf(z)} - \alpha\frac{z^2D_q^2f(z)}{[-p]_q[-p-1]_qf(z)}\right)} \right| \\
&= \left| \frac{[-p-1]_qzD_qf(z) - \alpha z^2D_q^2f(z) - (1-\alpha)[-p]_q[-p-1]_qf(z)}{(1-\alpha)A[-p]_q[-p-1]_qf(z) - B([-p-1]_qzD_qf(z) - \alpha z^2D_q^2f(z))} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty}([n]_q([-p-1]_q - \alpha[n-1]_q) - (1-\alpha)[-p]_q[-p-1]_q)a_nz^n}{(1-\alpha)(A-B)[-p]_q[-p-1]_qz^{-p} - \sum_{n=1}^{\infty}(B[n]_q([-p-1]_q - \alpha[n-1]_q) - (1-\alpha)A[-p]_q[-p-1]_q)a_nz^n} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty}([n]_q([-p-1]_q - \alpha[n-1]_q) - (1-\alpha)[-p]_q[-p-1]_q)a_nz^{n+p}}{(1-\alpha)(A-B)[-p]_q[-p-1]_q - \sum_{n=1}^{\infty}(B[n]_q([-p-1]_q - \alpha[n-1]_q) - (1-\alpha)A[-p]_q[-p-1]_q)a_nz^{n+p}} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty}((1-\alpha)[-p]_q[-p-1]_q - [n]_q([-p-1]_q - \alpha[n-1]_q))a_nz^{n+p}}{(1-\alpha)(B-A)[-p]_q[-p-1]_q - \sum_{n=1}^{\infty}((1-\alpha)A[-p]_q[-p-1]_q - B[n]_q([-p-1]_q - \alpha[n-1]_q))a_nz^{n+p}} \right| \\
&< 1.
\end{aligned}$$

This shows that $f(z) \in \Sigma_{p,q}(\alpha, A, B)$.
Conversely, let $f(z) = z^{-p} + \sum_{n=1}^{\infty} a_nz^n \quad (a_n \geqq 0) \in \Sigma_{p,q}(\alpha, A, B)$. From (1), we obtain

$$\begin{aligned}
&\left| \frac{\frac{zD_qf(z)}{[-p]_qf(z)} - \alpha\frac{z^2D_q^2f(z)}{[-p]_q[-p-1]_qf(z)} - (1-\alpha)}{(1-\alpha)A - B\left(\frac{zD_qf(z)}{[-p]_qf(z)} - \alpha\frac{z^2D_q^2f(z)}{[-p]_q[-p-1]_qf(z)}\right)} \right| \\
&= \left| \frac{\sum_{n=1}^{\infty}((1-\alpha)[-p]_q[-p-1]_q - [n]_q([-p-1]_q - \alpha[n-1]_q))a_nz^{n+p}}{(1-\alpha)(B-A)[-p]_q[-p-1]_q - \sum_{n=1}^{\infty}((1-\alpha)A[-p]_q[-p-1]_q - B[n]_q([-p-1]_q - \alpha[n-1]_q))a_nz^{n+p}} \right| \\
&< 1.
\end{aligned} \tag{3}$$

The inequality (3) is true for all $z \in D^*$. Thus, we choose $z = \text{Re}z \to 1^-$ and obtain the inequality (2). The proof of Theorem 1 is completed. □

From Theorem 1, we can easily obtain the following coefficient estimates.

Corollary 1. *Let* $-1 < B < A \leqq 1$ *and* $1 < \alpha < 1 - \frac{1+B}{(1+A)[-p]_q}$. *If*

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geqq 0) \in \Sigma_{p,q}(\alpha, A, B),$$

then

$$a_n \leqq \frac{(1-\alpha)(B-A)[-p]_q[-p-1]_q}{(1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q)} \quad (n=1,2,\cdots).$$

The results are sharp for the function given by

$$f(z)=z^{-p}+\frac{(1-\alpha)(B-A)[-p]_q[-p-1]_q}{(1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q)}z^n.$$

Theorem 2. *Let* $-1<B<A\leqq 1$ *and* $1<\alpha<1-\frac{1+B}{(1+A)[-p]_q}$. *If*

