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Asymptotic Properties of Discrete Minimal $s, \log^t$-Energy Constants and Configurations

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Abstract: We investigated the energy of N points on an infinite compact metric space (A, d) of a diameter less than 1 that interact through the potential $(1/d^s)(\log 1/d)^t$, where $s, t \geq 0$ and d is the metric distance. With $\mathcal{E}^s_{\log^t}(A, N)$ denoting the minimal energy for such N-point configurations, we studied certain continuity and differentiability properties of $\mathcal{E}^s_{\log^t}(A, N)$ in the variable s. Then, we showed that in the limits, as $s \to \infty$ and as $s \to s_0 > 0$, N-point configurations that minimize the $s, \log^t$-energy tends to an N-point best-packing configuration and an N-point configuration that minimizes the $s_0, \log^t$-energy, respectively. Furthermore, we considered when A are circles in the Euclidean space $\mathbb{R}^2$. In particular, we proved the minimality of N distinct equally spaced points on circles in $\mathbb{R}^2$ for some certain s and t. The study on circles shows a possibility for the utilization of N points generated through such new potential to uniformly discretize on objects with very high symmetry.

Keywords: discrete minimal energy; best-packing; Riesz energy; logarithmic energy

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1. Introduction

The general setting of the discrete minimal energy problem is the following. Let (A, d) be an infinite compact metric space and $K : A \times A \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous kernel. Note that in some contexts, the kernel $K(x, y)$ is called a *potential*. For a fixed set of N points $\omega_N \subset A$, we define the *K-energy of ω_N* as follows:

$$E_K(\omega_N) := \sum_{\substack{x \neq y \\ x,y \in \omega_N}} K(x, y).$$

The *minimal N-point K-energy of the set A* is defined by

$$\mathcal{E}_K(A, N) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} E_K(\omega_N),$$

where $\#\omega_N$ stands for the cardinality of the set ω_N. A *minimal N-point K-energy configuration* is a configuration ω_N^K of N points in A that minimizes such energy, namely

$$E_K(\omega_N^K) = \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} E_K(\omega_N).$$

It is known that ω_N^K always exists and in general ω_N^K may not be unique.

Two important kernels in the theory of minimal energy are Riesz and logarithmic kernels. The *(Riesz) s-kernel* and log-*kernel* are defined by

$$K^s(x,y) := \frac{1}{d(x,y)^s}, \quad s \geq 0, \tag{1}$$

and

$$K_{\log}(x,y) := \log \frac{1}{d(x,y)},$$

for all $(x,y) \in A \times A$, respectively. It is not difficult to check that both kernels are lower semicontinuous on $A \times A$. The *s-energy of* ω_N and the *minimal N-point s-energy of the set A* are

$$E^s(\omega_N) := E_{K^s}(\omega_N) \quad \text{and} \quad \mathcal{E}^s(A,N) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} E^s(\omega_N)$$

and we denote by $\omega_N^s := \omega_N^{K^s}$ and call this configuration a *minimal N-point s-energy configuration*. Similarly, the log-*energy of* ω_N and the *N-point* log-*energy of the set A* are

$$E_{\log}(\omega_N) := E_{K_{\log}}(\omega_N) \quad \text{and} \quad \mathcal{E}_{\log}(A,N) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} E_{\log}(\omega_N)$$

and we denote by $\omega_N^{\log} := \omega_N^{K_{\log}}$ and call this configuration a *minimal N-point* log-*energy configuration*.

Let us provide a short survey of these two energy problems.

The study of s-energy constants and configurations has a long history in physics, chemistry, and mathematics. Finding the arrangements of ω_N^s where the set A is the unit sphere $\mathbb{S}^2$ in the Euclidean space $\mathbb{R}^3$ has been an active area since the beginning of the 19th century. The problem is known as the generalized Thomson problem (see [1] and [2] (Chapter 2.4)). Candidates for ω_N^s for several numbers of N are available (see, e.g., [3]). However, the solutions (with rigorous proof) are obtainable for a handful of values of N (see, e.g., [4–7]). For example, when $N = 5$, the generalized Thomson problem becomes surprisingly difficult. Schwartz, using computer-aided proof, showed that such ω_5^s on $\mathbb{S}^2$ can be the vertices of the triangular bipyramid or a square-based pyramid (depending on s) in a single monograph of 180 pages [8] (see also a synopsis of his work [7]). For a general compact set A in the Euclidean space $\mathbb{R}^m$, the study of the distribution of a minimal N-point s-energy configurations of A as $N \to \infty$ can be found in [9,10]. In [10], it was shown that when s is any fixed number greater than the Hausdorff dimension of A, the minimal N-point s-energy configurations of A are "good points" to represent the set A. This is because such configurations are asymptotically uniformly distributed over the set A as $N \to \infty$ (see the precise statement in [10] (Theorems 2.1 and 2.2)). The results in [10] have wide ranging applications in various fields of computational science, such as computer-aided geometric design, finite element tesselations, statistical sampling, etc.

The log-energy problem has been heavily studied when A is a subset of the Euclidean space $\mathbb{R}^2$ (or $\mathbb{C}$) because it has had a profound influence in approximation theory (see, e.g., [11–15]). For $A \subset \mathbb{C}$, the points in $\omega_N^{\log}$ are commonly known as Fekete points or Chebyshev points which can be used as interpolation and integration nodes (see [16]). The log-energy problem received other special attention when Steven Smale posed Problem #7 in his book chapter entitled "Mathematical problems for the next century" [17]. Problem #7 asks for a construction of an algorithm which on input $N \geq 2$ outputs a configuration $\omega_N = \{x_1, \ldots, x_N\}$ of distinct points on $\mathbb{S}^2$ embedded in $\mathbb{R}^3$ such that

$$E_{\log}(\omega_N) - \mathcal{E}_{\log}(\mathbb{S}^2, N) \leq c \log N$$

(where c is a constant independent of N and ω_N) and requires that its running time grows at most polynomially in N. This problem arose from his joint work with Shub [18] on complexity theory. In order to answer this question, it is natural to understand the asymptotic expansion of $\mathcal{E}_{\log}(\mathbb{S}^2, N)$ in the variable N (see [19] for conjectures and the progress). The problem concerning the arrangements of $\omega_N^{\log}$ on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is posed by Whyte [20] in 1952. Whyte's problem is also attractive and intractable. We refer to [21] for a glimpse of this problem.

In [2], Borodachov, Hardin, and Saff investigated asymptotic properties of minimal N-point s-energy constants and configurations for fixed N and varying s. Since this is our main focus in this paper, we will state these results below.

The first theorem (Ref. [2] (Theorem 2.7.1 and Theorem 2.7.3)) concerns the continuity and differentiability of the function

$$f(s) := \mathcal{E}^s(A, N), \quad s \geq 0. \tag{2}$$

In order to state such a theorem, let us define a set

$$G^s_{\log}(A, N) := \left\{ \sum_{\substack{x \neq y \\ x,y \in \omega_N}} K^s(x, y) K_{\log}(x, y) : \omega_N \subset A \text{ and } E^s(\omega_N) = \mathcal{E}^s(A, N) \right\},$$

for $s \geq 0$.

