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New Geometric Constants in Banach Spaces Related to the Inscribed Equilateral Triangles of Unit Balls

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Abstract: Geometric constant is one of the important tools to study geometric properties of Banach spaces. In this paper, we will introduce two new geometric constants $J_L(X)$ and $Y_J(X)$ in Banach spaces, which are symmetric and related to the side lengths of inscribed equilateral triangles of unit balls. The upper and lower bounds of $J_L(X)$ and $Y_J(X)$ as well as the values of $J_L(X)$ and $Y_J(X)$ for Hilbert spaces and some common Banach spaces will be calculated. In addition, some inequalities for $J_L(X)$, $Y_J(X)$ and some significant geometric constants will be presented. Furthermore, the sufficient conditions for uniformly non-square and normal structure, and the necessary conditions for uniformly non-square and uniformly convex will be established.

Keywords: Banach space; geometric constant; normal structure; uniformly non-square; uniformly convex

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1. Introduction

The study of geometric constants can be traced back to the concept of the modulus of convexity introduced by Clarkson [1] in the study of uniformly convex spaces. In the following years of research, scholars found that some abstract geometric properties of Banach spaces can be quantitatively described by some special constants. For example, the modulus of convexity introduced by Clarkson [1] can be used to characterize uniformly convex spaces ([2], Lemma 2), the modulus of smoothness proposed by Day [3] can be used to characterize uniformly smooth spaces ([4], Theorem 2.5), the von Neumann–Jordan constant proposed by Clarkson [5] and the James constant proposed by Gao and Lau [6] can be used to characterize uniformly non-square spaces ([7], Theorem 2 and [8], Proposition 1). In the past two or three decades, many properties of these constants have been studied, such as the relations between these constants and normal structure or uniform normal structure ([9–12]) and the equalities or inequalities for these constants ([12–15]) and so on. On the other hand, an amount of new geometric constants have also been introduced to study the geometric properties of Banach spaces (see [9,13,15]), one of the best known is the Dunkl–Williams constant (see [16]). Nowadays, geometric constants have become one of the important tools to study the geometric properties of Banach spaces. For readers interested in this research direction, we recommend reading references [12,17–19] as well as the references mentioned in this paper.

In 2000, Baronti, Casini and Papini defined two new, simple constants in real normed spaces which related to the perimeters of triangles inscribed in semicircles of normed spaces, and they also gave a few connections between the values of constants and the geometry of real normed spaces. For more details, please refer to [20]. In 2008, Alonso and Llorens-Fuster defined two geometric constants for Banach spaces by using the geometric means of the variable lengths of the sides of triangles with vertices x, $-x$ and y, where x, y are points on the unit sphere of a normed space. These constants are closely related to the modulus of convexity of some spaces, and they seem to represent a useful tool to estimate the exact values of the James and von Neumann–Jordan constants of some Banach spaces. More details are suggested in [21].

Motivated by the fact that the problem about circles and their inscribed triangles is an important research topic of the Euclidean geometry and the works of the above two articles, we will introduce two new geometric constants $J_L(X)$ and $Y_J(X)$, in this paper, which are symmetric and related to the side lengths of inscribed equilateral triangles of unit balls in Banach spaces. Firstly, we will explain why these two constants are symmetric and related to the side lengths of inscribed equilateral triangles of unit balls in Banach spaces. Secondly, we will discuss the upper and lower bounds of constants $J_L(X)$ and $Y_J(X)$, and calculate the values of $J_L(X)$ and $Y_J(X)$ for Hilbert spaces and some common Banach spaces. Thirdly, we will give some inequalities for $J_L(X)$, $Y_J(X)$ and some significant geometric constants, including the modulus of convexity $\delta_X(\varepsilon)$, the modulus of smoothness $\rho_X(\tau)$, James constant $J(X)$ and von Neumann–Jordan constant $C_{\mathrm{NJ}}(X)$. Finally, the sufficient conditions for uniformly non-square and normal structure, and the necessary conditions for uniformly non-square and uniformly convex will be given by these two geometric constants $J_L(X)$ and $Y_J(X)$.

2. Notations and Preliminaries

Throughout the paper, let X be a real Banach space with dim $X \geq 2$. The unit ball and the unit sphere of X are denoted by B_X and S_X, respectively.