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_nz^n \qquad (a_n\geqq 0)\in\Sigma_{p,q}(\alpha,A,B),$$

then, for $0<|z|=r<1$, *it is asserted that*

$$\frac{1}{r^p}-r\tau_1\leqq|f(z)|\leqq\frac{1}{r^p}+r\tau_1,$$

where

$$\tau_1=\frac{(1-\alpha)(B-A)[-p]_q}{(1-\alpha)(1+A)[-p]_q-(1+B)}. \tag{4}$$

The results are sharp for the function

$$f(z)=z^{-p}+\frac{(1-\alpha)(B-A)[-p]_q}{(1-\alpha)(1+A)[-p]_q-(1+B)}z.$$

Proof. Let

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_nz^n \qquad (a_n\geqq 0)\in\Sigma_{p,q}(\alpha,A,B).$$

Then, by applying the triangle inequality, we have

$$|f(z)|=\left|z^{-p}+\sum_{n=1}^{\infty}a_nz^n\right|\leqq\frac{1}{|z|^p}+\sum_{n=1}^{\infty}a_n|z|^n.$$

Since $|z|=r<1$, we can see that $r^n\leqq r$. Thus, we have

$$|f(z)\leqq\frac{1}{r^p}+r\sum_{n=1}^{\infty}a_n \tag{5}$$

and

$$|f(z)\geqq\frac{1}{r^p}-r\sum_{n=1}^{\infty}a_n. \tag{6}$$

From Theorem 1, we know that

$$\sum_{n=1}^{\infty}\big((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q\big([-p-1]_q-\alpha[n-1]_q\big)\big)a_n$$
$$\leqq(1-\alpha)(B-A)[-p]_q[-p-1]_q.$$

It is easy to see that the sequence

$$\big\{(1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q\big([-p-1]_q-\alpha[n-1]_q\big)\big\}$$

is an increasing sequence with respect to $n(n \geqq 1)$. Thus,

$$\begin{aligned}&\big((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[-p-1]_q\big)\sum_{n=1}^{\infty}a_n\\&\leqq\sum_{n=1}^{\infty}\big((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q)\big)a_n\\&\leqq(1-\alpha)(B-A)[-p]_q[-p-1]_q,\end{aligned}$$

which shows that

$$\sum_{n=1}^{\infty}a_n\leqq\frac{(1-\alpha)(B-A)[-p]_q}{(1-\alpha)(1+A)[-p]_q-(1+B)}. \tag{7}$$

Substituting from (7) into the inequalities (5) and (6), we obtain the required results. The proof of Theorem 2 is completed. □

Theorem 3. *Let* $-1 < B < A \leqq 1$ *and* $1 < \alpha < 1 - \frac{1+B}{(1+A)[-p]_q}$. *If*

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_nz^n \qquad (a_n\geqq 0)\in\Sigma_{p,q}(\alpha,A,B),$$

then, for $0 < |z| = r < 1$, *it is asserted that*

$$-[-p]_q\frac{1}{r^{p+1}}-\tau_1\leqq|D_qf(z)|\leqq-[-p]_q\frac{1}{r^{p+1}}+\tau_1,$$

where τ_1 *is given by (4).*

Proof. Let

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_nz^n \qquad (a_n\geqq 0)\in\Sigma_{p,q}(\alpha,A,B).$$

Then, from Definition 1, we can write

$$D_qf(z)=[-p]_qz^{-p-1}+\sum_{n=1}^{\infty}[n]_qa_nz^{n-1}.$$

For $|z| = r < 1$, we have

$$|D_qf(z)|=\left|[-p]_qz^{-p-1}+\sum_{n=1}^{\infty}[n]_qa_nz^{n-1}\right|\leqq-[-p]_q\frac{1}{|r|^{p+1}}+\sum_{n=1}^{\infty}[n]_qa_n. \tag{8}$$

Similarly, we obtain

$$|D_qf(z)|\geqq-[-p]_q\frac{1}{r^{p+1}}-\sum_{n=1}^{\infty}[n]_qa_n. \tag{9}$$

Since $f(z) \in \Sigma_{p,q}(\alpha, A, B)$, we know from Theorem 1 that

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q}{[n]_q}-(1+B)([-p-1]_q-\alpha[n-1]_q)\right)[n]_qa_n\\&\leqq(1-\alpha)(B-A)[-p]_q[-p-1]_q.\end{aligned}$$

As we know that the sequence

$$\left\{\frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q}{[n]_q}-(1+B)([-p-1]_q-\alpha[n-1]_q)\right\}$$

is an increasing sequence with respect to n $(n \geqq 1)$. Thus, we have

$$\begin{aligned}
&((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)([-p-1]_q)\sum_{n=1}^{\infty}[n]_q a_n \\
&\leqq \sum_{n=1}^{\infty}\left(\frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q}{[n]_q}-(1+B)([-p-1]_q-\alpha[n-1]_q)\right)[n]_q a_n \\
&\leqq (1-\alpha)(B-A)[-p]_q[-p-1]_q,
\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty}[n]_q a_n \leqq \frac{(1-\alpha)(B-A)[-p]_q}{(1-\alpha)(1+A)[-p]_q-(1+B)}. \tag{10}$$