Theorem 1. *Let (A, d) be an infinite compact metric space and let $N \geq 2$ be fixed. Then:*

(a) the function $f(s)$ defined in (2) is continuous on $[0, \infty)$;

(b) the function $f(s)$ is right differentiable on $[0, \infty)$ and left differentiable on $(0, \infty)$ with:

$$f'_+(s) := \lim_{r \to s^+} \frac{f(r) - f(s)}{r - s} = \inf G^s(A, N), \quad s \geq 0,$$

and:

$$f'_-(s) := \lim_{r \to s^-} \frac{f(r) - f(s)}{r - s} = \sup G^s(A, N), \quad s > 0.$$

We will see in Theorems 2 and 3 below that there are certain relations between minimal s-energy problems as $s \to \infty$ and the best-packing problem defined as follows. The *N-point best-packing distance of the set A* is defined as

$$\delta_N(A) := \max\{\delta(\omega_N) : \omega_N \subset A\}, \tag{3}$$

where

$$\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} d(x_i, x_j)$$

denotes the *separation distance of an N-point configuration* $\omega_N = \{x_1, \ldots, x_N\}$, and *$N$-point best-packing configurations* are N-point configurations attaining the maximum in (3). For further details on the best-packing problem, we refer the reader to [2] (Chapter 3).

The following theorem [2] (Corollary 2.7.5 and Proposition 3.1.2) explains the behavior of $\mathcal{E}^s(A, N)$ as $s \to 0^+$ and $s \to \infty$.

Theorem 2. *For $N \geq 2$ and an infinite compact metric space (A, d),*

$$\lim_{s \to 0^+} \frac{\mathcal{E}^s(A, N) - N(N-1)}{s} = \mathcal{E}_{\log}(A, N)$$

and

$$\lim_{s \to \infty} (\mathcal{E}^s(A, N))^{1/s} = \frac{1}{\delta_N(A)}.$$

Before we state more results, let us define a cluster configuration. Let $s_0 \in [0, \infty]$. We say that

- An N-point configuration $\omega_N \subset A$ is a *cluster configuration of ω_N^s as $s \to s_0^+$* if there is a sequence $\{s_k\}_{k=1}^{\infty} \subset (s_0, \infty)$ such that $\lim\limits_{k\to\infty} s_k = s_0$ and $\lim\limits_{k\to\infty} \omega_N^{s_k} = \omega_N$ in the topology of A^N induced by the metric d.
- An N-point configuration $\omega_N \subset A$ is a *cluster configuration of ω_N^s as $s \to s_0^-$* if there is a sequence $\{s_k\}_{k=1}^{\infty} \subset [0, s_0)$ such that $\lim\limits_{k\to\infty} s_k = s_0$ and $\lim\limits_{k\to\infty} \omega_N^{s_k} = \omega_N$ in the topology of A^N induced by the metric d.
- An N-point configuration $\omega_N \subset A$ is a *cluster configuration of ω_N^s as $s \to s_0$* if there is a sequence $\{s_k\}_{k=1}^{\infty} \subset [0, \infty)$ such that $\lim\limits_{k\to\infty} s_k = s_0$ and $\lim\limits_{k\to\infty} \omega_N^{s_k} = \omega_N$ in the topology of A^N induced by the metric d.

The properties of the cluster configurations of minimal N-point s-energy configurations as s varies (see [2] (Theorem 2.7.1 and Proposition 3.1.2)) are described i Theorem 3.

Theorem 3. *Let (A, d) be an infinite compact metric space and for $s \geq 0$ and $N \geq 2$, let ω_N^s denote a minimal N-point s-energy configuration on A. Then,*

(a) For $s_0 > 0$, any cluster configuration of ω_N^s as $s \to s_0$ is a minimal N-point s_0-energy configuration;
(b) Any cluster configuration of ω_N^s as $s \to 0^+$ is a minimal N-point log*-energy configuration;*
(c) Any cluster configuration of ω_N^s as $s \to \infty$ is a N-point best-packing configuration.

In this paper, we consider the following $s, \log^t$*-kernel:*

$$K^s_{\log^t}(x,y) = \frac{1}{d(x,y)^s}\left(\log \frac{1}{d(x,y)}\right)^t, \quad s \geq 0, \quad t \geq 0. \tag{4}$$

with a corresponding $s, \log^t$*-energy of* ω_N and *minimal N-point $s, \log^t$-energy of the set A:*

$$E^s_{\log^t}(\omega_N) := E_{K^s_{\log^t}}(\omega_N) \quad \text{and} \quad \mathcal{E}^s_{\log^t}(A,N) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} E^s_{\log^t}(\omega_N),$$

respectively. We set

$$\omega_N^{s,\log^t} := \omega_N^{K^s_{\log^t}},$$

and call it a *minimal N-point $s, \log^t$-energy configuration*. Note that the kernel $K^s_{\log^t}(x,y)$ is lower semicontinuous on $A \times A$ and this $s, \log^t$-energy can be viewed as a generalization of both s-energy and log-energy. The kernel in (4) first appeared in the study of the differentiability of the function $f(s) = \mathcal{E}^s(A,N)$ in [2] (Theorem 2.7.3). To the authors' knowledge, no study involving $s, \log^t$-energy constants and configurations has appeared in the literature previously.

The first goal of this paper was to prove the analogues of Theorems 1–3 for $s, \log^t$-energy constants and configurations. We would like to emphasize that we will limit our interest to the sets A with $\operatorname{diam}(A) < 1$, where

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x,y)$$

denotes the diameter of A. For the case where $\operatorname{diam}(A) \geq 1$, the values of the kernel $K^s_{\log^t}(x,y)$ can be 0 or negative and the analysis becomes laborious.

The second goal was to investigate the arrangements of $\omega_N^{s,\log^t}$ on circles in $\mathbb{R}^2$. Using an available tool in (Ref. [2] (Theorem 2.3.1)), we show that $\omega_N^{s,\log^t}$ on any circle with a diameter less than 1 are N distinct equally spaced points. The motivation of this study for objects with very high symmetry comes from the study of the limiting distributions of $\omega_N^s, s > 0$ and $\omega_N^{\log}$ on the m-dimensional sphere $\mathbb{S}^m$ in the Euclidean space $\mathbb{R}^{m+1}$ in [2] (Theorem 6.1.7). In [2] (Theorem 6.1.7), it was shown that $\omega_N^s, s > 0$ and $\omega_N^{\log}$ on $\mathbb{S}^m$ are asymptotically uniformly distributed with respect to the surface area measured on $\mathbb{S}^m$ as $N \to \infty$ (see also [22] for applications of this result). Our study on circles exhibits a possibility for the utilization of $\omega_N^{s,\log^t}$ to uniformly discretize m-dimensional spheres in $\mathbb{R}^{m+1}$.

The remainder of this article is organized as follows. The main results of this paper are stated in Section 2. The proof of the main results are in Section 3. We keep all auxiliary lemmas in Section 4. Finally, conclusions and future work are discussed in Section 5.