Definition 1. *Ref. [22] A Banach space X is said to be uniformly non-square, if there exists a $\delta \in (0,1)$ such that if x, $y \in S_X$ then*

$$\left\|\frac{x+y}{2}\right\| \leq 1-\delta \ \ or \ \ \left\|\frac{x-y}{2}\right\| \leq 1-\delta.$$

The James constant $J(X)$ which can characterize the uniformly non-square spaces was introduced by Gao and Lau [6]

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) \ : \ x, y \in S_X\}.$$

It is well known (cf. [6,8]) that

(1) $\sqrt{2} \leq J(X) \leq 2$, for all Banach spaces X;

(2) $J(X)S(X) = 2$, for all Banach spaces X (i.e., $S(X) = \inf\{\max(\|x+y\|, \|x-y\|) \ : \ x, y \in S(X)\}$);

(3) X is uniformly non-square if and only if $J(X) < 2$ or $S(X) > 1$.

The following constant $C_{\mathrm{NJ}}(X)$, which named von Neumann–Jordan constant, was proposed by Clarkson [5]. To some extent, this constant characterizes the parallelogram rule of Banach spaces.

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \ : \ x, y \in X, (x,y) \neq (0,0)\right\}.$$

Some famous conclusions about $C_{\mathrm{NJ}}(X)$ are listed below

(1) $C_{\mathrm{NJ}}(X) \leq J(X)$, for all Banach spaces X (see [14]);

(2) $1 \leq C_{\mathrm{NJ}}(X) \leq 2$, for all Banach spaces X (see [23]);

(3) X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$ (see [23]);

(4) X is uniformly non-square if and only if $C_{\mathrm{NJ}}(X) < 2$ (see [7]).

Definition 2. *Ref. [1] A Banach space X is said to be uniformly convex, whenever given $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that if x, $y \in S_X$ and $\|x-y\| \geq \varepsilon$, then*

$$\left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

Definition 3. *Ref. [3] A Banach space X is said to be uniformly smooth, whenever given* $0 < \varepsilon \leq 2$, *there exists* $\delta > 0$ *such that if* $x \in S_X$ *and* $\|y\| \leq \delta$, *then*

$$\|x+y\| + \|x-y\| < 2 + \varepsilon\|y\|.$$

In order to study uniformly convex and uniformly smooth, the modulus of convexity $\delta_X(\varepsilon) : [0,2] \to [0,1]$ and the modulus of smoothness $\rho_X(\tau) : [0,+\infty) \to [0,+\infty)$ were introduced by Clarkson [1] and Day [3], respectively.

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x,y \in S_X, \|x-y\| = \varepsilon\right\} (0 \leq \varepsilon \leq 2);$$

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x,y \in S_X\right\} (\tau \geq 0).$$

It is known to all that

(1) (Ref. [2], Lemma 2) X is uniformly convex if and only if the characteristic of convexity $\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\} = 0$;

(2) (Ref. [4], Theorem 2.5) X is uniformly smooth if and only if

$$\lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0;$$

(3) (see [24]) For all Banach spaces X, then

$$\delta_X(\varepsilon) \leq 1 - \sqrt{\left(1 - \frac{\varepsilon^2}{4}\right)}; \tag{1}$$

(4) (Ref. [25], Lemma 1.e.10) For any $x, y \in X$ such that $\|x\|^2 + \|y\|^2 = 2$, then

$$\|x+y\|^2 \leq 4 - 4\delta_X\left(\frac{\|x-y\|}{2}\right); \tag{2}$$

(5) (Ref. [2], Lemma 3) X is uniformly non-square if and only if

$$\varepsilon_0(X) = \sup\{\varepsilon : \delta_X(\varepsilon) = 0\} < 2. \tag{3}$$

Definition 4. *Ref. [26] A bounded convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point* $x_0 \in H$ *such that* $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, *where* $d(H) = \sup\{\|x-y\| : x,y \in H\}$ *denotes the diameter of H. A Banach space X is said to have normal structure if every bound convex subsets K of X has normal structure. A Banach space X is said to have weak normal structure if each weakly compact convex set K in X that contains more than one point has normal structure.*

Remark 1. *For a reflexive Banach space X, the normal structure and weak normal structure coincide.*

Normal structure is closely related to the fixed point property. Kirk [27] proved that if a weakly compact convex subset K of X has normal structure, then any non-expansive mapping on K has a fixed point. Recall that T is a non-expansive mapping on K if $\|Tx - Ty\| \leq \|x-y\|$ for every $x, y \in K$.

