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On Certain Differential Subordination of Harmonic Mean Related to a Linear Function

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Abstract: In this paper we study a certain differential subordination related to the harmonic mean and its symmetry properties, in the case where a dominant is a linear function. In addition to the known general results for the differential subordinations of the harmonic mean in which the dominant was any convex function, one can study such differential subordinations for the selected convex function. In this case, a reasonable and difficult issue is to look for the best dominant or one that is close to it. This paper is devoted to this issue, in which the dominant is a linear function, and the differential subordination of the harmonic mean is a generalization of the Briot–Bouquet differential subordination.

Keywords: differential subordination; harmonic mean; arithmetic mean; geometric mean; convex function

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1. Introduction

Given $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and let $\mathbb{D} := \mathbb{D}_1$. Let $\mathcal{H}(D)$ be the set of all analytic functions in a domain D in $\mathbb{C}$ and let $\mathcal{H} := \mathcal{H}(\mathbb{D})$. A function $f \in \mathcal{H}$ is said to be subordinate to a function $F \in \mathcal{H}$ if there exists $\omega \in \mathcal{H}$ such that $\omega(0) := 0$, $\omega(\mathbb{D}) \subset \mathbb{D}$ and $f = F \circ \omega$ in $\mathbb{D}$. We write then that $f \prec F$. If F is univalent, then

$$f \prec F \Leftrightarrow (f(0) = F(0) \wedge f(\mathbb{D}) \subset F(\mathbb{D})). \tag{1}$$

Assume that $\psi : \mathbb{C}^2 \to \mathbb{C}$ and $h \in \mathcal{H}$ is univalent. We say that a function $p \in \mathcal{H}$ satisfies the first-order differential subordination if the function $\mathbb{D} \ni z \mapsto \psi(p(z), zp'(z))$ is analytic and

$$\psi(p(z), zp'(z)) \prec h(z), \quad z \in \mathbb{D}. \tag{2}$$

Then, we also say that p is a solution of (2). A univalent function $q \in \mathcal{H}$ is called a dominant of solutions of differential subordination (2) (shortly, a dominant) if $p \prec q$ for all solutions $p \in \mathcal{H}$ of (2). A dominant $\tilde{q}$ of (2) is called the best dominant of (2) if $\tilde{q} \prec q$ for all dominants q of (2) ([1–3], see ([4] p. 16)).

Note that the differential subordination (2) can be written as the differential equation

$$\psi(p(z), zp'(z)) = h(\omega(z)), \quad z \in \mathbb{D},$$

where $\omega \in \mathcal{H}$ is such that $\omega(0) := 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$.

The question when (2) yields $p \prec h$ is the basis for the theory of differential subordinations (see Lewandowski et al. [5], Miller and Mocanu [6–8], and the book of Miller and Mocanu [4]).

Of particular interest are cases in which the subordinate function ψ in (2) is associated with the arithmetic, geometric, and harmonic means. Differential subordinations related to the arithmetic and geometric means have been investigated by various authors. The case of the arithmetic mean, that is, the differential subordinations of the form

$$\begin{aligned}\psi(p(z), zp'(z)) &= p(z) + \alpha zp'(z)\Phi(p(z)) \\ &= (1-\alpha)p(z) + \alpha\big(p(z) + zp'(z)\Phi(p(z))\big) \prec h(z), \quad z \in \mathbb{D},\end{aligned} \tag{3}$$

with $\alpha \in \mathbb{C}$, was discussed in [4] (pp. 121–131), with further references. The simplest form of the differential subordination of type (3) is the following:

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad z \in \mathbb{D},$$

where $\gamma \neq 0$. Such a subordination with $\gamma \in \mathbb{C}$, $\operatorname{Re}\gamma > 0$, was examined by Hallenbeck and Ruscheweyh [9]. The differential subordinations related to the geometric mean were introduced by Kanas et al. [10] (for further references see [11,12]).

Research on the differential subordinations related to the harmonic mean is a fresh idea. It was started by Chojnacka et al. [1] and Cho et al. [2].

Let $\beta \in [0,1]$ and a, $b \in \mathbb{C}$. For $b + \beta(b-a) \neq 0$, the harmonic mean of a and b is defined as

$$\frac{ab}{b + \beta(a-b)}.$$

Definition 1. *Let $\beta \in [0,1]$ and $\Phi \in \mathcal{H}(D)$. By $\mathcal{H}(\beta, \Phi)$ we denote the subclass of $\mathcal{H}$ of all nonconstant functions p such that $p(\mathbb{D}) \subset D$, and the function*

$$\mathbb{D} \ni z \mapsto \frac{p(z)(p(z) + zp'(z)\Phi(p(z)))}{p(z) + (1-\beta)zp'(z)\Phi(p(z))}$$

is either analytic or has only removable singularities with an analytic extension on $\mathbb{D}$.