2. Main Results

2.1. Asymptotic Properties of Discrete Minimal $s, \log^t$-Energy

The asymptotic behavior of minimal N-point $s, \log^t$-energy constants and configurations as $s \to \infty$ can be explained in the following theorem.

Theorem 4. *Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that (A, d) is an infinite compact metric space with $\operatorname{diam}(A) < 1$. Then,*

$$\lim_{s\to\infty} \left(\mathcal{E}^s_{\log^t}(A, N) \right)^{1/s} = \frac{1}{\delta_N(A)}.$$

Furthermore, every cluster configuration of $\omega_N^{s,\log^t}$ as $s \to \infty$ is an N-point best-packing configuration on A.

For a fixed $t \geq 0$, we define

$$g(s) := \mathcal{E}^s_{\log^t}(A, N), \qquad s \geq 0.$$

The continuity of $g(s)$ is stated below.

Theorem 5. *Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that (A, d) is an infinite compact metric space with $\operatorname{diam}(A) < 1$. Then, the function $g(s)$ is continuous on $[0, \infty)$.*

As a consequence of the continuity of $g(s)$, we analyze a property of cluster configurations of $\omega_N^{s,\log^t}$ as $s \to s_0 > 0$ in the following theorem.

Theorem 6. *Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that (A, d) is an infinite compact metric space with $\operatorname{diam}(A) < 1$. Denote by $\omega_N^{s,\log^t}$ a minimal N-point $s, \log^t$-energy configuration on A. Then, for any $s_0 > 0$, any cluster configuration of $\omega_N^{s,\log^t}$, as $s \to s_0$, is a minimal N-point $s_0, \log^t$-energy configuration on A.*

For $s \geq 0$ and $t \geq 0$, we set

$$G^s_{\log^{t+1}}(A, N) := \{E^s_{\log^{t+1}}(\omega_N) : \omega_N \subset A \text{ and } E^s_{\log^t}(\omega_N) = \mathcal{E}^s_{\log^t}(A, N)\}.$$

The differentiability properties of $g(s)$ are described in Theorems 7 and 8.

Theorem 7. *Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that (A, d) is an infinite compact metric space with $\mathrm{diam}(A) < 1$. Then, the function $g(s)$ is right differentiable on $[0, \infty)$ and left differentiable on $(0, \infty)$ with*

$$g'_{+}(s) := \lim_{r \to s^{+}} \frac{g(r) - g(s)}{r - s} = \inf G^{s}_{\log^{t+1}}(A, N), \quad s \geq 0, \tag{5}$$

and

$$g'_{-}(s) := \lim_{r \to s^{-}} \frac{g(r) - g(s)}{r - s} = \sup G^{s}_{\log^{t+1}}(A, N), \quad s > 0. \tag{6}$$

Observing that

$$\inf G^{0}_{\log}(A, N) = \mathcal{E}_{\log}(A, N) \quad \text{and} \quad g(0) = N(N-1),$$

when $s, t = 0$, Theorem 7 simply implies that

$$\lim_{s \to 0^{+}} \frac{\mathcal{E}^{s}(A, N) - N(N-1)}{s} = \mathcal{E}_{\log}(A, N).$$

Theorem 8. *Let $N \geq 2$ and $t \geq 0$ be fixed. Assume that (A, d) is an infinite compact metric space with $\mathrm{diam}(A) < 1$. Then,*

(a) *The function $g(s)$ is differentiable at $s = s_0 > 0$ if and only if*

$$\inf G^{s_0}_{\log^{t}}(A, N) = \sup G^{s_0}_{\log^{t}}(A, N);$$

(b) *If ω_N^{*} is a cluster point of $\omega_N^{s,\log^{t}}$ as $s \to s_0^{+} \geq 0$, then*

$$E^{s_0}_{\log^{t+1}}(\omega_N^{*}) = \inf G^{s_0}_{\log^{t+1}}(A, N) = g'_{+}(s_0);$$

(c) *If ω_N^{**} is a cluster point of $\omega_N^{s,\log^{t}}$ as $s \to s_0^{-} > 0$, then*

$$E^{s_0}_{\log^{t+1}}(\omega_N^{**}) = \sup G^{s_0}_{\log^{t+1}}(A, N) = g'_{-}(s_0);$$

(d) *For $s_0 > 0$, if there exists a configuration ω_N^{*} that is both cluster configurations of $\omega_N^{s,\log^{t}}$ as $s \to s_0^{+}$ and $s \to s_0^{-}$, then the function $g(s)$ is differentiable at $s = s_0$ with*

$$E^{s_0}_{\log^{t+1}}(\omega_N^{*}) = g'(s_0).$$

In Theorem 8, we provide criteria for the differentiability of $g(s)$. In particular, the part (a) in Theorem 8 implies that if all minimal N-point $s_0, \log^{t}$-energy configurations on A have the same distribution of distances, then $g(s)$ is differentiable at s_0.

2.2. Minimality of $\omega_N^{s,\log^{t}}$ on Circles

Let d_u be the 2-dimensional Euclidean metric of $\mathbb{R}^2$. For $\alpha > 0$, we denote by

$$\mathbb{S}^1_{\alpha} := \{x \in \mathbb{R}^2 : d_u(0, x) = \alpha\}$$

the circle centered at 0 of radius α. We let $L(x, y)$ be the geodesic distance between the points x and y on $\mathbb{S}^1_{\alpha}$; that is, the length of the shorter arc of $\mathbb{S}^1_{\alpha}$ connecting the points x and y.

The minimality of N distinct equally spaced points on $\mathbb{S}^1_{\alpha}$ with the Euclidean metric d_u or the geodesic distance L for the certain $s, \log^{t}$-energy problems is stated in Propositions 1–3.

Proposition 1. *Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < \pi^{-1}$. Then, ω_N is a minimal N-point $s, \log^t$-energy configuration on $\mathbb{S}^1_\alpha$ with the geodesic distance L if and only if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}^1_\alpha$.*

Proposition 2. *Let $N \geq 2$, $0 < \alpha < (e\pi)^{-1}$, and s, t satisfy $s > 0, t \geq 0$ or $s = 0, t > 0$. Then, ω_N is a minimal N-point $s, \log^t$-energy configuration on $\mathbb{S}^1_\alpha$ with the geodesic distance L if and only if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}^1_\alpha$.*

Proposition 3. *Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < 1/2$. Then, ω_N is a minimal N-point $s, \log^t$-energy configuration on $\mathbb{S}^1_\alpha$ with the Euclidean metric d_u if and only if ω_N is a configuration of N distinct equally spaced points on $\mathbb{S}^1_\alpha$.*

Note that our approach works only for the case $\operatorname{diam}(\mathbb{S}^1_\alpha) < 1$ and the conditions $0 < \alpha < \pi^{-1}$ in Proposition 1 and $0 < \alpha < 1/2$ in Proposition 3 are required for $\operatorname{diam}(\mathbb{S}^1_\alpha) < 1$. The case $\operatorname{diam}(\mathbb{S}^1_\alpha) \geq 1$ remains open for further investigation.

3. Proofs of Main Results

We keep all proof of the main results in this section. In our proof, we may sometimes refer to lemmas. In order to avoid any interruption, we keep all lemmas in Section 4.