3. The Constants $J_L(X)$ and $Y_J(X)$

As we all know, the problem about circles and their inscribed triangles is an important research topic in Euclidean plane geometry. The research results of this problem reveal many important geometric properties of the Euclidean plane, such as the Law of Sines. Inspired by this, we will introduce the following geometric constants related to the side lengths of the inscribed equilateral triangles of unit balls to study the geometric properties of Banach spaces.

$$J_L(X) = \inf\{\|x-y\|, \|y-z\|, \|x-z\| \ : \ \|x\| = \|y\| = \|z\| = 1, \ x+y+z=0\};$$

$$Y_J(X) = \sup\{\|x-y\|, \|y-z\|, \|x-z\| \ : \ \|x\| = \|y\| = \|z\| = 1, \ x+y+z=0\}.$$

The geometric meanings of $J_L(X)$ and $Y_J(X)$:

It is well known that, in $\mathbb{R}^2$, any three different vectors x, y and z on the unit circle can form an inscribed triangle $\triangle xyz$. The lengths of the three sides of the triangle $\triangle xyz$ are equal to the lengths of vectors $x-y$, $y-z$ and $x-z$, respectively. In particular, it is easy to prove that, by using the knowledge of the Euclidean geometry, the triangle $\triangle xyz$ is an equilateral triangle if and only if $x+y+z=0$ (i.e., the vectors x, y and z can form a triangle by translation). After knowing the above facts, it is not difficult to understand that $J_L(X)$ and $Y_J(X)$ can be regarded as the infimum and supremum of the side lengths of the inscribed equilateral triangles of unit balls in Banach spaces.

In addition, the reason why we say these two constants $J_L(X)$ and $Y_J(X)$ are symmetric is actually that the positions of x, y and z are interchangeable. For this reason, $J_L(X)$ and $Y_J(X)$ can be rewritten as follows

$$J_L(X) = \inf\{\|x-y\| \ : \ \|x\| = \|y\| = \|x+y\| = 1\};$$

$$Y_J(X) = \sup\{\|x-y\| \ : \ \|x\| = \|y\| = \|x+y\| = 1\}.$$

The upper and lower bounds of $J_L(X)$ and $Y_J(X)$ can be estimated as follows

Theorem 1. *Let X be a Banach space, then $1 \le J_L(X) \le 2$ and $\sqrt{3} \le Y_J(X) \le 2$.*

Proof. Clearly, $1 \le J_L(X) \le 2$ and $Y_J(X) \le 2$ can be given by the following two inequalities

$$\|x-y\| \le \|x\| + \|y\| = 2;$$

$$2 = 2\|x\| = \|2x\| = \|2x+y-y\| \le \|x+y\| + \|x-y\| = 1 + \|x-y\|,$$

where x, y such that $\|x\| = \|y\| = \|x+y\| = 1$.

In addition, for any $x, y \in S_X$ such that $\|x-y\| = 1$. Let $\bar{x} = x$, $\bar{y} = -y$, it is obvious that $\bar{x}, \bar{y}$ satisfying $\|\bar{x}\| = 1$, $\|\bar{y}\| = 1$ and $\|\bar{x}+\bar{y}\| = 1$. Then we have

$$1 - \frac{1}{2}Y_J(X) \le 1 - \left\|\frac{\bar{x}-\bar{y}}{2}\right\| = 1 - \left\|\frac{x+y}{2}\right\|,$$

which implies that $1 - \frac{1}{2}Y_J(X) \le \delta_X(1)$. In terms of (1), we obtain

$$1 - \frac{1}{2}Y_J(X) \le \delta_X(1) \le 1 - \sqrt{\left(1 - \frac{1}{4}\right)} = 1 - \frac{\sqrt{3}}{2},$$

which deduces that $\sqrt{3} \le Y_J(X)$. □

4. Some Examples

In this section, the values of $J_L(X)$ and $Y_J(X)$ for Hilbert spaces and some common Banach spaces will be calculated. Moreover, these values can demonstrate that 1 and $\sqrt{3}$ are the best lower bounds of $J_L(X)$ and $Y_J(X)$, respectively, and 2 is the best upper bound of $Y_J(X)$ (see Examples 1–3).

