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New Results for Oscillation of Solutions of Odd-Order Neutral Differential Equations

Clemente Cesarano ^{1,*}, Osama Moaaz ^{1,2,*} , Belgees Qaraad ^{2,3}, Nawal A. Alshehri ⁴ , Sayed K. Elagan ⁴ and Mohammed Zakarya ^{5,6}

- ¹ Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
- ² Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; belgeesmath2016@students.mans.edu.eg
- ³ Department of Mathematics, Faculty of Science, Amran University, Amran 999101, Yemen
- ⁴ Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; nalshehri@tu.edu.sa (N.A.A.); skhalil@tu.edu.sa (S.K.E.)
- ⁵ Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia; mzibrahim@kku.edu.sa
- ⁶ Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
- * Correspondence: c.cesarano@uninettunouniversity.net (C.C.); o_moaaz@mans.edu.eg (O.M.)

Abstract: Differential equations with delay arguments are one of the branches of functional differential equations which take into account the system's past, allowing for more accurate and efficient future prediction. The symmetry of the equations in terms of positive and negative solutions plays a fundamental and important role in the study of oscillation. In this paper, we study the oscillatory behavior of a class of odd-order neutral delay differential equations. We establish new sufficient conditions for all solutions of such equations to be oscillatory. The obtained results improve, simplify and complement many existing results.

Keywords: quasilinear; odd-order; functional differential equations; delay; oscillation criteria



Citation: Cesarano, C.; Moaaz, O.; Qaraad, B.; Alshehri, N.A.; Elagan, S.K.; Zakarya, M. New Results for Oscillation of Solutions of Odd-Order Neutral Differential Equations. *Symmetry* **2021**, *13*, 1095. <https://doi.org/10.3390/sym13061095>

Academic Editors: Włodzimierz Fechner and Jacek Chudziak

Received: 11 May 2021
Accepted: 18 June 2021
Published: 21 June 2021

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1. Introduction

Consider the odd-order neutral delay differential equation (NDDE)

$$\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)' + q(t) f(x(h(t))) = 0, \quad t \geq t_0, \quad (1)$$

where $n > 1$ is odd, α is a quotient of odd positive integers, and

$$Y(t) := x(t) + p(t)x(\zeta(t)).$$

Throughout this paper, we assume the following :

- (I₁) r, h and ζ are continuously differentiable real-valued functions on $[t_0, \infty)$, and p, q and f are continuous real-valued functions on $[t_0, \infty)$.
- (I₂) $r(t) > 0, r'(t) \geq 0, 0 < p(t) \leq p_0 < \infty, q \geq 0$ does not vanish identically, and $\eta(t_0) = \infty$, where

$$\eta(t) := \int_t^\infty r^{-1/\alpha}(s) ds;$$

- (I₃) $h(t) < t, \zeta(t) < t$, and $\lim_{t \rightarrow \infty} h(t) = \lim_{t \rightarrow \infty} \zeta(t) = \infty$.
- (I₄) $f(x) \geq kx^\alpha$ for all $x \neq 0$, where k is a positive constant (note that $x^\alpha = -|x|^\alpha$ for $x < 0$).

By a solution of (1), we mean a continuous real-valued function $x(t)$ for $t \geq t_x \geq t_0$, which has the property: Y is continuously differentiable n times for $t \geq t_x, r \left(Y^{(n-1)} \right)^\alpha$ is

continuously differentiable for $t \geq t_x$, and x satisfies (1) on $[t_x, \infty)$. We consider only the nontrivial solutions of (1) is present on some half-line $[t_x, \infty)$ and satisfying the condition $\sup\{|x(t)| : t \leq t < \infty\} > 0$ for any $t \geq t_x$.

On many occasions, symmetries have appeared in mathematical formulations that have become essential for solving problems or delving further into research. High quality studies that use nontrivial mathematics and their symmetries applied to relevant problems from all areas were presented. In fact, in recent years, many monographs and a lot of research papers have been devoted to the behavior of solutions of delay differential equations. This is due to its relevance for different life science applications and its effectiveness in finding solutions of real world problems such as natural sciences, technology, population dynamics, medicine dynamics, social sciences and genetic engineering. For some of these applications, we refer to [1–3]. A study of the behavior of solutions to higher order differential equations yield much fewer results than for the least order equations although they are of the utmost importance in a lot of applications, especially neutral delay differential equations. In the literature, there are many papers and books which study the oscillatory and asymptotic behavior of solutions of neutral delay differential equations by using different technique in order to establish some sufficient conditions which ensure oscillatory behavior of the solutions of (1), see [4–6].