Now, the theorem is proven. □

Theorem 4. *Let* $1<\alpha<1-\frac{1+B}{(1+A)[-p]_q}$, $-1<B<A\leqq 1$ *and* $0\leqq\sigma<p$. *If*

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_n z^n \quad (a_n\geqq 0)\in\Sigma_{p,q}(\alpha,A,B),$$

then $f(z)$ *is meromorphic p-valent starlike function of order* σ *in* $0<|z|<r_1$, *where*

$$r_1=\min\left\{\inf_{n\geqq 1}\left(\frac{(p-\sigma)((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q))}{(n+\sigma)(1-\alpha)(B-A)[-p]_q[-p-1]_q}\right)^{\frac{1}{n+p}},1\right\}.$$

Proof. In order to prove that $f(z)$ is the meromorphic p-valent starlike function of order σ in $0<|z|<r_1$, we need only to show that

$$\frac{\frac{-zf'(z)}{f(z)}-\sigma}{p-\sigma}\prec\frac{1+z}{1-z}, \quad 0\leqq\sigma<p.$$

The subordination above is equivalent to $\left|\frac{-zf'(z)-pf(z)}{-zf'(z)+(p-2\sigma)f(z)}\right|<1$. After some calculations and simplifications, we have

$$\sum_{n=1}^{\infty}\frac{n+\sigma}{p-\sigma}a_n|z|^{n+p}<1. \tag{11}$$

From (2), we can see that

$$\sum_{n=1}^{\infty}\frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q)}{(1-\alpha)(B-A)[-p]_q[-p-1]_q}a_n<1.$$

The inequality (11) will be true if

$$\frac{n+\sigma}{p-\sigma}a_n|z|^{n+p}<\frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q)}{(1-\alpha)(B-A)[-p]_q[-p-1]_q}a_n$$

or

$$|z|<\left(\frac{(p-\sigma)((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q))}{(n+\sigma)(1-\alpha)(B-A)[-p]_q[-p-1]_q}\right)^{\frac{1}{n+p}}.$$

Let

$$r_1 = \min\left\{ \inf_{n\geqq 1}\left(\frac{(p-\sigma)((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q))}{(n+\sigma)(1-\alpha)(B-A)[-p]_q[-p-1]_q} \right)^{\frac{1}{n+p}}, 1 \right\}.$$

Then, clearly, we obtain the required condition. The proof of Theorem 4 is completed. □

Theorem 5. *Let* $1 < \alpha < 1 - \frac{1+B}{(1+A)[-p]_q}$, $-1 < B < A \leqq 1$ *and* $0 \leqq \sigma < p$. *If*

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geqq 0) \in \Sigma_{p,q}(\alpha, A, B),$$

then $f(z)$ *is the meromorphic* p*-valent convex function of order* σ *in* $0 < |z| < r_2$, *where*

$$r_2 = \min\left\{ \inf_{n\geqq 1}\left(\frac{p(p-\sigma)((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q))}{n(n+\sigma)(1-\alpha)(B-A)[-p]_q[-p-1]_q} \right)^{\frac{1}{n+p}}, 1 \right\}.$$

Proof. To prove that $f(z)$ is the meromorphic p-valent convex function of order σ in $0 < |z| < r_2$, we need only to show that

$$\frac{-\left(1 + \frac{zf''(z)}{f'(z)}\right) - \sigma}{p-\sigma} \prec \frac{1+z}{1-z}, \quad 0 \leqq \sigma < p.$$

This subordination relation is equivalent to the inequality $\left|\frac{-zf''(z)-(p+1)f'(z)}{-zf''(z)+(p-1-2\sigma)f'(z)}\right| < 1$. After some calculations and simplifications, we have

$$\sum_{n=1}^{\infty} \frac{n(n+\sigma)}{p(p-\sigma)} a_n |z|^{n+p} < 1. \tag{12}$$

From the inequality (2), we obtain that

$$\sum_{n=1}^{\infty} \frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q)}{(1-\alpha)(B-A)[-p]_q[-p-1]_q} a_n < 1.$$