Example 1. *Let X be a Hilbert space, then $J_L(X) = Y_J(X) = \sqrt{3}$.*

Proof. By applying the parallelogram law, we have the following equality holds for any x, y such that $\|x\| = \|y\| = \|x+y\| = 1$,

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 = 3,$$

which means $J_L(X) = Y_J(X) = \sqrt{3}$. $\square$

Example 2. *Let $1 \le p < \infty$, l_p be the linear space of all sequences in $\mathbb{R}$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ with the norm defined by*

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}},$$

l_∞ be the linear space of all bounded sequences in $\mathbb{R}$ with the norm defined by

$$\|x\| = \sup_{1 \le i < \infty} |x_i|,$$

then the following statements hold

(1) $J_L(l_1) = 1$ and $Y_J(l_1) = 2$;

(2) $J_L(l_p) = (2^p - 1)^{\frac{1}{p}}$ and $Y_J(l_p) \le (2^q - 1)^{\frac{1}{q}}$, $1 < p \le 2$; $(2^q - 1)^{\frac{1}{q}} \le J_L(l_p)$ and $Y_J(l_p) = (2^p - 1)^{\frac{1}{p}}$, $p \ge 2$;

(3) $J_L(l_\infty) = 1$ and $Y_J(l_\infty) = 2$.

Proof. (1) Let $x = \left(-\frac{1}{2}, -\left(\frac{1}{2}\right)^2, \cdots, -\left(\frac{1}{2}\right)^n, \cdots\right)$, $y = \left(-\frac{1}{2}, \left(\frac{1}{2}\right)^2, \cdots, \left(\frac{1}{2}\right)^n, \cdots\right)$, then $x + y = (-1, 0, \cdots)$ and $\|x\| = \|y\| = \|x+y\| = 1$. So, via the Theorem 1,

$$1 \le J_L(l_1) \le \|x-y\| = \left\|\left(0, -\frac{1}{2}, \cdots, -\left(\frac{1}{2}\right)^{n-1}, \cdots\right)\right\| = 1,$$

which implies that $J_L(l_1) = 1$.

Let $\overline{x} = \left(\frac{1}{2}, -\left(\frac{1}{2}\right)^2, \cdots, -\left(\frac{1}{2}\right)^n, \cdots\right)$, $\overline{y} = (-1, 0, \cdots)$, then

$$\overline{x} + \overline{y} = \left(-\frac{1}{2}, -\left(\frac{1}{2}\right)^2, \cdots, -\left(\frac{1}{2}\right)^n, \cdots\right)$$

and $\|\overline{x}\| = \|\overline{y}\| = \|\overline{x} + \overline{y}\| = 1$. Now, from Theorem 1, we obtain

$$2 \ge Y_J(l_1) \ge \|\overline{x} - \overline{y}\| = \left\|\left(\frac{3}{2}, 0, -\left(\frac{1}{2}\right)^2, \cdots, -\left(\frac{1}{2}\right)^n, \cdots\right)\right\| = 2,$$

which follows that $Y_J(l_1) = 2$.

(2) Let $\frac{1}{p} + \frac{1}{q} = 1$, by the Clarkson inequalities ([1], Theorem2):

$$\|x+y\|^p + \|x-y\|^p \le 2^{p-1}(\|x\|^p + \|y\|^p), \ \ p \ge 2,$$

$$2(\|x\|^p + \|y\|^p)^{q-1} \le \|x+y\|^q + \|x-y\|^q, \ \ p \ge 2,$$

$$\|x+y\|^p + \|x-y\|^p \ge 2^{p-1}(\|x\|^p + \|y\|^p), \ \ 1 < p \le 2,$$

$$2(\|x\|^p + \|y\|^p)^{q-1} \ge \|x+y\|^q + \|x-y\|^q, \ \ 1 < p \le 2,$$

show that

$$Y_J(l_p) \le (2^p - 1)^{\frac{1}{p}}, \ (2^q - 1)^{\frac{1}{q}} \le J_L(l_p), \ p \ge 2,$$

$$J_L(l_p) \geq (2^p - 1)^{\frac{1}{p}},\ Y_J(l_p) \leq (2^q - 1)^{\frac{1}{q}},\ 1 < p \leq 2.$$

Therefore, we only need to prove that, for any $p > 1$, there exist $x_0, y_0 \in S_{l_p}$ such that $x_0 + y_0 \in S_{l_p}$ and $\|x_0 - y_0\| = (2p-1)^{\frac{1}{p}}$.