The authors in [1,3,7] have studied the oscillatory behavior of the higher-order differential equation

$$\left((x^{(n-1)}(t))^\alpha \right)' + q(t)x^\beta(h(t)) = 0.$$

And the author of [8] extended the results to the following equation

$$\left(r(t) \left(x^{(n-1)}(t) \right)^\alpha \right)' + q(t)x^\alpha(h(t)) = 0.$$

Agarwal, Li and Rath [9–12] investigated the oscillatory behavior of quasi-linear neutral differential equation

$$\left(r(t) \left((x(t) + p(t)x(\zeta(t)))^{(n-1)} \right)^\alpha \right)' + q(t)x^\alpha(h(t)) = 0, \text{ for } t \geq t_0,$$

under the condition

$$0 \leq p(t) < 1.$$

The latter differential equation was studied by Xing et al. in [13] under the condition

$$0 \leq p(t) < \infty.$$

The aim of this paper is to study the oscillatory behavior of the solutions of odd-order NDDE (1). By using Riccati transformation, we establish some sufficient conditions which ensure that every solution of (1) is either oscillatory or tends to zero.

2. Auxiliary Results

In order to prove our main results, we will employ the following lemmas.

Lemma 1 ([14] Lemma (2.3)). *Let $G(v) = Cv - Dv^{(\alpha+1)/\alpha}$ where $C, D > 0$. Then G attains its maximum value on \mathbb{R} at $v^* = (\alpha C / (\alpha + 1)D)^\alpha$ and*

$$\max_{v \in \mathbb{R}} G(v) = G(v^*) = \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}} \frac{C^{\alpha+1}}{D^\alpha}. \quad (2)$$

Lemma 2 ([15]). *Assume that $c_1, c_2 \in [0, \infty)$ and $\gamma > 0$. Then*

$$\left(\sum_{i=1}^2 c_i \right)^\gamma \leq \mu_{i=1}^2 c_i^\gamma, \quad (3)$$

where

$$\mu(\gamma) := \begin{cases} 1 & \text{if } \gamma \leq 1, \\ 2^{\gamma-1} & \text{if } \gamma > 1. \end{cases}$$

Lemma 3. Let $f \in C^n([t_0, \infty), (0, \infty))$. Assume that $f^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$ and that there exists a $t_1 \geq t_0$ such that $f^{(n-1)}(t)f^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t \rightarrow \infty} f(t) \neq 0$, then for every $\mu \in (0, 1)$ there exists $t_\mu \geq t_1$ such that

$$f(t) \geq \frac{\mu}{(n-1)!} t^{n-1} |f^{(n-1)}(t)| \text{ for } t \geq t_\mu.$$

Lemma 4. Let the function x be a positive solution to (1) on the interval $[t_0, \infty)$. Then there exists $t_1 \geq t_0$ such that, for $t \geq t_1$, $Y(t) > 0$, $\left(r\left((Y)^{(n-1)}\right)^\alpha\right)' \leq 0$ and there occur two cases for the derivatives of the function Y :

- (I) $Y'(t) > 0, Y''(t) > 0, Y^{(n-1)}(t) > 0, Y^{(n)}(t) \leq 0;$
- (II) $Y'(t) < 0, Y''(t) > 0, Y^{(n-1)}(t) > 0, Y^{(n)}(t) \leq 0.$

Proof. By the definition of a positive solution to (1) there exists a $t_1 \in [t_0, \infty)$ such that $x(t) > 0, x(h(t)) > 0$ and $x(\zeta(t)) > 0$, for $t \geq t_1$. By the definition of Y , it is easy to see that $Y(t) > 0$. Furthermore, from (1), we have $\left(r\left((Y)^{(n-1)}\right)^\alpha\right)' \leq 0$. The rest of the proof is similar to proof of ([3] Lemma 2). Thus, the proof completed. \square

Lemma 5. Let x be a positive solution of (1), Y satisfy (II) and put

$$\tilde{\eta}(t) = \frac{1}{r^{\frac{1}{\alpha}}(t)} \left(\int_t^\infty q(s) ds \right)^{\frac{1}{\alpha}}.$$

If

$$\int_{t_0}^\infty \tilde{\eta}(s) s^{n-2} ds = \infty, \tag{4}$$

then $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} Y(t) = 0$.

Proof. Let x be a positive solution of (1). Using (I₄) in (1), we have

$$\left(r\left((Y)^{(n-1)}\right)^\alpha\right)'(t) + kq(t)x^\alpha(h(t)) \leq 0. \tag{5}$$

From (II), we note that $\lim_{t \rightarrow \infty} Y(t) = c \geq 0$, due to $Y(t) > 0$ and $Y'(t) < 0$. Assume that $c > 0$. Then for any $\epsilon > 0$, we have $\epsilon + c > Y(t) > c$, eventually. By definition of $Y(t)$, we have

$$x(t) = Y(t) - p(t)x(\zeta(t)) \geq Y(t) - p(t)Y(\zeta(t)),$$

thus,

$$x(t) \geq c - p_0(\epsilon + c) = \frac{c - p_0(\epsilon + c)}{\epsilon + c}(\epsilon + c).$$