In [2], for $\beta \in (0,1]$, $\Phi \in \mathcal{H}(D)$, $p \in \mathcal{H}(\beta, \Phi)$ and a univalent function $h \in \mathcal{H}$, the differential subordination of the harmonic mean of the type

$$\frac{p(z)(p(z) + zp'(z)\Phi(p(z)))}{p(z) + (1-\beta)zp'(z)\Phi(p(z))} \prec h(z), \quad z \in \mathbb{D}, \tag{4}$$

was examined. The above differential subordination with $\beta := 1/2$ and selected functions Φ and h was also considered in [13].

A function $f \in \mathcal{H}$ is said to be convex if it is univalent (analytic and injective) and $f(\mathbb{D})$ is a convex domain.

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For a set $A \subset \mathbb{C}$, its closure will be denoted as $\overline{A}$.

For details on the corners of curves, see, for example, [14] (pp. 51–65).

Definition 2. *By $\mathcal{Q}$ we denote the family of convex functions h with the following properties:*

(a) *$h(\mathbb{D})$ is bounded by finitely many smooth arcs which form corners at their end points (including corners at infinity);*

(b) *$E(h)$ is the set of all points $\zeta \in \mathbb{T}$ which corresponds to corners $h(\zeta)$ of $\partial h(\mathbb{D})$;*

(c) *$h'(\zeta) \neq 0$ exists at every $\zeta \in \mathbb{T} \setminus E(h)$.*

In [2], the following was shown.

Theorem 1. *Let $\beta \in (0,1]$, $h \in \mathcal{Q}$ with $0 \in \overline{h(\mathbb{D})}$, and $\Phi \in \mathcal{H}(D)$ be such that $D \supset h(\mathbb{T} \setminus E(h))$ and*

$$\operatorname{Re}\Phi(h(\zeta)) \geq 0, \quad \Phi(h(\zeta)) \neq 0, \quad \zeta \in \mathbb{T} \setminus E(h).$$

If $p \in \mathcal{H}(\beta, \Phi)$, $p(0) = h(0)$ *and*

$$\frac{p(z)(p(z) + zp'(z)\Phi(p(z)))}{p(z) + (1-\beta)zp'(z)\Phi(p(z))} \prec h(z), \quad z \in \mathbb{D},$$

then

$$p \prec h.$$

Let us mention that the proof of the above theorem was based on the symmetry properties of the harmonic mean related also to the inversion mapping of the complex plane. In a similar way, the symmetry properties of the geometric mean were applied to reprove in a new way the main theorem on the differential subordinations of the geometric mean [12], first shown in [10].

In this paper we continue the research on the differential subordination of the form (4). Now we assume that Φ is the composition of a linear function with the inversion function, and that h is a linear function. We also generalize the first-order Euler differential subordination (see [4] (pp. 334–336)) for the nonlinear case.

The lemma below is the special case of Lemma 2.2d [4] (p. 22) and it is needed for the proof of the main result.

Lemma 1. *Let* $h \in \mathcal{Q}$ *and* $p \in \mathcal{H}$ *be a nonconstant function with* $p(0) := h(0)$. *If* p *is not subordinate to* h, *then there exist* $z_0 \in \mathbb{D} \setminus \{0\}$ *and* $\zeta_0 \in \mathbb{T} \setminus E(h)$ *such that* $p\left(\mathbb{D}_{|z_0|}\right) \subset h(\mathbb{D})$,

$$p(z_0) = h(\zeta_0) \tag{5}$$

and

$$m_0 := \frac{z_0 p'(z_0)}{\zeta_0 h'(\zeta_0)} \geq 1. \tag{6}$$

2. Main Result

Given $\delta > 0$ and $\gamma > 0$, let

$$\Phi_{\delta,\gamma}(w) := \frac{1}{\delta w + \gamma}, \quad w \in \mathbb{C} \setminus \{-\gamma/\delta\},$$

and

$$\Phi_{0,\gamma}(w) := \frac{1}{\gamma}, \quad w \in \mathbb{C}.$$

For $M > 0$ let $h_M(z) := Mz$, $z \in \mathbb{D}$. Clearly, $E(h_M) = \emptyset$. Moreover, for $\delta > 0$,

$$\operatorname{Re} \Phi_{\delta,\gamma}(h_M(\zeta)) = \operatorname{Re} \frac{1}{\delta M \zeta + \gamma}$$

$$= \frac{1}{|\delta M \zeta + \gamma|^2} (\delta M \operatorname{Re}(\zeta) + \gamma) > 0, \quad \zeta \in \mathbb{T},$$

if and only if $M < \gamma/\delta$. Clearly, for $M > 0$,

$$\operatorname{Re} \Phi_{0,\gamma}(h_M(\zeta)) > 0, \quad \zeta \in \mathbb{T}.$$

Let $\mathcal{H}(\beta, \delta, \gamma) := \mathcal{H}(\beta, \Phi_{\delta,\gamma})$. Thus, the following conclusion follows from Theorem 1.