Proof of Theorem 4. Let $t \geq 0$ be fixed, $s > 0$, $\omega_N^{s,\log^t}$ be a minimal N-point $s, \log^t$-energy configuration on A, and let ω_N^∞ be an N-point best-packing configuration on A. Since $\operatorname{diam}(A) < 1$ and the points in $\omega_N^{s,\log^t}$ are distinct, there is a constant $c > 0$ such that

$$0 < \delta(\omega_N^{s,\log^t}) \leq c < 1$$

where the constant c only depends on the set A. This implies that

$$\left(\log \frac{1}{c}\right)^t \leq \left(\log \frac{1}{\delta(\omega_N^{s,\log^t})}\right)^t.$$

Then,

$$\begin{aligned} \frac{1}{\delta_N(A)}\left(\log \frac{1}{c}\right)^{t/s} &\leq \frac{1}{\delta(\omega_N^{s,\log^t})}\left(\log \frac{1}{c}\right)^{t/s} \leq \frac{1}{\delta(\omega_N^{s,\log^t})}\left(\log \frac{1}{\delta(\omega_N^{s,\log^t})}\right)^{t/s} \\ &\leq \left(E^s_{\log^t}(\omega_N^{s,\log^t})\right)^{1/s} = \left(\mathcal{E}^s_{\log^t}(A,N)\right)^{1/s} \leq \left(E^s_{\log^t}(\omega_N^\infty)\right)^{1/s} \leq \frac{1}{\delta_N(A)}\left(E_{\log^t}(\omega_N^\infty)\right)^{1/s}. \end{aligned} \tag{7}$$

Since

$$\lim_{s\to\infty} \frac{1}{\delta_N(A)}\left(\log \frac{1}{c}\right)^{t/s} = \frac{1}{\delta_N(A)}$$

and

$$\lim_{s\to\infty} \frac{1}{\delta_N(A)}\left(E_{\log^t}(\omega_N^\infty)\right)^{1/s} = \frac{1}{\delta_N(A)},$$

it follows that

$$\lim_{s\to\infty} \left(\mathcal{E}^s_{\log^t}(A,N)\right)^{1/s} = \frac{1}{\delta_N(A)}.$$

Let ω_N^* be a cluster configuration of $\omega_N^{s,\log^t}$ as $s \to \infty$. This implies that there is a sequence $\{s_k\}_{k=1}^\infty \subset \mathbb{R}$ such that $s_k \to \infty$ and $\omega_N^{s_k,\log^t} \to \omega_N^*$ as $k \to \infty$. Arguing as in (7), we have

$$\frac{1}{\delta(\omega_N^{s_k,\log^t})}\left(\log \frac{1}{c}\right)^{t/s_k} \leq \left(E^{s_k}_{\log^t}(\omega_N^{s_k,\log^t})\right)^{1/s_k} = \left(\mathcal{E}^{s_k}_{\log^t}(A,N)\right)^{1/s_k} \leq \left(E^{s_k}_{\log^t}(\omega_N^\infty)\right)^{1/s_k}$$

$$\leq \frac{1}{\delta(\omega_N^\infty)}\Big(E_{\log^t}(\omega_N^\infty)\Big)^{1/s_k}.$$

Taking $k \to \infty$, we obtain

$$\delta_N(A) = \delta(\omega_N^\infty) \leq \delta(\omega_N^*).$$

This means that ω_N^* is also an N-point best-packing configuration on A. □

Proof of Theorem 5. First of all, we show that $g(s)$ is continuous on $(0,\infty)$. Let $s > 0$ and let $\omega_N^{s,\log^t}$ be a minimal N-point $s,\log^t$-energy configuration on A. Using Lemma 4, we obtain for any $\omega_N^{s,\log^t}$,

$$\liminf_{r\to s^-}\frac{g(r)-g(s)}{r-s} \geq \liminf_{r\to s^-}\frac{E^r_{\log^t}(\omega_N^{s,\log^t}) - E^s_{\log^t}(\omega_N^{s,\log^t})}{r-s}$$

$$\geq \lim_{r\to s^-} E^r_{\log^{t+1}}(\omega_N^{s,\log^t}) = E^s_{\log^{t+1}}(\omega_N^{s,\log^t}) \geq \sup G^s_{\log^{t+1}}(A,N) > 0, \tag{8}$$

and

$$\limsup_{r\to s^-}\frac{g(r)-g(s)}{r-s} \leq \limsup_{r\to s^-}\frac{E^r_{\log^t}(\omega_N^{r,\log^t}) - E^s_{\log^t}(\omega_N^{r,\log^t})}{r-s} \leq \limsup_{r\to s^-} E^s_{\log^{t+1}}(\omega_N^{r,\log^t}), \tag{9}$$

where the second inequality in (8) follows from the arbitrariness of $\omega_N^{s,\log^t}$ and the last inequality in (8) follows from Lemma 3.

Let ω_N be a fixed configuration of N distinct points of A. Note that $0 < \delta(\omega_N) < 1$. For all $r \in (s/2, s)$, we have

$$\left(\frac{1}{\delta(\omega_N^{r,\log^t})}\right)^{s/2}\left(\log\frac{1}{\delta(\omega_N^{r,\log^t})}\right)^t \leq \left(\frac{1}{\delta(\omega_N^{r,\log^t})}\right)^r\left(\log\frac{1}{\delta(\omega_N^{r,\log^t})}\right)^t \leq E^r_{\log^t}(\omega_N^{r,\log^t})$$

$$\leq E^r_{\log^t}(\omega_N) \leq \left(\frac{1}{\delta(\omega_N)}\right)^r\left(\log\frac{1}{\delta(\omega_N)}\right)^t N(N-1)$$

$$\leq \left(\frac{1}{\delta(\omega_N)}\right)^s\left(\log\frac{1}{\delta(\omega_N)}\right)^t N(N-1).$$

That is,

$$(\delta(\omega_N^{r,\log^t}))^{s/2}\left(\log\frac{1}{\delta(\omega_N^{r,\log^t})}\right)^{-t} \geq (\delta(\omega_N))^s\left(\log\frac{1}{\delta(\omega_N)}\right)^{-t}(N(N-1))^{-1}.$$

This implies that for all $r \in (s/2, s)$,

$$\delta(\omega_N^{r,\log^t})\left(\log\frac{1}{\delta(\omega_N^{r,\log^t})}\right)^{-2t/s} \geq (\delta(\omega_N))^2\left(\log\frac{1}{\delta(\omega_N)}\right)^{-2t/s}(N(N-1))^{-2/s} =: c_1 > 0.$$

Since by Lemma 1,

$$h(x) := x\left(\log\frac{1}{x}\right)^{-\beta}, \quad \beta > 0,$$

is a strictly increasing function on $(0,1)$, there exists a constant $c_2 > 0$ such that for all $r \in (s/2, s)$,

$$\delta(\omega_N^{r,\log^t}) \geq c_2 > 0.$$

Therefore, $E^s_{\log^{t+1}}(\omega_N^{r,\log^t})$ are bounded above where $r \in (s/2, s)$. From this and (9),

$$\limsup_{r\to s^-} \frac{g(r)-g(s)}{r-s} \leq \limsup_{r\to s^-} E^s_{\log^{t+1}}(\omega_N^{r,\log^t}) < \infty. \tag{10}$$