This implies that

$$x(t) \geq \varrho Y(t), \tag{6}$$

where $\varrho = \frac{c - p_0(\epsilon + c)}{\epsilon + c} > 0$. Using (6) in (5), we obtain

$$\left(r\left((Y)^{(n-1)}\right)^\alpha\right)'(t) + k\varrho^\alpha q(t)Y^\alpha(h(t)) \leq 0.$$

Integrating the above inequality from t to ∞ , we obtain

$$r(t) \left(Y^{(n-1)}(t) \right)^\alpha \geq k \varrho^\alpha \int_t^\infty q(s) Y^\alpha(h(s)) ds.$$

By $\lim_{t \rightarrow \infty} Y(t) > c$, it follows that

$$Y^{(n-1)}(t) \geq \varrho c k^{\frac{1}{\alpha}} \tilde{\eta}(t). \tag{7}$$

Integrating (7) twice from t to ∞ , we have

$$Y^{(n-3)}(t) \geq \varrho c k^{\frac{1}{\alpha}} \int_t^\infty \int_u^\infty \tilde{\eta}(s) ds du = \varrho c k^{\frac{1}{\alpha}} \int_t^\infty \tilde{\eta}(s) (s-t) ds.$$

Repeating this procedure, we arrive at

$$-Y'(t) \geq \frac{\varrho c k^{\frac{1}{\alpha}}}{(n-3)!} \int_t^\infty \tilde{\eta}(s) (s-t)^{n-3} ds.$$

Now, integrating from t_1 to ∞ , we see that

$$Y(t_1) \geq \frac{\varrho c k^{\frac{1}{\alpha}}}{(n-2)!} \int_{t_1}^\infty \tilde{\eta}(s) (s-t_1)^{n-2} ds \geq \frac{\varrho c k^{\frac{1}{\alpha}}}{2^{n-2} (n-2)!} \int_{2t_1}^\infty \tilde{\eta}(s) s^{n-2} ds,$$

which contradicts (4), and so we have verified that $\lim_{t \rightarrow \infty} Y(t) = 0$. \square

3. Main Results

In the following lemma, we will use the notation $\tilde{q}(t) := \min\{q(t), q(h(t))\}$, $\tilde{q}_2(t) := \min\{q(h^{-1}(t)), q(h^{-1}(\zeta(t)))\}$ and

$$\zeta' \geq \zeta_0 > 0; \tag{8}$$

$$\left(h^{-1}(t) \right)' \geq h_0 > 0. \tag{9}$$

Lemma 6. *Let x be a positive solution of the equation in (1). If (8) and the equality $h \circ \zeta = \zeta \circ h$ hold, then the following inequality is valid*

$$\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha + \frac{p_0^\alpha}{\zeta_0} r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)' + \frac{k}{\mu} \tilde{q}(t) Y^\alpha(h(t)) \leq 0. \tag{10}$$

Moreover, if (8) and (9) hold, then

$$\begin{aligned} & \frac{\left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)'}{h_0} + \frac{p_0^\alpha \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)'}{h_0 \zeta_0} \\ & + \frac{k}{\mu} \tilde{q}_2(t) Y^\alpha(t) \leq 0. \end{aligned} \tag{11}$$

Proof. Let x be a positive solution of (1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(h(t)) > 0$ and $x(\zeta(t)) > 0$ for $t \geq t_1$. By the equality $Y(t) = x(t) + p(t)x(\zeta(t))$ together with Lemma 2, we obtain the inequality

$$Y^\alpha(t) \leq \mu(x^\alpha(t) + p_0^\alpha x^\alpha(\zeta(t))). \tag{12}$$

From (5) and the properties $h \circ \zeta = \zeta \circ h$ and $\zeta' \geq \zeta_0$, we obtain

$$\begin{aligned} 0 &\geq \frac{p_0^\alpha}{\zeta'(t)} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)' + p_0^\alpha kq(\zeta(t)) x^\alpha(h(\zeta(t))) \\ &\geq \frac{p_0^\alpha}{\zeta_0} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)' + p_0^\alpha kq(\zeta(t)) x^\alpha(\zeta(h(t))). \end{aligned} \quad (13)$$

Using the latter inequalities and taking those in (5) and (13) into account as well, we obtain

$$\begin{aligned} 0 &\geq \left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)' + \frac{p_0^\alpha}{\zeta_0} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)' \\ &\quad + kq(t) x^\alpha(h(t)) + p_0^\alpha kq(\zeta(t)) x^\alpha(\zeta(h(t))) \\ &\geq \left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)' + \frac{p_0^\alpha}{\zeta_0} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)' \\ &\quad + k\tilde{q}(t) (x^\alpha(h(t)) + p_0^\alpha x^\alpha(\zeta(h(t))))), \end{aligned}$$

which with (12) gives

$$\begin{aligned} 0 &\geq \left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)' + \frac{p_0^\alpha}{\zeta_0} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)' + \frac{k}{\mu} \tilde{q}(t) Y^\alpha(h(t)) \\ &\geq \left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha + \frac{p_0^\alpha}{\zeta_0} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right) \right)' + \frac{k}{\mu} \tilde{q}(t) Y^\alpha(h(t)). \end{aligned}$$