The inequality (12) will be true if

$$\frac{n(n+\sigma)}{p(p-\sigma)} a_n |z|^{n+p} < \frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q)}{(1-\alpha)(B-A)[-p]_q[-p-1]_q} a_n,$$

or

$$|z| < \left(\frac{p(p-\sigma)((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q))}{n(n+\sigma)(1-\alpha)(B-A)[-p]_q[-p-1]_q} \right)^{\frac{1}{n+p}}.$$

Let

$$r_2 = \min\left\{ \inf_{n\geqq 1}\left(\frac{p(p-\sigma)((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q))}{n(n+\sigma)(1-\alpha)(B-A)[-p]_q[-p-1]_q} \right)^{\frac{1}{n+p}}, 1 \right\}.$$

Then, we obtain the required condition. Now, Theorem 5 is proven. □

Theorem 6. *Let* $1 < \alpha \leqq \frac{(1+2A-B)[-p]_q - (1+B)}{(1+2A-B)[-p]_q}$ *and* $-1 < B < A \leqq 1$. *If*

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_n \geqq 0) \in \Sigma_{p,q}(\alpha, A, B)$$

and

$$f_k(z) = z^{-p} + \sum_{n=1}^{k} a_n z^n \quad (a_n \geqq 0;\ k \geqq 1),$$

then

$$\mathrm{Re}\left(\frac{f(z)}{f_k(z)}\right) \geqq 1 - \frac{1}{\varphi_{k+1}} \tag{13}$$

and

$$\mathrm{Re}\left(\frac{f_k(z)}{f(z)}\right) \geqq \frac{\varphi_{k+1}}{1+\varphi_{k+1}}, \tag{14}$$

where

$$\varphi_{k+1} = \frac{(1-\alpha)(1+A)[-p]_q[-p-1]_q) - (1+B)[k+1]_q([-p-1]_q - \alpha[k]_q)}{(1-\alpha)(B-A)[-p]_q[-p-1]_q}. \tag{15}$$

Proof. In order to prove the inequality (13), we set

$$\varphi_{k+1}\left[\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{\varphi_{k+1}}\right)\right] = \frac{1 + \sum_{n=1}^{k} a_n z^{n+p} + \varphi_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n+p}}{1 + \sum_{n=1}^{k} a_n z^{n+p}} = \frac{1+w(z)}{1-w(z)}.$$

After some simplifications, we have

$$w(z) = \frac{\varphi_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n+p}}{2 + 2\sum_{n=1}^{k} a_n z^{n+p} + \varphi_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n+p}}$$

and

$$|w(z)| \leqq \frac{\varphi_{k+1}\sum_{n=k+1}^{\infty} a_n}{2 - 2\sum_{n=1}^{k} a_n - \varphi_{k+1}\sum_{n=k+1}^{\infty} a_n}.$$

From (2), we know that $\sum_{n=1}^{\infty} \varphi_n a_n \leqq 1$. The sequence $\{\varphi_n\}$ given by (15) is an increasing sequence with respect to n and $\varphi_n \geqq 1$ $(n = 1, 2, 3, \cdots)$. Therefore,

$$\sum_{n=1}^{k} a_n + \varphi_{k+1}\sum_{n=k+1}^{\infty} a_n \leqq \sum_{n=1}^{k} \varphi_n a_n + \sum_{n=k+1}^{\infty} \varphi_n a_n = \sum_{n=1}^{\infty} \varphi_n a_n \leqq 1.$$

This shows that $|w(z)| < 1 (z \in D)$. Now, the proof of the inequality (13) is completed.

To prove the inequality (14), we put

$$(1+\varphi_{k+1})\left[\frac{f_k(z)}{f(z)} - \frac{\varphi_{k+1}}{1+\varphi_{k+1}}\right] = \frac{1 + \sum_{n=1}^{k} a_n z^{n+p} - \varphi_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n+p}}{1 + \sum_{n=1}^{\infty} a_n z^{n+p}} = \frac{1+w(z)}{1-w(z)}.$$

After some simplifications , we find that

$$w(z) = \frac{-(1+\varphi_{k+1})\sum_{n=k+1}^{\infty} a_n z^{n+p}}{2 + 2\sum_{n=1}^{k} a_n z^{n+p} - (\varphi_{k+1} - 1)\sum_{n=k+1}^{\infty} a_n z^{n+p}}$$

and

$$|w(z)| \leqq \frac{(1+\varphi_{k+1})\sum_{n=k+1}^{\infty} a_n}{2 - 2\sum_{n=1}^{k} a_n - (\varphi_{k+1} - 1)\sum_{n=k+1}^{\infty} a_n}.$$

Now, we can see that $|w(z)| < 1 (z \in D)$ if

$$\sum_{n=1}^{k} a_n + \varphi_{k+1}\sum_{n=k+1}^{\infty} a_n \leqq 1.$$

The proof of Theorem 6 is completed. □

Theorem 7. *Let* $1 < \alpha \leqq 1 - \frac{1}{[-p]_q}$. *If*

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqq 0;\ j = 1,2) \in \Sigma_{p,q}(\alpha, A, B),$$

then, for $0 \leqq \lambda \leqq 1$, *the function* $H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \in \Sigma_{p,q}(\alpha, A, B)$.