Let $x = \left(-\frac{1}{2}, \left(\frac{1}{2}\right)^2, \cdots, \left(\frac{1}{2}\right)^n, \cdots\right)$, $y = \left(\frac{1}{2}, \left(\frac{1}{2}\right)^2, \cdots, \left(\frac{1}{2}\right)^n, \cdots\right)$. Obviously, x, $y \in l_1 \subset l_p$, therefore $\frac{x}{\|x\|}, \frac{y}{\|y\|} \in S_{l_p}$. Let $x_0 = \frac{x}{\|x\|} = (2^p - 1)^{\frac{1}{p}} x$, $y_0 = \frac{y}{\|y\|} = (2^p - 1)^{\frac{1}{p}} y$. Then, we have

$$x_0 = \frac{x}{\|x\|} = (2^p - 1)^{\frac{1}{p}} \left(-\frac{1}{2}, \left(\frac{1}{2}\right)^2, \cdots, \left(\frac{1}{2}\right)^n, \cdots\right),$$

$$y_0 = \frac{y}{\|y\|} = (2^p - 1)^{\frac{1}{p}} \left(\frac{1}{2}, \left(\frac{1}{2}\right)^2, \cdots, \left(\frac{1}{2}\right)^n, \cdots\right),$$

$$x_0 + y_0 = (2^p - 1)^{\frac{1}{p}} \left(0, 2\left(\frac{1}{2}\right)^2, \cdots, 2\left(\frac{1}{2}\right)^n, \cdots\right),$$

$$x_0 - y_0 = (2^p - 1)^{\frac{1}{p}} (-1, 0, \cdots).$$

Further, we can obtain

$$\|x_0 + y_0\| = 2 \cdot (2^p - 1)^{\frac{1}{p}} \left(\frac{\left(\frac{1}{2}\right)^{2p}}{1 - \left(\frac{1}{2}\right)^p}\right)^{\frac{1}{p}} = 1,$$

$$\|x_0 - y_0\| = (2^p - 1)^{\frac{1}{p}}.$$

(3) Let $x = (1, 0, 0, \cdots)$, $y = (0, -1, 0, \cdots)$, then $x + y = (1, -1, 0, \cdots)$ and $\|x\| = \|y\| = \|x + y\| = 1$. So, via the Theorem 1,

$$1 \leq J_L(l_\infty) \leq \|x - y\| = \|(1, 1, 0, \cdots)\| = 1,$$

which implies that $J_L(l_\infty) = 1$.

Let $\overline{x} = (0, -1, 0, \cdots)$, $\overline{y} = (-1, 1, 0, \cdots)$, then $\overline{x} + \overline{y} = (-1, 0, 0, \cdots)$ and $\|\overline{x}\| = \|\overline{y}\| = \|\overline{x} + \overline{y}\| = 1$. Now, from Theorem 1, we obtain

$$2 \geq Y_J(l_\infty) \geq \|\overline{x} - \overline{y}\| = \|(1, -2, 0, \cdots)\| = 2,$$

which follows that $Y_J(l_\infty) = 2$. □

Example 3. *Let $C[a,b]$ be the linear space of all real valued continuous functions on $[a,b]$ with the norm defined by*

$$\|x\| = \sup_{t \in [a,b]} |x(t)|,$$

then $J_L(C[a,b]) = 1$ and $Y_J(C[a,b]) = 2$.

Proof. Let $x(t) = \frac{1}{a-b}(t-b)$, $y(t) = \frac{-1}{a-b}(t-b) + 1$, then $x(t) + y(t) = 1$ and $\|x(t)\| = \|y(t)\| = \|x(t) + y(t)\| = 1$. So, via the Theorem 1,

$$1 \leq J_L(C[a,b]) \leq \|x(t) - y(t)\| = \sup_{t \in [a,b]} \left| \frac{2}{a-b}(t-b) - 1 \right| = 1,$$

which implies that $J_L(C[a,b]) = 1$.