Corollary 1. *Let $\beta \in (0,1]$, $\delta \geq 0$, and $\gamma > 0$. Let $0 < M < \gamma/\delta$ when $\delta > 0$, and $0 < M < \infty$ when $\delta = 0$. If $p \in \mathcal{H}(\beta, \delta, \gamma)$, $p(0) = 0$, and*

$$\left| \frac{p(z)\left(p(z) + \dfrac{zp'(z)}{\delta p(z) + \gamma}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z) + \gamma}} \right| < M, \quad z \in \mathbb{D},$$

then

$$|p(z)| < M, \quad z \in \mathbb{D}.$$

Now we will improve the above result, so in the same way we will improve Theorem 1 to that special selected Φ.

Theorem 2. *Let $\beta \in (0,1]$, $\delta \geq 0$, and $\gamma > 0$. Let $0 < M < (\gamma - \beta + 1)/\delta$ when $\delta > 0$, and $0 < M < \infty$ when $\delta = 0$. If $p \in \mathcal{H}(\beta, \delta, \gamma)$, $p(0) := 0$, and*

$$\left| \frac{p(z)\left(p(z) + \dfrac{zp'(z)}{\delta p(z) + \gamma}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z) + \gamma}} \right| < M\frac{\delta M + \gamma + 1}{\delta M + \gamma - \beta + 1}, \quad z \in \mathbb{D}, \tag{7}$$

then

$$|p(z)| < M, \quad z \in \mathbb{D}. \tag{8}$$

Proof. Since h_M is univalent, $p(0) = h_M(0) = 0$ and (8) can be replaced by the inclusion $p(\mathbb{D}) \subset h_M(\mathbb{D})$; by using (1) the condition (8) is equivalent to the subordination $p \prec h_M$.

Suppose, on the contrary that p is not subordinate to h_M. Since $h_M \in \mathcal{Q}$ with $E(h_M) = \emptyset$, by Lemma 1 there exist $z_0 \in \mathbb{D} \setminus \{0\}$ and $\zeta_0 \in \mathbb{T}$ such that (5) and (6) hold. Thus

$$p(z_0) = M\zeta_0$$

and for some $m_0 \geq 1$,

$$z_0 p'(z_0) = m_0 M \zeta_0.$$

Hence

$$\begin{aligned} &\left| \frac{p(z_0)\left(p(z_0) + \dfrac{z_0p'(z_0)}{\delta p(z_0) + \gamma}\right)}{p(z_0) + (1-\beta)\dfrac{z_0p'(z_0)}{\delta p(z_0) + \gamma}} \right| \\ &= |p(z_0)| \left| \frac{p(z_0)(\delta p(z_0) + \gamma) + z_0p'(z_0)}{p(z_0)(\delta p(z_0) + \gamma) + (1-\beta)z_0p'(z_0)} \right| \\ &= M \left| \frac{M\zeta_0(\delta M\zeta_0 + \gamma) + m_0M\zeta_0}{M\zeta_0(\delta M\zeta_0 + \gamma) + (1-\beta)m_0M\zeta_0} \right| \\ &= M \left| \frac{\delta M\zeta_0 + \gamma + m_0}{\delta M\zeta_0 + \gamma + (1-\beta)m_0} \right|. \end{aligned} \tag{9}$$

Consider first the case $\delta > 0$. Since then $0 < M < (\gamma - \beta + 1)/\delta$ and $m_0 \geq 1$, it follows that $\delta M\zeta_0 + \gamma + m_0 \neq 0$ and $\delta M\zeta_0 + \gamma + (1-\beta)m_0 \neq 0$ for $\zeta \in \mathbb{T}$. Define

$$q(\zeta) := \frac{\delta M\zeta + \gamma + m_0}{\delta M\zeta + \gamma + (1-\beta)m_0}, \quad \zeta \in \mathbb{T}. \tag{10}$$

As q is a linear-fractional mapping having real coefficients, $q(\mathbb{T})$ is a circle symmetrical with respect to the real axis. Moreover, it is easy to check that

$$q(1) = \frac{\delta M + \gamma + m_0}{\delta M + \gamma + (1-\beta)m_0} < \frac{-\delta M + \gamma + m_0}{-\delta M + \gamma + (1-\beta)m_0} = q(-1).$$

Thus, particularly

$$|q(\zeta_0)| \geq q(1) = \frac{\delta M + \gamma + m_0}{\delta M + \gamma + (1-\beta)m_0}, \quad \zeta \in \mathbb{T}. \tag{11}$$