Let $s \geq 0$. Using Lemma 4, we also obtain for any $\omega_N^{s,\log^t}$,

$$\limsup_{r\to s^+} \frac{g(r)-g(s)}{r-s} \leq \limsup_{r\to s^+} \frac{E^r_{\log^t}(\omega_N^{s,\log^t}) - E^s_{\log^t}(\omega_N^{s,\log^t})}{r-s}$$

$$\leq \lim_{r\to s^+} E^r_{\log^{t+1}}(\omega_N^{s,\log^t}) = E^s_{\log^{t+1}}(\omega_N^{s,\log^t}) \leq \inf G^s_{\log^{t+1}}(A,N) < \infty, \tag{11}$$

and

$$\liminf_{r\to s^+} \frac{g(r)-g(s)}{r-s} \geq \liminf_{r\to s^+} \frac{E^r_{\log^t}(\omega_N^{r,\log^t}) - E^s_{\log^t}(\omega_N^{r,\log^t})}{r-s} \geq \liminf_{r\to s^+} E^s_{\log^{t+1}}(\omega_N^{r,\log^t}) > 0, \tag{12}$$

where the second inequality in (11) follows from the arbitrariness of $\omega_N^{s,\log^t}$ and the last inequality in (12) follows from Lemma 3.

The inequalities (8) and (10)–(12) imply that for all $s > 0$,

$$0 < \liminf_{r\to s^-} \frac{g(r)-g(s)}{r-s} \leq \limsup_{r\to s^-} \frac{g(r)-g(s)}{r-s} < \infty \tag{13}$$

and for all $s \geq 0$,

$$0 < \liminf_{r\to s^+} \frac{g(r)-g(s)}{r-s} \leq \limsup_{r\to s^+} \frac{g(r)-g(s)}{r-s} < \infty. \tag{14}$$

The inequalities in (13) and (14) further imply that $g(s)$ is continuous for all $s > 0$ and is right continuous at $s = 0$. □

Proof of Theorem 6. Let $s_0 > 0$. In order to show Theorem 6, it suffices to show that any cluster configuration of $\omega_N^{s,\log^t}$ as $s \to s_0^+$ or as $s \to s_0^-$ is a minimal N-point $s_0, \log^t$-energy configuration on A.

Let ω_N^* be a cluster configuration of $\omega_N^{s,\log^t}$, as $s \to s_0^+$. Then, there is a sequence $\{s_k\}_{k=1}^{\infty} \subset (s_o, \infty)$ such that $s_k \to s_0$ and $\omega_N^{s_k,\log^t} \to \omega_N^*$ as $k \to \infty$. Let $\alpha = \operatorname{diam}(A)$. For any configuration of N distinct points ω_N on A, notice that $\alpha^s E^s_{\log^t}(\omega_N)$ is an increasing function of s. Applying the continuity of $g(s) := \mathcal{E}^s_{\log^t}(A,N)$ at s_0, we have

$$\alpha^{s_0} E^{s_0}_{\log^t}(\omega_N^*) = \lim_{k\to\infty} \alpha^{s_0} E^{s_0}_{\log^t}(\omega_N^{s_k,\log^t}) \leq \lim_{k\to\infty} \alpha^{s_k} E^{s_k}_{\log^t}(\omega_N^{s_k,\log^t})$$

$$= \lim_{k\to\infty} \alpha^{s_k} \mathcal{E}^{s_k}_{\log^t}(A,N) = \alpha^{s_0} \mathcal{E}^{s_0}_{\log^t}(A,N).$$

This implies that $E^{s_0}_{\log^t}(\omega_N^*) = \mathcal{E}^{s_0}_{\log^t}(A,N)$. Hence, ω_N^* is a minimal N-point $s_0, \log^t$-energy configuration on A.

Let ω_N^{**} be a cluster configuration of $\omega_N^{s,\log^t}$, as $s \to s_0^-$. Then, there is a sequence $\{s_k\}_{k=1}^{\infty} \subset [0, s_0)$ such that $s_k \to s_0$ and $\omega_N^{s_k,\log^t} \to \omega_N^{**}$ as $k \to \infty$. Without loss of generality, we may assume that $s_0/2 < s_k < s_0$ for all k. For any configuration of N distinct points ω_N of A, observe that $\delta(\omega_N)^s E_{\log^t}^s(\omega_N)$ is a decreasing function of s. It follows from the continuity of the function $g(s)$ that $g(s)$ is bounded above by some number $M > 1$ for all $s \in (s_0/2, s_0)$. For every $s_0/2 < s_k < s_0$,

$$(\delta(\omega_N^{s_k,\log^t}))^{-s_0/2}\left(\log\frac{1}{\delta(\omega_N^{s_k,\log^t})}\right)^t \le (\delta(\omega_N^{s_k,\log^t}))^{-s_k}\left(\log\frac{1}{\delta(\omega_N^{s_k,\log^t})}\right)^t$$

$$\le E_{\log^t}^{s_k}(\omega_N^{s_k,\log^t}) \le M.$$

Then,

$$\delta(\omega_N^{s_k,\log^t})\left(\log\frac{1}{\delta(\omega_N^{s_k,\log^t})}\right)^{-2t/s_0} \ge M^{-2/s_0} > 0.$$

Using Lemma 1, there is a constant $c > 0$ such that

$$\delta(\omega_N^{s_k,\log^t}) \ge c > 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Using the continuity of $g(s) := \mathcal{E}_{\log^t}^s(A, N)$ at s_0, we have

$$(\delta(\omega_N^{**}))^{s_0} E_{\log^t}^{s_0}(\omega_N^{**}) = \lim_{k\to\infty} (\delta(\omega_N^{s_k,\log^t}))^{s_0} E_{\log^t}^{s_0}(\omega_N^{s_k,\log^t})$$

$$\le \lim_{k\to\infty} (\delta(\omega_N^{s_k,\log^t}))^{s_k} E_{\log^t}^{s_k}(\omega_N^{s_k,\log^t}) = \lim_{k\to\infty} (\delta(\omega_N^{s_k,\log^t}))^{s_k} \mathcal{E}_{\log^t}^{s_k}(A, N)$$
$$= (\delta(\omega_N^{**}))^{s_0} \mathcal{E}_{\log^t}^{s_0}(A, N).$$

This implies that $E_{\log^t}^{s_0}(\omega_N^{**}) = \mathcal{E}_{\log^t}^{s_0}(A, N)$. Hence, ω_N^{**} is a minimal N-point $s_0, \log^t$-energy configuration on A. □

Proof of Theorem 7. Firstly, we show (5). Let $s \ge 0$ be fixed and $\{r_k\}_{k=1}^{\infty} \subset (s, \infty)$ be a sequence such that $r_k \to s$ as $k \to \infty$ and

$$\lim_{k\to\infty} E_{\log^{t+1}}^s(\omega_N^{r_k,\log^t}) = \liminf_{r\to s^+} E_{\log^{t+1}}^s(\omega_N^{r,\log^t}). \tag{15}$$

