


Article

Heisenberg Doubles for Snyder-Type Models

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Abstract: A Snyder model generated by the noncommutative coordinates and Lorentz generators closes a Lie algebra. The application of the Heisenberg double construction is investigated for the Snyder coordinates and momenta generators. This leads to the phase space of the Snyder model. Further, the extended Snyder algebra is constructed by using the Lorentz algebra, in one dimension higher. The dual pair of extended Snyder algebra and extended Snyder group is then formulated. Two Heisenberg doubles are considered, one with the conjugate tensorial momenta and another with the Lorentz matrices. Explicit formulae for all Heisenberg doubles are given.

Keywords: quantum groups; Heisenberg doubles; Snyder model



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1. Introduction

Noncommutative coordinates and noncommutative spacetimes lead to a modification of their corresponding relativistic symmetries, which are described by quantum groups (Hopf algebras). Having a dual pair of Hopf algebras (one describing the quantum symmetry group A and its dual quantum Lie algebra A^*), one can construct the so-called Heisenberg double. The Heisenberg double, although interesting from a mathematical point of view, can also be seen as a generalization of the quantum mechanics phase space (Heisenberg algebra). Such constructions have been of interest especially for the κ -Minkowski noncommutative spacetime and its deformed relativistic symmetry κ -Poincaré quantum group [1–5], as well as for the θ -deformation [5] and for the general Lie algebra-type noncommutative spaces [6].

In the present paper, we focus on the Heisenberg double construction applied to a Snyder model. In the 1940s, Snyder proposed the model of Lorentz invariant discrete spacetime [7] as the first example of the noncommutative spacetime.

The Snyder model has attracted much attention in the literature [8–24]. Field theory on this space was considered, for example, in [8–10], its extension to a cosmological [11] and a curved [12,13] background was proposed; deformed Heisenberg uncertainty relations were investigated [14]; different applications to quantum gravity phenomenology have been considered as well; see, e.g., [11,15,16]. Different Snyder phase spaces arising within the projective geometry context were investigated in [17,18], and the κ -Snyder space with the non-associative star product was proposed in [19,20].

In the Snyder model, the coordinates do not commute, and their commutation relation is proportional to the Lorentz generators. For this reason, noncommutative coordinates by themselves do not close a Lie algebra and cannot be equipped in the Hopf algebra structure. In this paper, we investigated various ways of extending the Snyder space so that we can define the Lie algebra containing the Snyder coordinates. Then, the Hopf algebra structure arises naturally, and the Heisenberg double construction may be attempted.

We start with the algebra generated by Snyder coordinates \hat{x}_i and Lorentz generators M_{jk} , so that it becomes a Lie algebra, and then, we equip its universal enveloping algebra with the Hopf algebra structure, where the Lie algebra generators have primitive

coproducts. For the Heisenberg double construction, we need the dual Hopf algebra. Since the Hopf algebra of Snyder coordinates and Lorentz generators is associative, the dual Hopf algebra should be coassociative. However, we are interested in investigating the possibility of constructing the phase space relations between Snyder coordinates and the commuting momenta p_i with non-coassociative coproducts. We try to follow the steps of the Heisenberg double construction and investigated its limitations in obtaining the commutation relations between the Snyder coordinates and momenta generators. The cross-commutation relations obtained correspond to the known versions of the Snyder phase space used in various applications, e.g., in [12,13,15,16]. Then, the analogous cross-commutation relations are obtained for (non-coassociative) coproducts of momenta in various realizations [7,8].

In the second part of the paper, to overcome the obstacles raised by non-coassociativity and to construct the full dual Hopf algebra and the full Heisenberg double for the Snyder model, we propose using the extended noncommutative Snyder coordinates in the following sections.

We analyze the extended version of the Snyder model, where the Snyder coordinates were identified as $\hat{x}_i \sim \hat{x}_{iN} = M_{iN} \in so(1, N)/so(1, N - 1)$ [9,21]. Thanks to this, we are able to construct two full Heisenberg doubles, firstly for the extended Snyder algebra generated by tensorial coordinates $\hat{x}_{\mu\nu}$ with its dual Hopf algebra generated by tensorial (conjugate) momenta $p_{\rho\sigma}$. This way, we find the Heisenberg double for the extended Snyder space, which may be considered as the extended Snyder phase space. Secondly, we consider another Heisenberg double for the extended Snyder algebra with its dual Hopf algebra of functions on a group $\Lambda_{\rho\sigma}$ (Lorentz matrices). We also present the Weyl realization for these Lorentz matrices in terms of tensorial momenta. We finish the paper with brief conclusions.

2. Issues with the Heisenberg Double for the Snyder Model

Snyder space is defined by the position operators \hat{x}_i obeying the following commutation relations:

$$[\hat{x}_i, \hat{x}_j] = i\beta M_{ij} \quad (1)$$

where M_{ij} are the generators of the Lorentz algebra $so(1, N - 1)$ and β is the Snyder parameter of length square dimension (usually assumed to be of the order of Planck length L_p^2) that sets the scale of noncommutativity (we used natural units $\hbar = c = 1$). Note that, here, $i, j = 0, \dots, N - 1$.

In agreement with Snyder [7], the symmetry of such a (noncommutative) space is described by the undeformed Lorentz algebra $so(1, N - 1)$. This requires that the M_{ij} generators satisfy the standard commutation relations:

$$[M_{ij}, M_{kl}] = i(\eta_{ik}M_{jl} - \eta_{il}M_{jk} + \eta_{jl}M_{ik} - \eta_{jk}M_{il}). \quad (2)$$

We also have the cross-commutation relations between Lorentz generators and Snyder coordinates:

$$[M_{ij}, \hat{x}_k] = i(\eta_{ik}\hat{x}_j - \eta_{jk}\hat{x}_i). \quad (3)$$

Relations (1)–(3) constitute a Lie algebra, which we embed into the associative universal enveloping algebra. We denote this universal enveloping algebra as the algebra A .