Let $\overline{x}(t) = \frac{1}{a-b}(t-b)$, $\overline{y}(t) = -1$, then $\overline{x}(t) + \overline{y}(t) = \frac{1}{a-b}(t-b) - 1$ and $\|\overline{x}(t)\| = \|\overline{y}(t)\| = \|\overline{x}(t) + \overline{y}(t)\| = 1$. Now, from Theorem 1, we obtain

$$2 \geq Y_J(C[a,b]) \geq \|\overline{x}(t) - \overline{y}(t)\| = \sup_{t\in[a,b]} \left| \frac{1}{a-b}(t-b) + 1 \right| = 2,$$

which follows that $Y_J(C[a,b]) = 2$. □

5. Some Inequalities for $J_L(X)$, $Y_J(X)$ and Some Significant Geometric Constants

In this section, we will give some inequalities for $J_L(X)$, $Y_J(X)$ and some significant geometric constants, including the modulus of convexity $\delta_X(\varepsilon)$, the modulus of smoothness $\rho_X(\tau)$, James constant $J(X)$ and von Neumann–Jordan constant $C_{\mathrm{NJ}}(X)$. Moreover, some of these inequalities will help us to discuss the relations among $J_L(X)$, $Y_J(X)$ and some geometric properties of Banach spaces in the next section.

Theorem 2. *Let X be a Banach space, then* $J_L(X) \leq 2 - 2\delta_X(1) \leq Y_J(X) \leq 2\sqrt{1 - \delta_X\left(\frac{1}{2}\right)}$.

Proof. In fact, for any $x, y \in S_X$ such that $\|x - y\| = 1$. Let $\bar{x} = x$, $\bar{y} = -y$, it is obvious that $\bar{x}, \bar{y}$ satisfying $\|\bar{x}\| = 1$, $\|\bar{y}\| = 1$ and $\|\bar{x} + \bar{y}\| = 1$. Hence, we obtain

$$\delta_X(1) \leq 1 - \left\| \frac{x+y}{2} \right\| = 1 - \left\| \frac{\bar{x} - \bar{y}}{2} \right\| \leq 1 - \frac{1}{2} J_L(X),$$

which deduces that $J_L(X) \leq 2 - 2\delta_X(1)$.

On the other hand, for any $x, y \in S_X$ such that $\|x + y\| = 1$. Let $\widetilde{x} = x$, $\widetilde{y} = -y$, it is clear that $\widetilde{x}, \widetilde{y}$ satisfying $\|\widetilde{x}\|^2 + \|\widetilde{y}\|^2 = 2$ and $\|\widetilde{x} - \widetilde{y}\| = 1$. According to (2),

$$\|x - y\| = \|\widetilde{x} + \widetilde{y}\| \leq \sqrt{4 - 4\delta_X\left(\frac{\|\widetilde{x} - \widetilde{y}\|}{2}\right)} = 2\sqrt{1 - \delta_X\left(\frac{1}{2}\right)},$$

which means that $Y_J(X) \leq 2\sqrt{1 - \delta_X\left(\frac{1}{2}\right)}$.

Finally, for the inequality $2 - 2\delta_X(1) \leq Y_J(X)$, it has been proved in Theorem 1. □

Theorem 3. *Let X be a Banach space, then the following statements hold*

(1) $Y_J(X) \leq \frac{2\rho_X(\tau) - \tau + 2}{\tau}$, $0 \leq \tau < 1$;

(2) $Y_J(X) \leq \frac{2\rho_X(\tau) + \tau}{\tau}$, $\tau \geq 1$.