This proves the inequality in (10). In order to show inequality (11) we proceed as follows. From (8) and (9), we obtain

$$\begin{aligned} 0 &\geq \frac{1}{(h^{-1}(t))'} \left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)' + kq(h^{-1}(t)) x^\alpha(t) \\ &\geq \frac{1}{h_0} \left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)' + kq(h^{-1}(t)) x^\alpha(t). \end{aligned} \quad (14)$$

Moreover,

$$\begin{aligned} 0 &\geq \frac{p_0^\alpha}{(h^{-1}(\zeta(t)))'} \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)' \\ &\quad + p_0^\alpha kq(h^{-1}(\zeta(t))) x^\alpha(\zeta(t)) \\ &\geq \frac{p_0^\alpha}{h_0 \zeta_0} \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)' + p_0^\alpha kq(h^{-1}(\zeta(t))) x^\alpha(\zeta(t)). \end{aligned} \quad (15)$$

Combining (14) with (15) and taking into account (12), we have

$$\begin{aligned} 0 &\geq \frac{1}{h_0} \left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)' + \frac{k}{\mu} \tilde{q}_2(t) (x(t) + p_0 x(\zeta(t)))^\alpha \\ &\quad + \frac{p_0^\alpha}{h_0 \zeta_0} \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)', \end{aligned}$$

that is

$$\begin{aligned} 0 &\geq \frac{1}{h_0} \left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha + \frac{p_0^\alpha}{\zeta_0} r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)' \\ &\quad + \frac{k}{\mu} \tilde{q}_2(t) Y^\alpha(t). \end{aligned}$$

This proves (11) and completes the proof of Lemma 6. \square

Theorem 1. Suppose that $h(t) \leq \zeta(t)$, $h \circ \zeta = \zeta \circ h$, $h'(t) > 0$ and (8) hold. Moreover, assume that (4) is satisfied and that there exists a function $\delta \in C^1([t_0, \infty), (0, \infty))$ with the property that for all sufficiently large $t_1 \geq t_0$, there exists $t_2 \geq t_1$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_1}^t \left[k\delta(s) \frac{\tilde{q}(s)}{\mu} - \frac{((n-2)!)^\alpha}{\mu^\alpha(\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\zeta_0}\right) \frac{r(s)(\delta'(s))^{\alpha+1}}{(\delta(s)h^{n-2}(s)h'(s))^\alpha} \right] ds = \infty. \quad (16)$$

Then, a solution $x(t)$ to (1) either oscillates or else tends to zero when $t \rightarrow \infty$.

Proof. Let x be a positive solution of (1). Then, there exist $t_1 \geq t_0$ such that $x(t) > 0$, $x(h(t)) > 0$ and $x(\zeta(t)) > 0$ for $t \geq t_1$. Define the positive function ω by

$$\omega(t) = \delta(t) \frac{r(t) \left(Y^{(n-1)}(t) \right)^\alpha}{Y^\alpha(h(t))}. \quad (17)$$

Hence, by differentiating (17), we obtain

$$\begin{aligned} \omega'(t) &= \delta'(t) \frac{r \left(Y^{(n-1)}(t) \right)^\alpha}{Y^\alpha(h(t))} + \delta(t) \frac{\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)'}{Y^\alpha(h(t))} \\ &\quad - \frac{\alpha \delta(t) r(t) \left(Y^{(n-1)}(t) \right)^\alpha Y^{\alpha-1}(h(t)) Y'(h(t)) h'(t)}{Y^{2\alpha}(h(t))}. \end{aligned} \quad (18)$$

Since $Y' > 0$, $Y'' > 0$, we see that $\lim_{t \rightarrow \infty} Y'(t) \neq 0$, using Lemma 3 with $f = Y'$, we obtain

$$Y'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} Y^{(n-1)}(t),$$

for every $\mu \in (0, 1)$. Thus, by $Y^{(n)}(t) \leq 0$, we obtain

$$Y'(h(t)) \geq \frac{\mu}{(n-2)!} (h(t))^{n-2} Y^{(n-1)}(h(t)) \geq \frac{\mu}{(n-2)!} (h(t))^{n-2} Y^{(n-1)}(t). \quad (19)$$