We are interested in constructing the Heisenberg double corresponding to the noncommutative Snyder space; therefore, first, we need to equip A with the Hopf algebra structure. It is enough to impose the primitive coalgebra structure on \hat{x}_i and M_{ij} . We investigate if it is possible to use the Heisenberg double construction to obtain the phase space built up from the Snyder noncommutative coordinates \hat{x}_i and the momenta p_i . We consider the algebra \tilde{A} generated by commuting momenta p_i equipped in the non-coassociative coalgebra structure. We choose the realization for the coproducts of momenta, which was proposed in [8] and called the Snyder realization therein.

The defining relations of \tilde{A} are:

$$[p_i, p_j] = 0 \quad (4)$$

and the (non-coassociative) coalgebra structure is the following:

$$\begin{aligned} \Delta p_i &= 1 \otimes p_i + \frac{1}{1 - \beta p_k \otimes p^k} \left(p_i \otimes 1 - \frac{\beta}{1 + \sqrt{1 + \beta p^2}} p_i p_j \otimes p^j + \sqrt{1 + \beta p^2} \otimes p_i \right), \\ \epsilon(p_i) &= 0, \quad S(p_i) = -p_i, \end{aligned} \quad (5)$$

where $p^2 = \eta^{ij} p_i p_j$ is Lorentz invariant and $i, j = 0, \dots, N - 1$.

We note that \tilde{A} is non-coassociative; therefore, momenta cannot be the proper dual generators to Snyder coordinates, which are the generators of the associative Hopf algebra A (see, e.g., [25], Section 1.2.8). Nevertheless, we can still propose the following duality relations $\langle \cdot, \cdot \rangle: \tilde{A} \times A \rightarrow \mathbb{C}$ (on the generators only¹):

$$\langle p_i, \hat{x}_j \rangle = -i\eta_{ij}, \quad (6)$$

and:

$$\langle p_i, M_{jk} \rangle = 0. \quad (7)$$

Then, we consider the analogue of the left Hopf action \triangleright of \tilde{A} on A defined as:

$$p_i \triangleright \hat{x}_j = \langle p_i, \hat{x}_{j(2)} \rangle \hat{x}_{j(1)} = -i\eta_{ij}. \quad (8)$$

(on the generators only) and use the usual cross-product construction mimicking that of the Heisenberg double (we refer the reader to the Appendix A for the details of the Heisenberg double construction in the Hopf algebra setting). The resulting cross-commutation relations are:

$$[p_i, \hat{x}_j] = \hat{x}_{j(1)} \langle p_{i(1)}, \hat{x}_{j(2)} \rangle p_{i(2)} - \hat{x}_j p_i = -i(\eta_{ij} + \beta p_i p_j). \quad (9)$$

We note that the relations (9) between momenta and Snyder coordinates obtained here (although with the limitations discussed) are in agreement with the commutation relations for the phase space of the Snyder model usually considered in the literature; see, e.g., [12,13,15,16].

Different Realization for Coproducts of Momenta

In the previous section, we used the coalgebra sector for momenta in the so-called Snyder realization [7,8]. There exist more possible realizations for momenta's (non-coassociative) coproducts; see e.g., [8,22,23]. Different realizations for coproducts can be related to each other by a change of basis in the momentum space.

However, there is a general way to write the (non-coassociative) coproduct for momenta corresponding to the Snyder model, i.e., the realization proposed in [8] and called "general realization" therein².

The formula we recall below [8] is calculated only up to the second order in the parameter β :

$$\begin{aligned} \Delta p_i &= 1 \otimes p_i + p_i \otimes 1 \\ &+ \beta \left(\left(c - \frac{1}{2} \right) p_i \otimes p^2 + \left(2c - \frac{1}{2} \right) p_i p_k \otimes p^k + c \left(p^2 \otimes p_i + 2p_k \otimes p^k p_i \right) \right) + O(\beta^2), \end{aligned} \quad (10)$$

$$\epsilon(p_i) = 0, \quad S(p_i) = -p_i. \quad (11)$$

This formula describes the general non-coassociative³ coproduct for Snyder momenta. The choice of the parameter c encodes different realizations. For $c = \frac{1}{2}$, the coproduct

(10) admits its finite form considered in the previous section. For $c = 0$, we obtain the realization investigated in [22,23]. If $c = \frac{1}{6}$, we obtain the realization investigated in [29,30].

We can now calculate the cross-relations between momenta and Snyder coordinates corresponding to the “general realization” of the momenta coproducts for the Snyder model (the proposed duality relations on the generators and the left Hopf action proposed in the previous section remain unchanged (6)–(8) for the linear power of the generators). This results in the following cross-commutation relations between momenta and Snyder coordinates:

$$\begin{aligned}
 [p_i, \hat{x}_k] &= \hat{x}_{k(1)} \langle p_{i(1)}, \hat{x}_{k(2)} \rangle p_{i(2)} - \hat{x}_k p_i \\
 &= -i\eta_{ik} \left(1 + \beta \left(c - \frac{1}{2} \right) p^2 \right) - 2ic\beta p_k p_i + O(\beta^2).
 \end{aligned}
 \tag{12}$$

This reduces to (for the specific, above-mentioned choices of the parameter c):

- for $c = \frac{1}{2}$ [8] to:

$$[p_i, \hat{x}_k] = -i(\eta_{ik} + \beta p_i p_k)
 \tag{13}$$

cf. (9);

- for $c = 0$ [22,23] to:

$$[p_i, \hat{x}_k] = -i\eta_{ik} \left(1 - \frac{\beta}{2} p^2 \right) + O(\beta^2);
 \tag{14}$$

- for $c = \frac{1}{6}$ [29,30] to:

$$[p_i, \hat{x}_k] = -i\eta_{ik} \left(1 - \frac{\beta}{3} p^2 \right) - \frac{i}{3} \beta p_k p_i + O(\beta^2).
 \tag{15}$$

It is worth noting that in the limit of $\beta \rightarrow 0$, the coproducts for the momenta (5) and (10) reduce to $\Delta p_i = 1 \otimes p_i + p_i \otimes 1$. For $\beta = 0$ (classical case), there exists full-duality between algebra A generators x_i, M_{ij} and group elements p_i, Λ_{ij} (where Λ_{ij} are the matrix elements of Lorentz matrices; see, e.g., [1–5]).