Proof. (1) For any $0 \leq \tau < 1$, $x, y \in S_X$, it follows that

$$\begin{aligned}
&2\rho_X(\tau) + 2 \\
\geq &\|x + \tau y\| + \|x - \tau y\| \\
= &\left\| \left(\frac{1+\tau}{2}\right)(x+y) + \left(\frac{1-\tau}{2}\right)(x-y) \right\| + \left\| \left(\frac{1-\tau}{2}\right)(x+y) + \left(\frac{1+\tau}{2}\right)(x-y) \right\| \\
\geq &\left| \left(\frac{1+\tau}{2}\right)\|x+y\| - \left|\frac{1-\tau}{2}\right|\|x-y\| \right| + \left| \left|\frac{1-\tau}{2}\right|\|x+y\| - \left(\frac{1+\tau}{2}\right)\|x-y\| \right| \\
\geq &\left(\frac{1+\tau}{2}\right)\|x+y\| - \left|\frac{1-\tau}{2}\right|\|x-y\| + \left(\frac{1+\tau}{2}\right)\|x-y\| - \left|\frac{1-\tau}{2}\right|\|x+y\| \\
\geq &\left(\frac{1+\tau}{2}\right)\|x+y\| - \left(\frac{1-\tau}{2}\right)\|x-y\| + \left(\frac{1+\tau}{2}\right)\|x-y\| - \left(\frac{1-\tau}{2}\right)\|x+y\|.
\end{aligned}$$

Thus, for any $0 \le \tau < 1$, $x, y \in S_X$ such that $x + y \in S_X$, we have

$$2\rho_X(\tau) + 2 \ge \left(\frac{1+\tau}{2}\right) - \left(\frac{1-\tau}{2}\right)\|x-y\| + \left(\frac{1+\tau}{2}\right)\|x-y\| - \left(\frac{1-\tau}{2}\right)$$
$$= \tau + \tau\|x-y\|,$$

which leads to

$$Y_J(X) \le \frac{2\rho_X(\tau) - \tau + 2}{\tau}, \ \ 0 \le \tau < 1.$$

(2) For any $\tau \ge 1$, $x, y \in S_X$, it follows that

$$\begin{aligned} 2\rho_X(\tau) + 2 &\ge \|x + \tau y\| + \|x - \tau y\| \\ &= \|x - \tau x + \tau x + \tau y\| + \|x - \tau x + \tau x - \tau y\| \\ &\ge |\tau\|x+y\| - |1-\tau|\|x\|| + |\tau\|x-y\| - |1-\tau|\|x\|| \\ &\ge \tau\|x+y\| - (\tau - 1) + \tau\|x-y\| - (\tau - 1). \end{aligned}$$

Hence, for any $\tau \ge 1$, $x, y \in S_X$ such that $x + y \in S_X$, we obtain

$$\begin{aligned} 2\rho_X(\tau) + 2 &\ge \tau - (\tau - 1) + \tau\|x-y\| - (\tau - 1) \\ &= \tau\|x-y\| - \tau + 2, \end{aligned}$$

which means that

$$Y_J(X) \le \frac{2\rho_X(\tau) + \tau}{\tau}, \ \ \tau \ge 1.$$

□

Proposition 1. *Let X be a Banach space, then $Y_J(X) \le \sqrt{4C_{\mathrm{NJ}}(X) - 1}$.*

Proof. Clearly, for any $x, y \in X$, we have

$$\|x+y\|^2 + \|x-y\|^2 \le 2C_{\mathrm{NJ}}(X)(\|x\|^2 + \|y\|^2).$$

Hence, for any $x, y \in S_X$ such that $x + y \in S_X$, we obtain

$$\|x-y\| \le \sqrt{4C_{\mathrm{NJ}}(X) - 1},$$

which implies $Y_J(X) \le \sqrt{4C_{\mathrm{NJ}}(X) - 1}$. □

Proposition 2. *Let X be a Banach space, then $\frac{2}{J(X)} \le J_L(X) \le Y_J(X) \le \sqrt{4J(X) - 1}$.*

Proof. From the definitions of $S(X)$ and $J_L(X)$, it is obvious that $S(X) \le J_L(X)$. Applying $J(X)S(X) = 2$, we can obtain $\frac{2}{J(X)} \le J_L(X)$. The second inequality $J_L(X) \le Y_J(X)$ clearly holds. Finally, the third inequality $Y_J(X) \le \sqrt{4J(X) - 1}$ can be obtained by the Proposition 1 and $C_{\mathrm{NJ}}(X) \le J(X)$. □

6. The Relations among $J_L(X)$, $Y_J(X)$ and Some Geometric Properties of Banach Spaces

In this section, we will discuss the relations among $J_L(X)$, $Y_J(X)$ and some geometric properties of Banach spaces. The conclusions in this section including the sufficient conditions for uniformly non-square and normal structure, and the necessary conditions for uniformly non-square and uniformly convex.