Since A^N is compact, there exists a subsequence $\{s_\ell\}_{\ell=1}^{\infty} \subset \{r_k\}_{k=1}^{\infty}$ such that

$$\lim_{\ell\to\infty} \omega_N^{s_\ell,\log^t} = \omega_N^* \tag{16}$$

and ω_N^* is a minimal N-point $s, \log^t$-energy configuration by Theorem 6. By

$$\lim_{k\to\infty} E_{\log^{t+1}}^s(\omega_N^{r_k,\log^t}) = \lim_{\ell\to\infty} E_{\log^{t+1}}^s(\omega_N^{s_\ell,\log^t}),$$

(11), (12), (15) and (16), we get

$$\liminf_{r\to s^+} \frac{g(r) - g(s)}{r - s} \ge \liminf_{r\to s^+} E_{\log^{t+1}}^s(\omega_N^{r,\log^t}) = \lim_{\ell\to\infty} E_{\log^{t+1}}^s(\omega_N^{s_\ell,\log^t})$$
$$= E_{\log^{t+1}}^s(\omega_N^*) \ge \inf G_{\log^{t+1}}^s(A, N) \ge \limsup_{r\to s^+} \frac{g(r) - g(s)}{r - s}. \tag{17}$$

Then,

$$g'_+(s) = \inf G^s_{\log^{t+1}}(A,N). \tag{18}$$

It is easy to check that from Lemma 3, the constant $\inf G^s_{\log^{t+1}}(A,N)$ in (18) is finite. This verifies (5).

Then, we prove (6). Let $s > 0$ be fixed and $\{r_k\}_{k=1}^\infty \subset [0,s)$ be a sequence such that $r_k \to s$ as $k \to \infty$ and

$$\lim_{k\to\infty} E^s_{\log^{t+1}}(\omega_N^{r_k,\log^t}) = \limsup_{r\to s^-} E^s_{\log^{t+1}}(\omega_N^{r,\log^t}). \tag{19}$$

Because A^N is compact, there exists a subsequence $\{s_\ell\}_{\ell=1}^\infty \subset \{r_k\}_{k=1}^\infty$ such that

$$\lim_{\ell\to\infty} \omega_N^{s_\ell,\log^t} = \omega_N^{**}$$

and ω_N^{**} is a minimal N-point $s, \log^t$-energy configuration by Theorem 6. Then, we get

$$\lim_{k\to\infty} E^s_{\log^{t+1}}(\omega_N^{r_k,\log^t}) = \lim_{\ell\to\infty} E^s_{\log^{t+1}}(\omega_N^{s_\ell,\log^t}). \tag{20}$$

Using (8), (9), (19) and (20), we obtain

$$\liminf_{r\to s^-} \frac{g(r)-g(s)}{r-s} \geq \sup G^s_{\log^{t+1}}(A,N) \geq E^s_{\log^{t+1}}(\omega_N^{**})$$

$$= \lim_{\ell\to\infty} E^s_{\log^{t+1}}(\omega_N^{s_\ell,\log^t}) = \limsup_{r\to s^-} E^s_{\log^{t+1}}(\omega_N^{r,\log^t}) \geq \limsup_{r\to t^-} \frac{g(r)-g(s)}{r-s}.$$

Then,

$$g'_-(s) = \sup G^s_{\log^{t+1}}(A,N). \tag{21}$$

Then, we want to show that $\sup G^s_{\log^{t+1}}(A,N)$ is finite. Let ω_N be a fixed configuration of N distinct points on A and let $\omega_N^{s,\log^t}$ be any minimal N-point $s, \log^t$ configurations. Then,

$$(\delta(\omega_N^{s,\log^t}))^{-s}\left(\log \frac{1}{\delta(\omega_N^{s,\log^t})}\right)^t \leq E^s_{\log^t}(\omega_N^{s,\log^t})$$

$$\leq E^s_{\log^t}(\omega_N) \leq (\delta(\omega_N))^{-s}\left(\log \frac{1}{\delta(\omega_N)}\right)^t N(N-1).$$

That is,

$$\delta(\omega_N^{s,\log^t})\left(\log \frac{1}{\delta(\omega_N^{s,\log^t})}\right)^{-t/s} \geq \delta(\omega_N)\left(\log \frac{1}{\delta(\omega_N)}\right)^{-t/s}(N(N-1))^{-1/s} =: c_1 > 0.$$

It follows from Lemma 1 that there is a constant $c_2 > 0$ such that for any $\omega_N^{s,\log^t}$,

$$\delta(\omega_N^{s,\log^t}) \geq c_2 > 0.$$

Since by Lemma 2,

$$p(x) := \frac{1}{x^s}\left(\log \frac{1}{x}\right)^{t+1},$$

is a strictly decreasing function on $(0,1)$, the set $G^s_{\log^{t+1}}(A,N)$ is bounded above. This implies that $\sup G^s_{\log^{t+1}}(A,N)$ in (21) is finite. Hence, (6) is proved. □

Proof of Theorem 8.
(a): This is a direct consequence of Theorem 7.
(b): Let $s_0 \geq 0$ and ω^*_N be a cluster configuration of $\{\omega_N^{s,\log^t}\}$ as $s \to s_0^+$. Then, there exists a sequence $\{s_k\}_{k=1}^\infty \subset (s_0,\infty)$ such that $\lim\limits_{k\to\infty} s_k = s_0$ and $\lim\limits_{k\to\infty} \omega_N^{s_k,\log^t} = \omega^*_N$. Then, ω^*_N is a minimal N-point $s_0, \log^t$-energy configuration by Theorem 6. Using (5) and the similar argument used to show (12), we have

$$E^{s_0}_{\log^{t+1}}(\omega^*_N) = \lim_{k\to\infty} E^{s_0}_{\log^{t+1}}(\omega_N^{s_k,\log^t}) \leq \lim_{k\to\infty} \frac{g(s_k)-g(s_0)}{s_k - s_0} = g'_+(s_0) = \inf G^{s_0}_{\log^{t+1}}(A,N).$$

Since $\inf G^{s_0}_{\log^{t+1}}(A,N) \leq E^{s_0}_{\log^{t+1}}(\omega^*_N)$,

$$E^{s_0}_{\log^{t+1}}(\omega^*_N) = \inf G^{s_0}_{\log^{t+1}}(A,N) = g'_+(s_0).$$

(c): Let $s_0 > 0$ and ω^{**}_N be a cluster configuration of $\{\omega_N^{s,\log^t}\}$ as $s \to s_0^-$. Then, there exists a sequence $\{s_k\}_{k=1}^\infty \subset [0,s_0)$ such that $\lim\limits_{k\to\infty} s_k = s_0$ and $\lim\limits_{k\to\infty} \omega_N^{s_k,\log^t} = \omega^{**}_N$. Then, ω^{**}_N is a minimal N-point $s_0, \log^t$-energy configuration by Theorem 6. Using (6) and the similar argument used to show (10), we have

$$E^{s_0}_{\log^{t+1}}(\omega^{**}_N) = \lim_{k\to\infty} E^{s_0}_{\log^{t+1}}(\omega_N^{s_k,\log^t}) \geq \lim_{k\to\infty} \frac{g(s_k)-g(s_0)}{s_k - s_0} = g'_-(s_0) = \sup G^{s_0}_{\log^{t+1}}(A,N).$$