Substituting (17) and (19) into (18) implies

$$\begin{aligned} \omega'(t) &\leq \delta'(t) \frac{r(t) \left(Y^{(n-1)}(t) \right)^\alpha}{Y^\alpha(h(t))} + \delta(t) \frac{\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)'}{Y^\alpha(h(t))} \\ &\quad - \left(\frac{Y^{(n-1)}(t)}{Y(h(t))} \right)^{\alpha+1} \frac{\alpha \delta(t) r(t) \mu h^{n-2}(t) h'(t)}{(n-2)!} \\ &\leq \delta(t) \frac{\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)'}{Y^\alpha(h(t))} + \frac{\delta'(t)}{\delta(t)} \omega(t) \\ &\quad - \frac{\alpha \delta(t) r(t) \mu h^{n-2}(t) h'(t)}{(n-2)!} \left(\frac{\omega(t)}{\delta(t) r(t)} \right)^{\frac{\alpha+1}{\alpha}}, \end{aligned}$$

that is,

$$\begin{aligned} \omega'(t) &\leq \delta(t) \frac{\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)'}{Y^\alpha(h(t))} + \frac{\delta'(t)}{\delta(t)} \omega(t) \\ &\quad - \frac{\alpha \mu h^{n-2}(t) h'(t)}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(t)} \omega^{(\alpha+1)/\alpha}(t). \end{aligned} \quad (20)$$

Now, define another positive function v by

$$v(t) = \delta(t) \frac{r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha}{Y^\alpha(h(t))}. \quad (21)$$

By differentiating (21), we obtain

$$v'(t) = \delta'(t) \frac{r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha}{Y^\alpha(h(t))} + \frac{\delta(t) \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)'}{Y^\alpha(h(t))} - \frac{\alpha \delta(t) r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha Y^{\alpha-1}(h(t)) Y'(h(t)) h'(t)}{Y^{2\alpha}(h(t))}. \quad (22)$$

From (19), $h(t) \leq \zeta(t)$ and $Y^{(n)}(t) \leq 0$, we have

$$Y'(h(t)) \geq \frac{\mu}{(n-2)!} (h(t))^{n-2} Y^{(n-1)}(h(t)) \geq \frac{\mu}{(n-2)!} (h(t))^{n-2} Y^{(n-1)}(\zeta(t)). \quad (23)$$

Substituting (23) and (21) into (22), implies

$$\begin{aligned} v'(t) &\leq \delta'(t) \frac{r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha}{Y^\alpha(h(t))} + \delta(t) \frac{\left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)'}{Y^\alpha(h(t))} \\ &\quad - \left(\frac{Y^{(n-1)}(\zeta(t))}{Y(h(t))} \right)^{\alpha+1} \frac{\alpha \delta(t) r(\zeta(t)) \mu h^{n-2}(t) h'(t)}{(n-2)!} \\ &\leq \delta(t) \frac{\left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)'}{Y^\alpha(h(t))} + \frac{\delta'(t)}{\delta(t)} v(t) \\ &\quad - \frac{\alpha \delta(t) r(\zeta(t)) \mu h^{n-2}(t) h'(t)}{(n-2)!} \left(\frac{v(t)}{\delta(t) r(\zeta(t))} \right)^{\frac{\alpha+1}{\alpha}}. \end{aligned}$$

By $r'(t) > 0$, we obtain

$$v'(t) \leq \delta(t) \frac{\left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)'}{Y^\alpha(h(t))} + \frac{\delta'(t)}{\delta(t)} v(t) - \frac{\alpha \mu h^{n-2}(t) h'(t)}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(t)} v^{(\alpha+1)/\alpha}(t). \quad (24)$$

Now, using inequalities (20) and (24), we obtain

$$\begin{aligned} \omega'(t) + \frac{p_0^\alpha}{\zeta_0} v'(t) &\leq \delta(t) \frac{\left(r(t) \left(Y^{(n-1)}(t) \right)^\alpha \right)' + \frac{\delta_0^\alpha}{\zeta_0} \left(r(\zeta(t)) \left(Y^{(n-1)}(\zeta(t)) \right)^\alpha \right)'}{Y^\alpha(h(t))} \\ &\quad + \frac{\delta'(t)}{\delta(t)} \omega(t) - \frac{\alpha \mu h^{n-2}(t) h'(t)}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(t)} \omega^{(\alpha+1)/\alpha}(t) \\ &\quad + \frac{\delta_0^\alpha}{\zeta_0} \left(\frac{\delta'(t)}{\delta(t)} v(t) - \frac{\alpha \mu h^{n-2}(t) h'(t)}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(t)} v^{(\alpha+1)/\alpha}(t) \right). \quad (25) \end{aligned}$$