In this section, we focused on the Snyder space, and we tried to investigate the possibility of using the Heisenberg double procedure to obtain the phase space for this model. We encountered the following issues. First, the noncommutative Snyder coordinates do not close the Lie algebra, and only after extending the algebra by the Lorentz generators, we have a Lie algebra that can be equipped in the Hopf algebra structure. Second, the momenta corresponding to the Snyder model are non-coassociative; hence, the corresponding structure is not a Hopf algebra. Nevertheless, we tried and proposed the duality between the Snyder coordinates and momenta, which is valid only on the generators (in the linear power). This allowed us to mimic the Heisenberg double construction and led to the cross-relations that are in agreement with the literature. Additionally, we cannot define the dual elements to the Lorentz generators M_{ij} for $\beta \neq 0$ (cf. (The general realization for the coproduct of momenta corresponds to the general realization for the Snyder coordinates; see Equations (6) and (7) in [8], where $\hat{x}_i \triangleright 1 = x_i, M_{ij} \triangleright 1 = 0$)). The non-coassociative momenta do not allow for expanding the Lorentz algebra (2) to the Poincaré algebra.

The only way to construct the full dual Hopf algebra for the Snyder model and the full Heisenberg double is described in Sections 3 and 4 and requires introducing the extended noncommutative coordinates \hat{x}_{ij} .

3. Unified Notation for the Snyder Algebra: Extended Snyder Model

In the previous section, to consider a Hopf algebra related to the Snyder model, we expanded the commutation relations between Snyder coordinates (1) by the Lorentz algebra (2) and included the cross-commutation relations (3), which allowed us to define a Hopf algebra related to the Snyder model (1)–(3). Then, we calculated the cross-commutation relations mimicking the Heisenberg double construction. However, in that framework,

it was not possible to define the dual elements to the Lorentz generators, and the proper treatment of the algebra of momenta \tilde{A} would require using the quasi-Hopf algebra framework; hence, we were not able to obtain the full Heisenberg double for the Snyder model. The full Heisenberg double construction for the Snyder space within the Hopf algebra setting requires an introduction of the extended noncommutative coordinates, which we discuss in this section.

Another way to obtain a Lie algebra from the Snyder model (1) is to extend it by identifying the Snyder coordinates as $\hat{x}_i \sim \hat{x}_{iN} = M_{iN} \in so(1, N)/so(1, N-1)$ [9,21]. This way, one can define a Lie algebra corresponding to the extended Snyder space, and further, one can also define the (coassociative) Hopf algebra structure [21]. In this approach, the Snyder coordinates are seen as generators of the Lorentz algebra, but the Lorentz algebra considered now has one dimension higher (i.e., $so(1, N)$ instead of $so(1, N-1)$)⁴ version of the Snyder model [9,21], the Heisenberg double construction completely mimics the construction of the undeformed Heisenberg double for the Lorentz algebra $so(1, N)$ with its dual algebra of functions on a group $SO(1, N)$ (Lorentz matrices). In this way, the noncommutativity parameter β related to the Snyder space (1) is implicitly included in the cross-commutation relations. This will be considered in Section 4.2. In Section 4.3, we will also present the realizations for the Lorentz matrices in the so-called Weyl realization [21,29,30]. However, to obtain the phase space from the Heisenberg double, we now can consider a dual Hopf algebra of momenta and find the corresponding extended Snyder phase space. In the remaining part of the paper, we want to focus on finding the explicit formulae for such Heisenberg doubles corresponding to the extended Snyder model.

We first need to define the Hopf algebra related to the extended Snyder model and also define the dual Hopf algebra of objects that would play the role of momenta. We start by embedding the Snyder algebra relations (1)–(3) in an algebra that is generated by the N position operators denoted by \hat{x}_i and $N(N-1)/2$ antisymmetric tensorial coordinates \hat{x}_{ij} , transforming as Lorentz generators [9,21]. This larger algebra has the following commutation relations:

$$[\hat{x}_i, \hat{x}_j] = i\lambda\beta\hat{x}_{ij}, \quad [\hat{x}_{ij}, \hat{x}_{kl}] = i\lambda(\eta_{ik}\hat{x}_{jl} - \eta_{il}\hat{x}_{jk} - \eta_{jk}\hat{x}_{il} + \eta_{jl}\hat{x}_{ik}), \quad (16)$$

$$[\hat{x}_{ij}, \hat{x}_k] = i\lambda(\eta_{ik}\hat{x}_j - \eta_{jk}\hat{x}_i), \quad (17)$$

where λ and β are real parameters. We can easily notice that these commutation relations reduce to those of the standard Lorentz algebra acting on commutative coordinates in the limit of $\beta \rightarrow 0$ and $\lambda \rightarrow 1$ and to the Lie algebra from Section 2 ((1)–(3)) in the limit $\lambda \rightarrow 1$.

To define the algebra in a unified way, one can exploit the isomorphism between the Snyder coordinates and the Lorentz generators of $so(1, N)$ and write the previous Formulas (16) and (17) more compactly, defining, for positive β ,

$$\hat{x}_i = \sqrt{\beta}\hat{x}_{iN}. \quad (18)$$

The extended Snyder algebra then takes the form [9,21] of the Lorentz algebra $so(1, N)$ (to be precise, it is $U_{so(1, N)}[[\lambda]]$ ⁵, given by one set of commutation relations as:

$$[\hat{x}_{\mu\nu}, \hat{x}_{\rho\sigma}] = i\lambda(\eta_{\mu\rho}\hat{x}_{\nu\sigma} - \eta_{\nu\rho}\hat{x}_{\mu\sigma} + \eta_{\nu\sigma}\hat{x}_{\mu\rho} - \eta_{\mu\sigma}\hat{x}_{\nu\rho}), \quad (19)$$

with $\eta_{NN} = 1$ and $\eta_{kN} = 0$; here, $\mu = 0, 1, \dots, N$ (Greek indices run from zero up to N , whereas Latin indices are $i, j = 0, 1, \dots, N-1$, as before).