Since $E^{s_0}_{\log^{t+1}}(\omega^{**}_N) \leq \sup G^{s_0}_{\log^{t+1}}(A,N)$,

$$E^{s_0}_{\log^{t+1}}(\omega^{**}_N) = \sup G^{s_0}_{\log^{t+1}}(A,N) = g'_-(s_0).$$

(d): This is a direct consequence of (b) and (c). □

Proof of Proposition 1. Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < \pi^{-1}$. We prove this proposition using Lemma 5. The function $k : (0,1) :\to \mathbb{R}$ in the lemma is

$$k(x) = \frac{1}{x^s}\left(\log\frac{1}{x}\right)^t.$$

By Lemma 2, $k(x)$ is strictly decreasing on $(0,1)$. Since for all $x \in (0,1)$,

$$k''(x) = \frac{1}{x^{s+2}}\left(\log\frac{1}{x}\right)^{-2+t}\left[(-1+t)t + (t+2st)\log\frac{1}{x} + s(1+s)\log^2\frac{1}{x}\right] > 0, \quad (22)$$

$k(x)$ is strictly convex on $(0,1)$. Hence, because the function $k(x)$ satisfies all required properties in Lemma 5, all minimal N-point K-energy configurations on $\mathbb{S}^1_\alpha$ are configurations of N distinct equally spaced points on $\mathbb{S}^1_\alpha$ with respect to the arc length and vice versa. □

Proof of Proposition 2. Let $N \geq 2$, $0 < \alpha < (e\pi)^{-1}$, and s,t satisfy $s > 0, t \geq 0$ or $s = 0, t > 0$. We can use the same lines of reasoning as in the proof of Proposition 1 except the function k is considered on $(0,1/e)$ and for all $x \in (0,1/e)$,

$$k''(x) = \frac{1}{x^{s+2}}\left(\log\frac{1}{x}\right)^{-2+t}\left[(-1+t)t + (t+2st)\log\frac{1}{x} + s(1+s)\log^2\frac{1}{x}\right]$$

$$\geq \frac{1}{x^{s+2}}\left(\log\frac{1}{x}\right)^{-2+t}\left[t^2+2st\log\frac{1}{x}+s(1+s)\log^2\frac{1}{x}+\left(\log\frac{1}{x}-1\right)t\right]>0.$$

Hence, because the function $k(x)$ satisfies all required properties in Lemma 5, Proposition 2 is proved. □

Proof of Proposition 3. Let $N \geq 2$, $s \geq 0$, $t \geq 1$, and $0 < \alpha < 1/2$. Again, we want to use Lemma 5. The function $k : (0, \pi\alpha] \to \mathbb{R}$ in the lemma is

$$k(x) = \left(\frac{1}{2\alpha\sin(x/2\alpha)}\right)^s\left(\log\frac{1}{2\alpha\sin(x/2\alpha)}\right)^t.$$

Since $2\alpha\sin(x/2\alpha)$ is strictly increasing on $(0, \pi\alpha]$ and $(1/x^s)(\log(1/x))^t$ is strictly decreasing on $(0, 1)$, $k(x)$ is strictly decreasing on $(0, \pi\alpha]$. Then, we want to show that $k(x)$ is strictly convex on $(0, \pi\alpha]$, meaning that

$$k''(x) > 0 \quad \text{for all } x \in (0, \pi\alpha). \tag{23}$$

To show (23), it suffices to show that $q''(x) > 0$ for all $x \in (0, \pi/2)$, where

$$q(x) := \left(\frac{1}{2\alpha\sin x}\right)^s\left(\log\frac{1}{2\alpha\sin x}\right)^t.$$

Because for all $x \in (0, \pi/2)$,

$$\begin{aligned}q''(x) &= s(\cot^2 x)(2\alpha\sin x)^{-s}\left(\log\left(\frac{1}{2\alpha\sin x}\right)\right)^{t-1}\\ &+(t-1)(\cot^2 x)(2\alpha\sin x)^{-s}\left(\log\left(\frac{1}{2\alpha\sin x}\right)\right)^{t-2}\left(s\log\left(\frac{1}{2\alpha\sin x}\right)+t\right)\\ &+(\csc^2 x+s\cot^2 x)(2\alpha\sin x)^{-s}\left(\log\left(\frac{1}{2\alpha\sin x}\right)\right)^{t-1}\left(s\log\left(\frac{1}{2\alpha\sin x}\right)+t\right)>0,\end{aligned}$$

$k(x)$ is strictly convex on $(0, \pi\alpha]$. Hence, the function $k(x)$ satisfies all required properties in Lemma 5. This completes the proof. □

4. Appendix: Auxiliary Lemmas

Lemmas 1–3 are very fundamental but highly important. For example, making use of Lemma 3 and the assumption that $\text{diam}(A) < 1$, we can conclude that

$$\mathcal{E}^s_{\log}(A, N) \geq \frac{N(N-1)}{(\text{diam}(A))^s}\left(\log\frac{1}{\text{diam}(A)}\right)^t > 0.$$

Lemma 1. *Let $\beta \geq 0$ and $h : (0, 1) \to (0, \infty)$ be a function defined by*

$$h(x) := x\left(\log\frac{1}{x}\right)^{-\beta} \quad \text{for all } x \in (0, 1).$$

Then, $h(x)$ is strictly increasing on $(0, 1)$.

Proof of Lemma 1.

Because

$$h'(x) = \beta\left(\log\frac{1}{x}\right)^{-(\beta+1)} + \left(\log\frac{1}{x}\right)^{-\beta}$$

and $(\log(1/x))^{-\beta} > 0$ for all $x \in (0, 1)$ and $\beta \geq 0$, $h'(x) > 0$ for all $x \in (0, 1)$. Therefore, $h(x)$ is strictly increasing on $(0, 1)$. □

Lemma 2. *Let* $(s,t) \in [0,\infty) \times [0,\infty)] \setminus \{(0,0)\}$ *and* $p : (0,1) \to (0,\infty)$ *be a function defined by*

$$p(x) := \frac{1}{x^s}\left(\log\frac{1}{x}\right)^t \quad \text{for all } x \in (0,1).$$

Then, $p(x)$ *is strictly decreasing on* $(0,1)$.

Proof of Lemma 2. Using Lemma 1, we set $\beta = t/s$ and

$$p(x) = \left(\frac{1}{h(x)}\right)^s = \frac{1}{x^s}\left(\log\frac{1}{x}\right)^t$$

is strictly decreasing on $(0,1)$. □

Lemma 3. *Let* (A,d) *be an infinite compact metric space with* $\operatorname{diam}(A) < 1$ *and* $s,t \geq 0$. *Then, for all N-point configurations* $\omega_N \subset A$,

$$E^s_{\log^t}(\omega_N) \geq \frac{N(N-1)}{(\operatorname{diam}(A))^s}\left(\log\frac{1}{\operatorname{diam}(A)}\right)^t.$$

Proof of Lemma 3. The proof relies on the fact that $p(x)$ in Lemma 2 is strictly decreasing on $(0,1)$. □

The following is the main lemma of this paper. It allows us to prove analogues of Theorems 1–3.