By inserting the inequality in (10) in (26), we obtain

$$\begin{aligned} \omega'(t) + \frac{p_0^\alpha}{\zeta_0} v'(t) \leq & -\delta(t) \left(\frac{k\tilde{q}(t)}{\mu} \right) \\ & + \frac{\delta'(t)}{\delta(t)} \omega(t) - \frac{\alpha\mu h^{n-2}(t)h'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)} \omega^{(\alpha+1)/\alpha}(t) \\ & + \frac{\delta_0^\alpha}{\zeta_0} \left(\frac{\delta'(t)}{\delta(t)} v(t) - \frac{\alpha\mu h^{n-2}(t)h'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(\zeta(t))} v^{(\alpha+1)/\alpha}(t) \right). \end{aligned} \tag{26}$$

By applying the Lemma 1 with

$$C = \frac{\delta'(t)}{\delta(t)} \text{ and } D = \frac{\alpha\mu h^{n-2}(t)h'(t)}{(n-2)!\delta^{1/\alpha}(t)r^{1/\alpha}(t)},$$

we obtain

$$\begin{aligned} \omega'(t) + \frac{p_0^\alpha}{\zeta_0} v'(t) \leq & -\delta(t) \left(\frac{k\tilde{q}(t)}{\mu} \right) + \frac{((n-2)!)^\alpha}{\mu^\alpha(\alpha+1)^{\alpha+1}} \frac{r(t)(\delta'(t))^{\alpha+1}}{(\delta(t)h^{n-2}(t)h'(t))^\alpha} \\ & + \frac{p_0^\alpha((n-2)!)^\alpha}{\zeta_0\mu^\alpha(\alpha+1)^{\alpha+1}} \frac{r(t)(\delta'(t))^{\alpha+1}}{(\delta(t)h^{n-2}(t)h'(t))^\alpha}. \end{aligned}$$

Integrating last the inequality from t_2 to t , we obtain

$$\int_{t_2}^t \left[k\delta(s) \frac{\tilde{q}(s)}{\mu} - \frac{((n-2)!)^\alpha}{\mu^\alpha(\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\zeta_0} \right) \frac{r(s)(\delta'(s))^{\alpha+1}}{(\delta(s)h^{n-2}(s)h'(s))^\alpha} \right] ds \leq \omega(t_2) + \frac{p_0^\alpha}{\zeta_0} v(t_2).$$

The proof is complete. \square

Theorem 2. Suppose that the functions h and ζ satisfy (8), (9) and $h(t) \leq \zeta(t)$ for t_0 . In addition, suppose that (4) is satisfied. If there exists a function $\delta \in C^1([t_0, \infty), (0, \infty))$ with the property that for all sufficiently large $t_1 \geq t_0$, there exists $t_2 \geq t_1$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_2}^t \left[k\delta(s) \frac{\tilde{q}_2(s)}{\mu} - \frac{((n-2)!)^\alpha}{\mu^\alpha h_0(\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\zeta_0} \right) \frac{r(h^{-1}(s))(\delta'(s))^{\alpha+1}}{(\delta(s)s^{n-2})^\alpha} \right] ds = \infty \tag{27}$$

is valid. Then a solution $x(t)$ of Equation (1) oscillates or tends to zero when $t \rightarrow \infty$.

Proof. Let x be a positive solution of (1). Then, there exist $t_1 \geq t_0$ such that $x(t) > 0$, $x(h(t)) > 0$ and $x(\zeta(t)) > 0$ for $t \geq t_1$. Define the positive function ω by

$$\omega(t) = \delta(t) \frac{r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha}{Y^\alpha(t)}. \tag{28}$$

Hence, by differentiating (28), we obtain

$$\begin{aligned} \omega'(t) = & \delta'(t) \frac{r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha}{Y^\alpha(t)} + \delta(t) \frac{\left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)'}{Y^\alpha(t)} \\ & - \frac{\alpha\delta(t)r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha Y^{\alpha-1}(t)Y'(t)}{Y^{2\alpha}(t)}. \end{aligned} \tag{29}$$

Since $Y' > 0$, $Y'' > 0$, we see that $\lim_{t \rightarrow \infty} Y' \neq 0$, using Lemma 3 with $f = Y'$, we obtain

$$Y'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} Y^{(n-1)}(t), \tag{30}$$

for every $\mu \in (0, 1)$. Thus, by $h^{-1}(t) > t$ and $Y^{(n)}(t) \leq 0$, we obtain

$$Y'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} Y^{(n-1)}(t) \geq Y'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} Y^{(n-1)}(h^{-1}(t)). \quad (31)$$