One can check explicitly that (19), via (18), reduces:

- to the Snyder noncommutative spacetime relations (16):

$$[\hat{x}_{jN}, \hat{x}_{iN}] = \left[\frac{1}{\sqrt{\beta}}\hat{x}_j, \frac{1}{\sqrt{\beta}}\hat{x}_i \right] = i\lambda(\eta_{ji}\hat{x}_{NN} - \eta_{Ni}\hat{x}_{jN} + \eta_{NN}\hat{x}_{ji} - \eta_{jN}\hat{x}_{Ni}) = i\lambda\hat{x}_{ji}$$

(note that $\hat{x}_{NN} = 0$ due to antisymmetry);

- to the commutation relations for Lorentz generators (16):

$$[\hat{x}_{ij}, \hat{x}_{kl}] = i\lambda(\eta_{ik}\hat{x}_{jl} - \eta_{il}\hat{x}_{jk} - \eta_{jk}\hat{x}_{il} + \eta_{jl}\hat{x}_{ik});$$

- and to cross-commutation relations of Lorentz generators acting on coordinates (17):

$$\begin{aligned} [\hat{x}_{jk}, \hat{x}_{iN}] &= [\hat{x}_{jk}, \frac{1}{\sqrt{\beta}}\hat{x}_i] = i\lambda(\eta_{ji}\hat{x}_{kN} - \eta_{ki}\hat{x}_{jN} + \eta_{kN}\hat{x}_{ji} - \eta_{jN}\hat{x}_{ki}) \\ &= i\lambda \frac{1}{\sqrt{\beta}}(\eta_{ji}\hat{x}_k - \eta_{ki}\hat{x}_j). \end{aligned}$$

In turn, these all reduce to (1)–(3) from Section 2, respectively, for $\lambda \rightarrow 1$.

3.1. Generalized Heisenberg Algebra

To discuss the extended phase space associated with this extended Snyder model (19) as a result of the Heisenberg double construction, we need to first recall a few facts about the generalized Heisenberg algebra and Weyl realization of the Lorentz algebra based on results presented in [29].

The generalized Heisenberg algebra can be introduced as a unital, associative algebra generated by (commutative) $x_{\mu\nu}$ and $p_{\mu\nu}$ (both antisymmetric), satisfying the following commutation relations:

$$[x_{\mu\nu}, x_{\alpha\beta}] = 0, \tag{20}$$

$$[p_{\mu\nu}, p_{\alpha\beta}] = 0, \tag{21}$$

$$[p_{\mu\nu}, x_{\rho\sigma}] = -i(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}). \tag{22}$$

Here, we considered the elements $p_{\mu\nu}$ as canonically conjugate to $x_{\mu\nu}$, which can be realized in a standard way as $p_{\mu\nu} = -i\frac{\partial}{\partial x^{\mu\nu}}$.

Commutative coordinates $x_{\mu\nu}$ can be viewed as the classical limit (when $\lambda \rightarrow 0$) of $\hat{x}_{\mu\nu}$ generators of $so(1, N)$ used in the extended version for the Snyder algebra (19). In other words, the Lie algebra $so(1, N)$, more specifically its universal enveloping algebra $U_{so(1, N)}[[\lambda]]$, generated by $\hat{x}_{\mu\nu}$ can be seen as a deformation of the underlying commutative space $x_{\mu\nu}$ with λ as the deformation parameter.

Now, since we are interested in the Snyder model (in its extended version) and the corresponding deformed phase space, first, let us notice that we can use the analogous relation to (18) for the commutative coordinates, i.e., take $x_i = \sqrt{\beta}x_{iN}$, and similarly, for the conjugate canonical momenta $p_{\mu\nu}$, we can introduce: $p_i = \frac{p_{iN}}{\sqrt{\beta}}$. This allowed us to reduce the above generalized Heisenberg algebra (20)–(22) to:

- the usual Heisenberg algebra sector, i.e., quantum mechanical phase space corresponding to the commutative (classical) spacetime:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0 \tag{23}$$

$$[p_j, x_i] = [p_{iN}, x_{jN}] = -i(\eta_{ij}\eta_{NN} - \eta_{iN}\eta_{Nj}) = -i\eta_{ij}; \tag{24}$$

- and “the remaining part”, consisting of commutation relations between x_{ij} —tensorial coordinates—and p_{ij} —their corresponding canonical momenta:

$$[x_{ij}, x_{kl}] = [x_{ij}, x_k] = 0, \tag{25}$$

$$[p_{ij}, p_{kl}] = [p_{ij}, p_k] = 0, \tag{26}$$

$$[p_{ij}, x_{kl}] = -i(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}), \tag{27}$$

$$[p_i, x_{kl}] = [p_{ij}, x_k] = 0. \tag{28}$$

Therefore, the relations (20)–(22) indeed describe the generalization of the quantum mechanical phase space as they contain, as a subalgebra, the relations (23) and (24).

The Weyl realization of the Lorentz algebra in terms of this generalized Heisenberg algebra (20)–(22) (as formal power series) were discussed in detail in [29].

4. Extended Snyder Space and Its Heisenberg Doubles

We are now ready to discuss how to construct two full Heisenberg doubles corresponding to the extended Snyder space (19), in its unified $so(1, N)$ -like version (with the generators $\hat{x}_{\mu\nu}$). For the purpose of this section, let us denote the extended Snyder algebra, defined by the relations (19), as an algebra B . To construct the Heisenberg double for the extended Snyder algebra B , we first need to equip it in the Hopf algebra structure (which is straightforward, as we can use the primitive coproducts for $\hat{x}_{\mu\nu}$), and then, we need to define the dual Hopf algebra.

We equip the Snyder algebra B (as $so(1, N)$), defined by the relations (19), with the Hopf algebra structure as follows:

$$\Delta(\hat{x}_{\mu\nu}) = \Delta_0(\hat{x}_{\mu\nu}), \quad (29)$$

$$\epsilon(\hat{x}_{\mu\nu}) = 0 \text{ and } S(\hat{x}_{\mu\nu}) = -\hat{x}_{\mu\nu}. \quad (30)$$

With the above relations algebra B , using (18) leads to the extended Snyder Hopf algebra.