Lemma 4. *Let* (A,d) *be an infinite compact metric space with* $\operatorname{diam}(A) < 1$ *and* $\omega_N = \{x_1,\ldots,x_N\}$ *be any configuration of N distinct points of A. Then, for any* $s > r \geq 0$ *and* $t \geq 0$,

$$E^r_{\log^{t+1}}(\omega_N) \leq \frac{E^s_{\log^t}(\omega_N) - E^r_{\log^t}(\omega_N)}{s-r} \leq E^s_{\log^{t+1}}(\omega_N).$$

Proof of Lemma 4. Let $x_i, x_j \in \omega_N$ where $1 \leq i \neq j \leq N$, let $s > r \geq 0$, and let $t \geq 0$. Then,

$$\frac{1}{d(x_i,x_j)^r}\log\frac{1}{d(x_i,x_j)} \leq \frac{\dfrac{1}{d(x_i,x_j)^s} - \dfrac{1}{d(x_i,x_j)^r}}{s-r} \leq \frac{1}{d(x_i,x_j)^s}\log\frac{1}{d(x_i,x_j)}.$$

Since $\left(\log\dfrac{1}{d(x_i,x_j)}\right)^t > 0$,

$$\frac{1}{d(x_i,x_j)^r}\left(\log\frac{1}{d(x_i,x_j)}\right)^{t+1} \leq \frac{\dfrac{1}{d(x_i,x_j)^s}\left(\log\dfrac{1}{d(x_i,x_j)}\right)^t - \dfrac{1}{d(x_i,x_j)^r}\left(\log\dfrac{1}{d(x_i,x_j)}\right)^t}{s-r}$$

$$\leq \frac{1}{d(x_i,x_j)^s}\left(\log\frac{1}{d(x_i,x_j)}\right)^{t+1}.$$

It follows that

$$E^r_{\log^{t+1}}(\omega_N) \leq \frac{E^s_{\log^t}(\omega_N) - E^r_{\log^t}(\omega_N)}{s-r} \leq E^s_{\log^{t+1}}(\omega_N).$$

□

Let Γ be a rectifiable simple closed curve in $\mathbb{R}^m, m \geq 2$, of length $|\Gamma|$ with a chosen orientation. We recall that $L(x,y)$ is the geodesic distance between the points x and y on Γ. With the help of the following lemma [2] (Theorem 2.3.1), we can prove propositions 1–3.

Lemma 5. *Let $k : (0, |\Gamma|/2] \to \mathbb{R}$ be a strictly convex and decreasing function defined at $u = 0$ by the (possibly infinite) value $\lim_{u\to 0^+} k(u)$ and let K be the kernel on $\Gamma \times \Gamma$ defined by $K(x,y) = k(L(x,y))$. Then, all minimal N-point K-energy configurations on Γ are configurations of N distinct equally spaced points on Γ with respect to the arc length and vice versa.*

5. Discussion and Conclusions

We introduce minimal N-point $s, \log^t$-energy constants and configurations of an infinite compact metric space (A,d). Such constants and configurations are generated using the kernel (or potential):

$$K^s_{\log^t}(x,y) = \frac{1}{d(x,y)^s}\left(\log\frac{1}{d(x,y)}\right)^t, \quad s \geq 0, \quad t \geq 0.$$

In this paper, we studied the asymptotic properties of minimal N-point $s, \log^t$-energy constants and configurations of A with $\mathrm{diam}(A) < 1$ where N, t are fixed and s is varying. In Theorem 4, we show that

$$\lim_{s\to\infty}\left(\mathcal{E}^s_{\log^t}(A,N)\right)^{1/s} = \frac{1}{\delta_N(A)}.$$

and minimal N-point $s, \log^t$-energy configurations on A tend to an N-point best-packing configuration on A as $s \to \infty$. Then, we show that the $s, \log^t$-energy

$$g(s) := \mathcal{E}^s_{\log^t}(A,N)$$

is continuous and right differentiable on $[0,\infty)$ and is left differentiable on $(0,\infty)$ in Theorems 5 and 7. Using the continuity of $\mathcal{E}^s_{\log^t}(A,N)$ in the variable s, we show in Theorem 6 that for any $s_0 > 0$, any cluster configuration of $\omega_N^{s,\log^t}$, as $s \to s_0$, is a minimal N-point $s_0, \log^t$-energy configuration on A.

We want to emphasize that when $t = 0$, our proof of Theorems 4–8 can handle the case $\mathrm{diam}(A) \geq 1$. However, when $t > 0$, we require that $\mathrm{diam}(A) < 1$. This is because our methods rely on the positivity of the kernel $K^s_{\log^t}(x,y)$ and the property that $K^s_{\log^t}(x,y)$ decreases as $d(x,y)$ increases. These limitations would leave room for future improvement (when $t > 0$ and $\mathrm{diam}(A) \geq 1$).

Note that the kernel $K^s_{\log^t}(x,y)$ is symmetric, namely $K^s_{\log^t}(x,y) = K^s_{\log^t}(y,x)$. When the metric space (A,d) has a great symmetry, we observe that such minimal N-point $s, \log^t$-energy configurations should be evenly distributed over the set A. The most prominent sets with a great symmetry are the spheres:

$$\mathbb{S}^m := \{x \in \mathbb{R}^{m+1} : d_u(x,0) = 1\},$$

where d_u is the $m+1$-dimensional Euclidean metric. As a motivated result, it is known that for $s > 0$, minimal N-point s-energy configurations and minimal N-point log-energy configurations on the metric space $(\mathbb{S}^m, d_u)$ are asymptotically uniformly distributed with respect to the surface area measure on $\mathbb{S}^m$ as $N \to \infty$ (see [2] (Theorem 6.1.7)). We refer the reader to the review article [22] for a number of applications of uniformly distributed points on the sphere $\mathbb{S}^m$. Our investigation in this paper on circles in $\mathbb{R}^2$ serves as a basis example of our observation. In Propositions 1–3, we prove that for certain values of s and t, all minimal N-point $s, \log^t$-energy configurations on the circle $\mathbb{S}^1_\alpha$ with $\mathrm{diam}(\mathbb{S}^1_\alpha) < 1$ are the configurations of N distinct equally spaced points $\mathbb{S}^1_\alpha$.

In addition to the problem on the sphere $\mathbb{S}^m$, explaining the limiting distribution as the $N \to \infty$ of minimal N-point $s, \log^t$-energy configurations on a compact set in a finite dimensional Euclidean space would be another interesting problem. We refer the reader to Chapters 4 and 8 in [2] or [9,10] for the study of such a problem for the minimal N-point s-energy and log-energy configurations. The study of such limiting distribution problem is important in both theoretical and computational sciences. For example, it shows applications in computer-aided geometric design, finite element tesselations, and statistical sampling.

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