Proof. For $0 \leqq \lambda \leqq 1$, we have

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) = z^{-p} + \sum_{n=1}^{\infty} (\lambda a_{n,1} + (1-\lambda) a_{n,2}) z^n.$$

Since $f_j(z)$ $(j = 1,2) \in \Sigma_{p,q}(\alpha, A, B)$, by Theorem 1, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \big((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q)\big)(\lambda a_{n,1} + (1-\lambda) a_{n,2}) \\
&= \lambda \sum_{n=1}^{\infty} \big((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q)\big) a_{n,1} \\
&+ (1-\lambda) \sum_{n=1}^{\infty} \big((1-\alpha)(1+A)[-p]_q[-p-1]_q - (1+B)[n]_q([-p-1]_q - \alpha[n-1]_q)\big) a_{n,2} \\
&\leqq \lambda(1-\alpha)(B-A)[-p]_q[-p-1]_q + (1-\lambda)(1-\alpha)(B-A)[-p]_q[-p-1]_q \\
&= (1-\alpha)(B-A)[-p]_q[-p-1]_q.
\end{aligned}$$

This shows that $H(z) \in \Sigma_{p,q}(\alpha, A, B)$. The theorem is provem. □

Corollary 2. *Let* $1 < \alpha \leqq 1 - \frac{1}{[-p]_q}$. *If*

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqq 0;\ j = 1,2,\cdots,t) \in \Sigma_{p,q}(\alpha, A, B),$$

then the function

$$F(z) = \sum_{j=1}^{t} \lambda_j f_j(z) \in \Sigma_{p,q}(\alpha, A, B),$$

where $\lambda_j \geqq 0$ *and* $\sum_{j=1}^{t} \lambda_j = 1$.

Theorem 8. *Let* $1 < \alpha \leqq 1 - \frac{1}{[-p]_q}$. *If*

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqq 0;\ j = 1,2) \in \Sigma_{p,q}(\alpha, A, B),$$

then, for $-1 \leqq m \leqq 1$, *the function*

$$Q_m(z) = \frac{(1-m) f_1(z) + (1+m) f_2(z)}{2} \in \Sigma_{p,q}(\alpha, A, B).$$

Proof. For $-1 \leqq m \leqq 1$, we have

$$Q_m(z) = \frac{(1-m) f_1(z) + (1+m) f_2(z)}{2} = z^{-p} + \sum_{n=1}^{\infty} \left(\frac{1-m}{2} a_{n,1} + \frac{1+m}{2} a_{n,2} \right) z^n.$$

In view of $f_1(z), f_2(z) \in \Sigma_{p,q}(\alpha, A, B)$, by Theorem 1, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q))\left(\frac{1-m}{2}a_{n,1}+\frac{1+m}{2}a_{n,2}\right)\\
&=\frac{1-m}{2}\sum_{n=1}^{\infty}((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q))a_{n,1}\\
&\quad+\frac{1+m}{2}\sum_{n=1}^{\infty}((1-\alpha)(1+A)[-p]_q[-p-1]_q-(1+B)[n]_q([-p-1]_q-\alpha[n-1]_q))a_{n,2}\\
&\leqq\frac{1-m}{2}(1-\alpha)(B-A)[-p]_q[-p-1]_q+\frac{1+m}{2}(1-\alpha)(B-A)[-p]_q[-p-1]_q\\
&=(1-\alpha)(B-A)[-p]_q[-p-1]_q,
\end{aligned}$$

which shows that $Q_m(z) \in \Sigma_{p,q}(\alpha, A, B)$. The proof of the theorem is completed. □

3. Conclusions

In this article, we introduce a new subclass $\Sigma_{p,q}(\alpha, A, B)$ of meromorphic multivalent functions by using the q-difference operator and Janowski functions. Some geometric properties of functions in $\Sigma_{p,q}(\alpha, A, B)$, such as sufficient and necessary conditions, coefficient estimates, growth and distortion theorems, radius of starlikeness and convexity, partial sums and closure theorems, are studied.

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