4.1. Extended Snyder Phase Space from the Heisenberg Double Construction

To discuss the phase space corresponding to the extended Snyder space, we consider the Hopf algebra generated by $p_{\mu\nu}$ (satisfying (21)), as a dual Hopf algebra B^* , which is equipped with the Hopf algebra structure introduced in [21] (see Equation (25) therein), where the coproducts, calculated up to the third order, have the following form:

$$\begin{aligned} \Delta p_{\mu\nu} &= \Delta_0 p_{\mu\nu} - \frac{\lambda}{2} (p_{\mu\alpha} \otimes p_{\nu\alpha} - p_{\nu\alpha} \otimes p_{\mu\alpha}) \\ &\quad - \frac{\lambda^2}{12} (p_{\mu\alpha} \otimes p_{\alpha\beta} p_{\nu\beta} - p_{\nu\alpha} \otimes p_{\alpha\beta} p_{\mu\beta} - 2p_{\alpha\beta} \otimes p_{\mu\alpha} p_{\nu\beta} \\ &\quad + p_{\mu\alpha} p_{\alpha\beta} \otimes p_{\nu\beta} - p_{\nu\alpha} p_{\alpha\beta} \otimes p_{\mu\beta} - 2p_{\mu\alpha} p_{\nu\beta} \otimes p_{\alpha\beta}) + O(\lambda^3). \end{aligned} \quad (31)$$

Counits are $\epsilon(p_{\mu\nu}) = 0$, and antipodes are $S(p_{\mu\nu}) = -p_{\mu\nu}$. This defines the coassociative⁶ Hopf algebra B^* as the dual to the extended Snyder Hopf algebra B . The above coproducts for momenta correspond to the so-called Weyl realization for the $\hat{x}_{\mu\nu}$ ⁷. One could use the coproducts for generic realization, but they depend on five free parameters and are calculated up to the second order in λ [21].

The duality relation $\langle \cdot, \cdot \rangle: B^* \times B \rightarrow \mathbb{C}$ is as follows:

$$\langle p_{\mu\nu}, \hat{x}_{\rho\sigma} \rangle = -i(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\sigma\mu}\eta_{\rho\nu}) \quad (32)$$

and can be extended to all elements of both Hopf algebras B and B^* . We take the left Hopf action \triangleright of B^* on B , which is defined by:

$$p_{\rho\sigma} \triangleright \hat{x}_{\mu\nu} = \langle p_{\rho\sigma}, \hat{x}_{\mu\nu(2)} \rangle \hat{x}_{\mu\nu(1)} = -i(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\sigma\mu}\eta_{\rho\nu}). \quad (33)$$

We can now construct the corresponding Heisenberg double resulting in the following cross-commutation relations:

$$\begin{aligned} [p_{\mu\nu}, \hat{x}_{\rho\sigma}] &= \hat{x}_{\rho\sigma(1)} \langle p_{\mu\nu(1)}, \hat{x}_{\rho\sigma(2)} \rangle p_{\mu\nu(2)} - \hat{x}_{\rho\sigma} p_{\mu\nu} \\ &= -i(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\sigma\mu}\eta_{\rho\nu}) + \frac{i\lambda}{2} (\eta_{\rho\mu} p_{\nu\sigma} - \eta_{\sigma\mu} p_{\nu\rho} - \eta_{\rho\nu} p_{\mu\sigma} + \eta_{\sigma\nu} p_{\mu\rho}) \\ &\quad + \frac{i\lambda^2}{12} [\eta_{\rho\mu} p_{\sigma\beta} p_{\nu\beta} - \eta_{\sigma\mu} p_{\rho\beta} p_{\nu\beta} - \eta_{\rho\nu} p_{\sigma\beta} p_{\mu\beta} + \eta_{\sigma\nu} p_{\rho\beta} p_{\mu\beta} - 2p_{\mu\rho} p_{\nu\sigma} + 2p_{\mu\sigma} p_{\nu\rho}] + O(\lambda^3). \end{aligned} \quad (34)$$

For the description of the phase space corresponding to the extended Snyder model (in Snyder coordinates \hat{x}_i, \hat{x}_{ij}), we use the isomorphism (18) $\hat{x}_i = \sqrt{\beta}\hat{x}_{iN}$ and $p_i = \frac{p_{iN}}{\sqrt{\beta}}$. The above duality (32) then becomes:

$$\langle p_j, \hat{x}_i \rangle = -i\eta_{ij}, \quad (35)$$

$$\langle p_k, \hat{x}_{ij} \rangle = 0, \quad (36)$$

$$\langle p_{kl}, \hat{x}_i \rangle = 0, \quad (37)$$

$$\langle p_{kl}, \hat{x}_{ij} \rangle = -i(\eta_{ik}\eta_{jl} - \eta_{jk}\eta_{il}). \quad (38)$$

The cross-commutation relations are as follows:

- commutation relations between Snyder coordinates and their coupled momenta:

$$[p_k, \hat{x}_i] = [p_{kN}, \hat{x}_{iN}] = -i\eta_{ik}(1 - \frac{\beta\lambda^2}{12}p_l p_l) - \frac{i\beta\lambda^2}{12}p_k p_i + \frac{i\lambda}{2}p_{ki} + \frac{i\lambda^2}{12}p_{il}p_{kl} + O(\lambda^3); \quad (39)$$

- commutation relations between tensorial coordinates and their coupled momenta:

$$[p_{kl}, \hat{x}_{ij}] = -i(\eta_{ik}\eta_{jl} - \eta_{jk}\eta_{il}) + i\frac{\lambda}{2}(\eta_{ik}p_{lj} - \eta_{jk}p_{li} - \eta_{il}p_{kj} + \eta_{jl}p_{ki}) + \frac{i\lambda^2}{12}[(\eta_{ik}p_{jm}p_{lm} - \eta_{jk}p_{im}p_{lm}) - (\eta_{il}p_{jm}p_{km} - \eta_{jl}p_{im}p_{km}) - 2p_{ki}p_{lj} + 2p_{kj}p_{li}] + O(\lambda^3); \quad (40)$$