Substituting (28) and (31) into (29) implies

$$\begin{aligned} \omega'(t) &\leq \delta'(t) \frac{r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha}{Y^\alpha(t)} + \delta(t) \frac{\left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)'}{Y^\alpha(t)} \\ &\quad - \left(\frac{Y^{(n-1)}(h^{-1}(t))}{Y(t)} \right)^{\alpha+1} \frac{\alpha \delta(t) r(h^{-1}(t)) \mu t^{n-2}}{(n-2)!} \\ &\leq \delta(t) \frac{\left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)'}{Y^\alpha(t)} + \frac{\delta'(t)}{\delta(t)} \omega(t) \\ &\quad - \frac{\alpha \delta(t) r(h^{-1}(t)) \mu t^{n-2}}{(n-2)!} \left(\frac{\omega(t)}{\delta(t) r(h^{-1}(t))} \right)^{\frac{\alpha+1}{\alpha}}, \end{aligned}$$

that is,

$$\begin{aligned} \omega'(t) &\leq \delta(t) \frac{\left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)'}{Y^\alpha(t)} + \frac{\delta'(t)}{\delta(t)} \omega(t) \\ &\quad - \frac{\alpha \mu t^{n-2}}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(t))} \omega^{(\alpha+1)/\alpha}(t). \end{aligned} \quad (32)$$

Now, define another positive function v by

$$v(t) = \delta(t) \frac{r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha}{Y^\alpha(t)}. \quad (33)$$

By differentiating (33), we obtain

$$\begin{aligned} v'(t) &= \delta'(t) \frac{r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha}{Y^\alpha(t)} \\ &\quad + \frac{\delta(t) \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)'}{Y^\alpha(t)} \\ &\quad - \frac{\alpha \delta(t) r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha Y^{\alpha-1}(t) Y'(t)}{Y^{2\alpha}(t)}. \end{aligned} \quad (34)$$

From (30), $h^{-1}(\zeta(t)) \geq t$ and $Y^{(n)}(t) \leq 0$, we have

$$Y'(t) \geq \frac{\mu}{(n-2)!} t^{n-2} Y^{(n-1)}(t) \geq \frac{\mu}{(n-2)!} t^{n-2} Y^{(n-1)}(h^{-1}(\zeta(t))). \quad (35)$$

A similar method has been used in the work [16]. Substituting (35) and (33) into (34), implies

$$\begin{aligned} v'(t) &\leq \delta'(t) \frac{r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha}{Y^\alpha(t)} \\ &\quad + \frac{\delta(t) \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)'}{Y^\alpha(t)} \\ &\quad - \left(\frac{Y^{(n-1)}(h^{-1}(\zeta(t)))}{Y(t)} \right)^{\alpha+1} \frac{\alpha \delta(t) r(h^{-1}(\zeta(t))) \mu t^{n-2}}{(n-2)!} \\ &\leq \frac{\delta(t) \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)'}{Y^\alpha(t)} + \frac{\delta'(t)}{\delta(t)} v(t) \\ &\quad - \frac{\alpha \delta(t) r(h^{-1}(\zeta(t))) \mu t^{n-2}}{(n-2)!} \left(\frac{v(t)}{\delta(t) r(h^{-1}(\zeta(t)))} \right)^{\frac{\alpha+1}{\alpha}}. \end{aligned}$$

By $r'(t) > 0$, we obtain

$$\begin{aligned} v'(t) &\leq \frac{\delta(t) \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)'}{Y^\alpha(t)} + \frac{\delta'(t)}{\delta(t)} v(t) \\ &\quad - \frac{\alpha \mu t^{n-2}}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(t))} v^{(\alpha+1)/\alpha}(t). \end{aligned} \quad (36)$$

Now, using inequalities (20) and (24), we obtain

$$\begin{aligned} &\frac{1}{h_0} \omega'(t) + \frac{p_0^\alpha}{h_0 \zeta_0} v'(t) \\ &\leq \frac{\delta(t)}{h_0 Y^\alpha(t)} \left(r(h^{-1}(t)) \left(Y^{(n-1)}(h^{-1}(t)) \right)^\alpha \right)' + \frac{p_0^\alpha}{h_0 \zeta_0} \left(r(h^{-1}(\zeta(t))) \left(Y^{(n-1)}(h^{-1}(\zeta(t))) \right)^\alpha \right)' \\ &\quad + \frac{\delta'(t)}{h_0 \delta(t)} \omega(t) - \frac{\alpha \mu t^{n-2}}{h_0 (n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(t))} \omega^{(\alpha+1)/\alpha}(t) \\ &\quad + \frac{p_0^\alpha}{h_0 \zeta_0} \left(\frac{\delta'(t)}{\delta(t)} v(t) - \frac{\alpha \mu t^{n-2}}{(n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(t))} v^{(\alpha+1)/\alpha}(t) \right). \end{aligned}$$

By (4), we obtain

$$\begin{aligned} &\frac{1}{h_0} \omega'(t) + \frac{p_0^\alpha}{h_0 \zeta_0} v'(t) \\ &\leq -\delta(t) \left(\frac{k \tilde{q}_2(t)}{\mu} \right) \\ &\quad + \frac{\delta'(t)}{h_0 \delta(t)} \omega(t) - \frac{\alpha \mu t^{n-2}}{h_0 (n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(t))} \omega^{(\alpha+1)/\alpha}(t) \\ &\quad + \frac{p_0^\alpha}{\zeta_0} \left(\frac{\delta'(t)}{h_0 \delta(t)} v(t) - \frac{\alpha \mu t^{n-2}}{h_0 (n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(\zeta(t)))} v^{(\alpha+1)/\alpha}(t) \right). \end{aligned}$$