Proposition 3. *Let X be a Banach space, if $Y_J(X) < 2$, then X is uniformly non-square.*

Proof. Notice that $\delta_X(\varepsilon)$ is a nondecreasing function of ε, so $\delta_X(1) > 0$ means that $\varepsilon_0(X) < 2$. Therefore, the result now follows from (3) and Theorem 2. $\square$

Lemma 1. *Ref. [28] Let X be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$, there exist x_1, x_2, x_3 in S_X satisfying*
(1) $x_2 - x_3 = x_1$;
(2) $\|(x_1 + x_2)/2\| > 1 - \varepsilon$;
(3) $\|(x_3 - x_1)/2\| > 1 - \varepsilon$.

Proposition 4. *Let X be a Banach space, if $Y_J(X) < 2$, then X has normal structure.*

Proof. Note that uniformly non-square spaces must be reflexive (see [22]). Thus, by Proposition 3 and Remark 1, we only need to prove X has weak normal structure. Suppose X without weak normal structure. From Lemma 1, we know that, for any $0 < \varepsilon < 1$, there exist x_1, x_2, x_3 in S_X satisfying $x_2 = x_3 + x_1$ and $\|(x_3 - x_1)/2\| > 1 - \varepsilon$. Thus,

$$2 - 2\varepsilon < \|x_3 - x_1\| < Y_J(X).$$

Since ε can be arbitrarily small, we obtain $2 \leq Y_J(X)$. This contradicts $Y_J(X) < 2$. $\square$

Proposition 5. *Let X be a Banach space, if X is uniformly non-square, then $J_L(X) > 1$.*

Proof. Suppose $J_L(X) = 1$, thus there exist $x_n, y_n \in S_X$ such that $\|x_n + y_n\| = 1$ and $\|x_n - y_n\| \to 1 (n \to \infty)$. Let $v_n = x_n + y_n$, $u_n = \frac{x_n - y_n}{\|x_n - y_n\|}$. Applying the continuity of norm, yield

$$\lim_{n\to\infty} \|v_n + u_n\| = \lim_{n\to\infty} \left\| x_n + y_n + \frac{x_n - y_n}{\|x_n - y_n\|} \right\| = 2;$$
$$\lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \left\| x_n + y_n - \frac{x_n - y_n}{\|x_n - y_n\|} \right\| = 2,$$

which means that we can not find a $\delta \in (0,1)$ such that if $x, y \in S_X$ then

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta \text{ or } \left\| \frac{x - y}{2} \right\| \leq 1 - \delta.$$

This completes the proof. $\square$

Lemma 2. *Ref. [29] A Banach space X is uniformly convex if and only if for any $x_n, y_n \in S_X$ such that $\|x_n + y_n\| \to 2$ implies that $\|x_n - y_n\| \to 0$.*

Proposition 6. *Let X be a Banach space, if X is uniformly convex, then $Y_J(X) < 2$.*

Proof. Suppose $Y_J(X) = 2$, so there exist $x_n, y_n \in S_X$ such that $\|x_n + y_n\| = 1$ and $\|x_n - y_n\| \to 2 (n \to \infty)$. Let $\overline{x}_n = x_n$, $\overline{y}_n = -y_n$, we compute

$$\|\overline{x}_n + \overline{y}_n\| = \|x_n - y_n\| \to 2;$$
$$\|\overline{x}_n - \overline{y}_n\| = \|x_n + y_n\| = 1 \neq 0,$$

which contradicts the Lemma 2. $\square$

7. Conclusions

In view of the fact that the problem of circles and their inscribed triangles is an important research topic in the Euclidean geometry and the works in [20,21], we define two constants, in this paper, which are symmetric and related to the side lengths of the inscribed equilateral triangles of unit balls in Banach spaces. In addition, we use them to study some geometric properties of Banach spaces. However, there are still many issues deserve further discussion. Are there simpler inequalities for $J_L(X)$, $Y_J(X)$ and some significant geometric constants? Can $J_L(X)$ and $Y_J(X)$ be used to characterize some geometric properties of Banach spaces? In the following research, we will continue to use these two constants to study the geometric properties of Banach spaces, and try to solve the problems mentioned above. At the same time, we will also consider introducing other new constants which can be related to the perimeter, area or some properties of inscribed triangles of unit balls. This will help us to judge whether it is feasible or powerful to study the geometric properties of Banach spaces by using some constants related to inscribed triangles of unit balls.

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