- and mixed relations:

$$[p_{kl}, \hat{x}_i] = [p_{kl}, \sqrt{\beta}\hat{x}_{iN}] = i\frac{\lambda}{2}\beta(\eta_{ik}p_l - \eta_{il}p_k) - i\frac{\lambda^2}{12}\beta[\eta_{ik}p_m p_{lm} - \eta_{il}p_m p_{km} + 2p_{ki}p_l - 2p_k p_{li}] + O(\lambda^3), \quad (41)$$

$$[p_k, \hat{x}_{ij}] = \left[\frac{1}{\sqrt{\beta}}p_{kN}, \hat{x}_{ij} \right] = -i\frac{\lambda}{2}(\eta_{ik}p_j - \eta_{jk}p_i) - i\frac{\lambda^2}{12}[(\eta_{ik}p_{jl}p_l - \eta_{jk}p_{il}p_l) - 2p_{ki}p_j + 2p_{kj}p_i] + O(\lambda^3). \quad (42)$$

One can notice that the commutator between the momenta generators and the Snyder coordinates (39) obtained from (34) actually resembles the Snyder model phase space for the Weyl realization (15) obtained in Section 2. The first three terms agree up to the factor $\frac{\lambda^2}{4}$, but the remaining terms include the tensorial momenta.

We now obtained the full extended Snyder phase space (39)–(42) resulting from the Heisenberg double construction.

It is worth mentioning that some authors (see e.g., [17]) also consider another version of the Snyder phase spaces, where momenta do not commute, but we did not consider this type of phase space in this work; the momenta sector is always commutative, for both the Snyder model considered in Section 2 and the extended Snyder model in Section 4.1.

4.2. Another Heisenberg Double for the Extended Snyder Algebra

To construct another Heisenberg double for the extended Snyder algebra B written in a unified $so(1, N)$ -like form (19), it is quite straightforward to mimic the Heisenberg double construction for the Lorentz algebra.

We take the extended Snyder Hopf algebra B (19) equipped with the Hopf algebra structure (29) and (30), as before. We define the dual Hopf algebra algebra D (as the dual to the extended Snyder Hopf algebra B) as an algebra of functions on a group $SO(1, N)$, which is generated by Lorentz matrices $\Lambda_{\alpha\beta}$, i.e.,

$$D = F(SO(1, N)) = \{\Lambda_{\alpha\beta} : [\Lambda_{\alpha\beta}, \Lambda_{\mu\nu}] = 0 : \Lambda^T \eta \Lambda = \eta\}, \quad (43)$$

$$\Delta(\Lambda_{\rho\sigma}) = \Lambda_{\rho\alpha} \otimes \Lambda_{\alpha\sigma}; \quad \epsilon(\Lambda_{\rho\sigma}) = \delta_{\rho\sigma}; \quad S(\Lambda_{\rho\sigma}) = (\Lambda^{-1})_{\rho\sigma} = \Lambda_{\sigma\rho}. \quad (44)$$

Note that the Greek indices run up to N , i.e., $\alpha, \beta = 0, 1, \dots, N$. The duality relation $\langle \cdot, \cdot \rangle: D \times B \rightarrow \mathbb{C}$ is given by:

$$\langle \Lambda_{\rho\sigma}, \hat{x}_{\mu\nu} \rangle = -i\lambda(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\nu}\eta_{\sigma\mu}). \quad (45)$$

We consider the left Hopf action \triangleright of D on B , which is defined by:

$$\Lambda_{\rho\sigma} \triangleright \hat{x}_{\mu\nu} = \langle \Lambda_{\rho\sigma}, \hat{x}_{\mu\nu(2)} \rangle \hat{x}_{\mu\nu(1)} = \eta_{\rho\sigma} \hat{x}_{\mu\nu} - i\lambda(\eta_{\rho\mu}\eta_{\sigma\nu} - \eta_{\rho\nu}\eta_{\sigma\mu}). \quad (46)$$

We calculate the cross-commutation relations defining the Heisenberg double as:

$$[\Lambda_{\rho\sigma}, \hat{x}_{\mu\nu}] = \hat{x}_{\mu\nu(1)} \langle \Lambda_{\rho\sigma(1)}, \hat{x}_{\mu\nu(2)} \rangle \Lambda_{\rho\sigma(2)} - \hat{x}_{\mu\nu} \Lambda_{\rho\sigma} \quad (47)$$

$$= -i\lambda(\eta_{\rho\mu}\Lambda_{\nu\sigma} - \eta_{\rho\nu}\Lambda_{\mu\sigma}). \quad (48)$$

For the description of the Heisenberg double for the extended Snyder model (in Snyder coordinates \hat{x}_i, \hat{x}_{ij}), we again use the isomorphism (18), and the above formulae lead to:

- duality for the Snyder coordinates with Lorentz matrices:

$$\langle \Lambda_{jk}, \hat{x}_i \rangle = \sqrt{\beta} \langle \Lambda_{jk}, \hat{x}_{iN} \rangle = -i\lambda\sqrt{\beta}(\eta_{ji}\eta_{kN} - \eta_{jN}\eta_{ki}) = 0, \quad (49)$$

$$\langle \Lambda_{jN}, \hat{x}_i \rangle = \sqrt{\beta} \langle \Lambda_{jN}, \hat{x}_{iN} \rangle = -i\lambda\sqrt{\beta}\eta_{ji}, \quad (50)$$

$$\langle \Lambda_{Nk}, \hat{x}_i \rangle = \sqrt{\beta} \langle \Lambda_{Nk}, \hat{x}_{iN} \rangle = i\lambda\sqrt{\beta}\eta_{ki}, \quad (51)$$

$$\langle \Lambda_{NN}, \hat{x}_i \rangle = \sqrt{\beta} \langle \Lambda_{NN}, \hat{x}_{iN} \rangle = 0; \quad (52)$$

- duality of the Lorentz generators with their dual Lorentz matrices:

$$\langle \Lambda_{jk}, \hat{x}_{ip} \rangle = -i\lambda(\eta_{ji}\eta_{kp} - \eta_{jp}\eta_{ki}), \quad (53)$$

$$\langle \Lambda_{jN}, \hat{x}_{ip} \rangle = 0 = \langle \Lambda_{Nk}, \hat{x}_{ip} \rangle, \quad (54)$$

$$\langle \Lambda_{NN}, \hat{x}_{ip} \rangle = 0. \quad (55)$$

Similarly, the cross-commutation relations from the Heisenberg double construction (47) then become as follows:

- cross-commutation relations between the Snyder coordinates and Lorentz matrices:

$$[\Lambda_{jk}, \hat{x}_i] = \sqrt{\beta} [\Lambda_{jk}, \hat{x}_{iN}] = -i\lambda\sqrt{\beta}\eta_{ji}\Lambda_{Nk}, \quad (56)$$

$$[\Lambda_{jN}, \hat{x}_i] = \sqrt{\beta} [\Lambda_{jN}, \hat{x}_{iN}] = -i\lambda\sqrt{\beta}\eta_{ji}\Lambda_{NN}, \quad (57)$$

$$[\Lambda_{Nk}, \hat{x}_i] = \sqrt{\beta} [\Lambda_{Nk}, \hat{x}_{iN}] = i\lambda\sqrt{\beta}\Lambda_{ik}, \quad (58)$$

$$[\Lambda_{NN}, \hat{x}_i] = \sqrt{\beta} [\Lambda_{NN}, \hat{x}_{iN}] = i\lambda\sqrt{\beta}\Lambda_{iN}; \quad (59)$$

- and the cross-commutation relations between the Lorentz generators (of $so(1, N-1)$) with the Lorentz matrices:

$$[\Lambda_{jk}, \hat{x}_{ip}] = -i\lambda(\eta_{ji}\Lambda_{pk} - \eta_{jp}\Lambda_{ik}), \quad (60)$$

$$[\Lambda_{jN}, \hat{x}_{ip}] = -i\lambda(\eta_{ji}\Lambda_{pN} - \eta_{jp}\Lambda_{iN}), \quad (61)$$

$$[\Lambda_{Nk}, \hat{x}_{ip}] = -i\lambda(\eta_{Ni}\Lambda_{pk} - \eta_{Np}\Lambda_{ik}) = 0, \quad (62)$$

$$[\Lambda_{NN}, \hat{x}_{ip}] = 0. \quad (63)$$

The primitive coproduct for $\hat{x}_{\mu\nu}$ reduces to the primitive coproduct for \hat{x}_{ip} (Lorentz generators of $so(1, N-1)$) and to the primitive coproduct for \hat{x}_i (Snyder coordinates), as in Section 2, respectively.

We also note that:

$$[\Lambda_{\mu\nu}, p_{\rho\sigma}] = 0. \quad (64)$$

4.3. Realizations for Lorentz Matrices

We can actually relate the dual momenta discussed in Section 4.1 to the dual Lorentz matrices discussed in Section 4.2.

This can be performed by introducing the realizations of the elements of the dual algebra D , i.e., functions on a group $SO(1, N)$ —Lorentz matrices $\Lambda_{\alpha\beta}$. These realizations can be expressed as a formal power series of the tensorial momenta introduced in Section 3.1. For more details, we refer the reader to [29], where the Lorentz algebra extension by its dual counterpart was discussed in detail. The formulas presented below were based on the Theorem III.1 from [29]. The Weyl realization [29,30] for the generators of the algebra D , satisfying (43), can be written in a very compact form as follows:

$$\Lambda_{\rho\sigma} = \left(e^{\lambda p} \right)_{\rho\sigma}. \quad (65)$$

If we want to calculate the explicit formulae for the realization of the elements dual to the Lorentz sector in the extended Snyder algebra written in the form of tensorial coordinates \hat{x}_{ij} (16) and (17), we use the above formula and obtain:

$$\begin{aligned} \Lambda_{kl} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (p^n)_{kl}, \\ \Lambda_{Nl} &= \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} p_{Nk} (p^{n-1})_{kl} = p_{Nk} \left(\frac{\Lambda - \eta}{p} \right)_{kl}, \\ \Lambda_{kN} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (p^n)_{kN} = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (p^{n-1})_{kl} p_{lN} = \left(\frac{\Lambda - \eta}{p} \right)_{kl} p_{lN} = (\Lambda^{-1})_{Nk}, \\ \Lambda_{NN} &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (p^n)_{NN} = \eta_{NN} + \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} p_{Nk} (p^{n-2})_{kl} p_{lN} = \eta_{NN} + p_{Nk} \left(\frac{\Lambda - \eta - \lambda p}{p^2} \right)_{kl} p_{lN}, \end{aligned}$$

where we also used the following notation $(p^0)_{\alpha\beta} = \eta_{\alpha\beta}$ and $(\Lambda^0)_{\rho\sigma} = \eta_{\rho\sigma}$.

5. Conclusions

In this paper, we investigated three Heisenberg doubles related to the two types of noncommutative Snyder models. The Heisenberg double construction was widely investigated for other noncommutative spacetimes [1–5], but not yet (to our knowledge) for the Snyder space. Therefore, this work offers the first study on Heisenberg doubles for the Snyder model, as well as for the extended Snyder model.

In Section 2, we discussed issues arising when applying the Heisenberg double construction to the Snyder model. We proposed a duality between Snyder coordinates and momenta with the (non-coassociative) coproducts related to the so-called Snyder realization (5). This duality is valid on the generators (in the linear powers) only. The cross-commutation relations obtained for the generators (9) were compared with the Snyder phase space relations considered in the literature. Then, we used the momenta with the (non-coassociative) coproducts in the general realization, which provided a more general version of the cross-commutation relations between the momenta and Snyder coordinates and reduced to all known cases for the certain choices of the parameter c (which parametrizes the non-coassociative coproduct for the momenta generators). However, at the end of Section 2, we pointed out that the construction of the full Heisenberg double for the Snyder space in the Hopf algebra setting requires the introduction of extended noncommutative coordinates.

Therefore, we used the fact that the Snyder model can be embedded in a larger algebra, $U_{so(1,N)}[[\lambda]]$, for which the dual algebra admits the coassociative coalgebra structure. We then constructed the Heisenberg double for this extended Snyder model in two ways: firstly, by introducing the dual tensorial momentum space; secondly, by using the Lorentz matrices, i.e., functions on the Lorentz group.