By using Lemma 1 with

$$C = \frac{\delta'(t)}{h_0 \delta(t)} \quad \text{and} \quad D = \frac{\alpha \mu t^{n-2}}{h_0 (n-2)! \delta^{1/\alpha}(t) r^{1/\alpha}(h^{-1}(t))},$$

we obtain

$$\begin{aligned} \frac{1}{h_0}\omega'(t) + \frac{p_0^\alpha}{h_0\zeta_0}v'(t) &\leq -\delta(t)\left(\frac{k\tilde{q}_2(t)}{\mu}\right) \\ &+ \frac{((n-2)!)^\alpha}{\mu^\alpha h_0(\alpha+1)^{\alpha+1}} \frac{r(h^{-1}(t))(\delta'(t))^{\alpha+1}}{(\delta(t)t^{n-2})^\alpha} \\ &+ \frac{p_0^\alpha((n-2)!)^\alpha}{\zeta_0 h_0 \mu^\alpha(\alpha+1)^{\alpha+1}} \frac{r(h^{-1}(t))(\delta'(t))^{\alpha+1}}{(\delta(t)t^{n-2})^\alpha}. \end{aligned}$$

Integrating both sides of the latter inequality from t_2 to t , we obtain

$$\begin{aligned} &\int_{t_2}^t \left[\frac{k\delta(s)}{\mu} \tilde{q}_2(s) - \frac{((n-2)!)^\alpha}{\mu^\alpha h_0(\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\zeta_0}\right) \frac{r(h^{-1}(s))(\delta'(s))^{\alpha+1}}{(\delta(s)s^{n-2})^\alpha} \right] ds \\ &\leq \frac{1}{h_0}\omega(t_2) + \frac{p_0^\alpha}{h_0\zeta_0}v(t_2). \end{aligned}$$

The proof is complete. \square

Example 1. Consider the odd order neutral delay differential equation

$$Y^{(n)}(t) + \frac{q_0}{t^n}x\left(\frac{t}{e^2}\right) = 0, \quad t \geq 1, \quad q_0 > 0, \quad n \geq 3, \tag{37}$$

where $Y(t) = x(t) + \frac{17}{18}x\left(\frac{t}{e}\right)$, and

$$k = \mu = \alpha = r(t) = 1, \quad \tilde{q}_2(s) = \frac{q_0}{(e^2t)^n}, \quad h(t) = \frac{t}{e^2}, \quad \zeta(t) = \frac{t}{e}, \quad \text{and set } \delta(t) = t^{n-1}.$$

Using Example 1 in [17], we find that every solution of (37) oscillates or tends to zero if

$$q_0 > 9(n-1)!e^{2n-3},$$

and using Example 2.11 in [13], we find that every solution of (37) oscillates or tends to zero if

$$q_0 > (n-1)! \left(e^{2n-3} + \frac{17}{18}e^{2n-2} \right).$$

From condition (27) in Theorem 2, we see that every solution of (37) oscillates or tends to zero if

$$\limsup_{t \rightarrow \infty} \left[\frac{q_0}{e^{2n}} - \frac{(n-2)!(n-1)}{4e^2} \left(1 + \frac{17}{18}e\right) \right] \ln t = \infty,$$

thus,

$$q_0 > \frac{e}{4}(n-1)! \left(e^{2n-3} + \frac{17}{18}e^{2n-2} \right).$$

Hence, we can see that our results are better than ([17] Example 1) and ([13] Example 2.11).

4. Conclusions

In this work, we established the oscillation criteria for a class of odd-order delay differential equations. By using Riccati transformation, we presented some sufficient conditions which ensure that every solution of (1) is either oscillatory or tends to zero. The approach used does not need to be restricted by the condition $0 < p(t) < 1$, unlike many previous work.

For interested researchers, results presented in this paper may be extended to more general equations than (1). Another interesting problem for further research is to obtain new criteria for oscillatory solutions for (1) without requiring (8).

Author Contributions: Conceptualization, C.C., O.M., B.Q., N.A.A., S.K.E. and M.Z.; Data curation, O.M., B.Q., N.A.A., S.K.E. and M.Z.; Formal analysis, C.C., O.M., B.Q., N.A.A., S.K.E. and M.Z.; Investigation, C.C., O.M., B.Q., N.A.A. and M.Z.; Methodology, C.C., O.M., B.Q., N.A.A., S.K.E. and M.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to Taif university and King Khalid University for funding support for this paper.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Program under Grant No. RGP. 2/51/42. Taif University Researchers Supporting Project number (TURSP-2020/247), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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