The Heisenberg double of the extended Snyder algebra with the corresponding dual momenta can be interpreted as the extended Snyder phase space. The formulation of Heisenberg doubles as extended Snyder phase spaces, proposed in this work, can be used in many further applications. An additional advantage is that, since the noncommutative coordinates generate the Lie algebra, then the corresponding coproducts of momenta are coassociative, and the related star products between coordinates are associative [21], which opens a way to a great number of applications where the associative star product is required. The only drawback of this approach is the unclear physical interpretation of tensorial coordinates and corresponding conjugated momenta appearing in this picture.

Nevertheless, by using the algebraic scheme of Heisenberg doubles, one can introduce the covariant Snyder phase spaces and further investigate the applications where the consistent definition of a phase space is crucial, for example: such a deformed (extended) Snyder phase space may lead to deformed Heisenberg uncertainty relations [14], or may be considered in the context of quantum gravity phenomenology [11,15,16], or when investigating cosmological [11] and curved [12,13] backgrounds coupled to Snyder spacetime. We consider our work presented here as a first step towards such applications.

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Appendix A. Heisenberg Double Construction

Let A and A^* be dual Hopf algebras. To construct the Heisenberg double $A \rtimes A^*$, we started with the left Hopf action \triangleright of A^* on A defined as:

$$a^* \triangleright a = \langle a^*, a_{(2)} \rangle a_{(1)}, \quad (\text{A1})$$

where we used the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for the coproduct and $a^* \in A^*, a \in A$.

Duality needs to satisfy the following compatibility conditions between algebras:

$$\langle a^*_{(1)}, a \rangle \langle a^*_{(2)}, a' \rangle = \langle a^*, a \cdot a' \rangle, \quad (\text{A2})$$

$$\langle a^*, a_{(1)} \rangle \langle a^{*'}, a_{(2)} \rangle = \langle a^* \cdot a^{*'}, a \rangle. \quad (\text{A3})$$

The Heisenberg double corresponding to these data can be then constructed as the cross-product algebra (also known as the “smash product”) $A \rtimes A^*$. The (left) product in the cross-product algebra (Heisenberg double) becomes:

$$(a \otimes a^*) \rtimes (a' \otimes a^{*'}) = a \left(a^*_{(1)} \triangleright a' \right) \otimes a^*_{(2)} a^{*'} = \langle a^*_{(1)}, a'_{(2)} \rangle a a'_{(1)} \otimes a^*_{(2)} a', \quad (\text{A4})$$

which leads to the following (left) products:

$$a^* \circ a = (1 \otimes a^*) \rtimes (a \otimes 1) = \left(a^*_{(1)} \triangleright a \right) \otimes a^*_{(2)} = \langle a^*_{(1)}, a_{(2)} \rangle a_{(1)} \otimes a^*_{(2)}, \quad (\text{A5})$$

$$a \circ a^* = (a \otimes 1) \rtimes (1 \otimes a^*). \quad (\text{A6})$$

Therefore, the cross-commutation relation becomes:

$$[a^*, a] = a_{(1)} \langle a_{(1)}^*, a_{(2)} \rangle a_{(2)}^* - a \circ a^*. \quad (\text{A7})$$

There are some special cases worth considering:

1. If the generators of A have a primitive coproduct, then the above formula, for the generators, reduces to:

$$[a^*, a] = \langle a_{(1)}^*, a \rangle a_{(2)}^*. \quad (\text{A8})$$

2. If the coproduct on the algebra A^* is opposite, i.e., $\Delta a^* = a_{(2)}^* \otimes a_{(1)}^*$, then the commutator becomes:

$$[a^*, a] = a_{(1)} \langle a_{(2)}^*, a_{(2)} \rangle a_{(1)}^* - a \circ a^*. \quad (\text{A9})$$

If, in addition, the generators of A have a primitive coproduct then, for the generators, we obtain:

$$[a^*, a] = \langle a_{(2)}^*, a \rangle a_{(1)}^*. \quad (\text{A10})$$

Notes

- ¹ The pairing we propose here is satisfied on the generators (in the first power) only, and it may not be possible to extend it to the full algebra due to the non-coassociative nature of \tilde{A} . We also note that the proper definition of \tilde{A} would require the quasi-Hopf algebra framework [26]. Defining the Heisenberg double within the quasi-Hopf algebra setting has been proposed, for example, in [27,28] and would be worth investigating further.
- ² The general realization for the coproduct of momenta corresponds to the general realization for the Snyder coordinates; see Equations (6) and (7) in [8], where $\hat{x}_i \triangleright 1 = x_i$, $M_{ij} \triangleright 1 = 0$.
- ³ The difference between the left-hand side and the right-hand side of the coassociativity condition for the coproduct of momenta can be explicitly calculated and is as follows, in the first order in β :

$$((id \otimes \Delta) \circ \Delta p_i) - ((\Delta \otimes id) \circ \Delta p_i) = -\frac{1}{2}\beta(p_i \otimes p_k \otimes p^k - p_k \otimes p_i \otimes p^k) + O(\beta^2).$$

- ⁴ Many authors use the word “generalized” for the version of Snyder space in a different meaning (see, e.g., [18] or [24]); therefore, following [21], we shall call the version used here “extended” instead of generalized—since it is unified with the additional tensorial coordinates transforming as the Lorentz generators.
- ⁵ A topological extension of the corresponding enveloping algebra $U_{so(1,N)}$ into an algebra of formal power series $U_{so(1,N)}[[\lambda]]$ in the formal parameter λ is required here. This provides the λ -adic topology (see, for example, Chapter 1.2.10 in [25]).
- ⁶ In general, if noncommutative coordinates close a Lie algebra, as is the case for $\hat{x}_{\mu\nu}$, then the corresponding coproducts of momenta are coassociative [30–32].
- ⁷ The Weyl realization for the extended Snyder space is defined as $e^{ik_i \hat{x}_i + \frac{1}{2} k_{ij} \hat{x}_{ij}} \triangleright 1 = e^{ik_i x_i + \frac{1}{2} k_{ij} x_{ij}}$ where $\hat{x}_i \triangleright 1 = x_i$, $\hat{x}_{ij} \triangleright 1 = x_{ij}$, see Equation (17) in [21]. Note that the action differs from the one described in Note 2.

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