Since $m_0 \geq 1$, so

$$\frac{\delta M + \gamma + m_0}{\delta M + \gamma + (1-\beta)m_0} \geq \frac{\delta M + \gamma + 1}{\delta M + \gamma - \beta + 1}. \tag{12}$$

Hence, from (11) and (9) we deduce that

$$\left| \frac{p(z_0)\left(p(z_0) + \dfrac{z_0 p'(z_0)}{\delta p(z_0) + \gamma}\right)}{p(z_0) + (1-\beta)\alpha \dfrac{z_0 p'(z_0)}{\delta p(z_0) + \gamma}} \right| \geq M \frac{\delta M + \gamma + 1}{\delta M + \gamma - \beta + 1}, \tag{13}$$

which contradicts (7).

When $\delta = 0$, q given by (10) is a constant function. It is clear that then the inequality (12) so the inequality (13) holds with $\delta = 0$. This ends the proof of the theorem. □

Remark 1. *Since* $0 \in \overline{h_M(\mathbb{D})}$ *and*

$$\frac{\delta M + \gamma + 1}{\delta M + \gamma - \beta + 1} \geq 1,$$

Corollary 1 follows from Theorem 2 for the so-selected Φ *and* $h := h_M$.

Note that Theorem 2 can be formulated as follows.

Theorem 3. *Let* $\beta \in (0,1]$, $\delta \geq 0$, *and* $\gamma > 0$. *Let* $0 < M < (\gamma - \beta + 1)/\delta$ *when* $\delta > 0$, *and* $0 < M < \infty$ *when* $\delta = 0$. *If* $p \in \mathcal{H}(\beta, \delta, \gamma)$, $p(0) := 0$, *and*

$$\frac{p(z)\left(p(z) + \dfrac{zp'(z)}{\delta p(z) + \gamma}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z) + \gamma}} \prec Mz, \quad z \in \mathbb{D}, \tag{14}$$

then

$$p(z) \prec \left(1 - \frac{\beta}{\delta M + \gamma + 1}\right) Mz, \quad z \in \mathbb{D}.$$

Remark 2. *It is interesting to ask which is the best dominant of* (14). *Applying Theorem 2.3e of* [4] *we can expect that the best dominant* $\tilde{q}$ *of* (14) *should be a univalent solution* $q := \tilde{q}$ *of the differential equation*

$$\frac{q(z)\left(q(z) + \dfrac{zq'(z)}{\delta q(z) + \gamma}\right)}{q(z) + (1-\beta)\dfrac{zq'(z)}{\delta q(z) + \gamma}} = Mz, \quad z \in \mathbb{D},$$

if it exists. As can be easily checked, the function

$$q(z) := \left(1 - \frac{\beta}{\delta M + \gamma + 1}\right) Mz, \quad z \in \mathbb{D},$$

with $\delta \neq 0$, *does not satisfy the above equation. Therefore, the problem of finding the best dominant* (14) *is open.*

Theorem 3 gives the sequence of corollaries listed below. The case $M = 1$ can be considered when $\gamma + 1 > \delta + \beta$. The last inequality obviously holds when $\delta = 0$.

Corollary 2. *Let $\beta \in (0,1]$, $\delta \geq 0$ and $\gamma > 0$ be such that $\gamma + 1 > \delta + \beta$. If $p \in \mathcal{H}(\beta,\delta,\gamma)$, $p(0) := 0$, and*

$$\frac{p(z)\left(p(z) + \dfrac{zp'(z)}{\delta p(z) + \gamma}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{\delta p(z) + \gamma}} \prec z, \quad z \in \mathbb{D},$$

then

$$p(z) \prec \left(1 - \frac{\beta}{\delta + \gamma + 1}\right)z, \quad z \in \mathbb{D}.$$

For $\delta = 1$ and $\gamma = 0$ Theorem 3 is reduced to the following conclusion.

Corollary 3. *Let $\beta \in (0,1]$ and $0 < M < 1 - \beta$. If $p \in \mathcal{H}(\beta,1,0)$, $p(0) := 0$, and*

$$\frac{p(z)\left(p(z) + \dfrac{zp'(z)}{p(z)}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{p(z)}} \prec Mz, \quad z \in \mathbb{D},$$

then

$$p(z) \prec \left(1 - \frac{\beta}{M+1}\right)M, \quad z \in \mathbb{D}.$$

For $\beta = 1$ and $\delta > 0$ Theorem 3 applies to the special case of the well-known Briot–Bouquet differential subordination of the first-order (see, e.g., [15]).

Corollary 4. *Let $\delta > 0$, $\gamma > 0$, and $0 < M < \gamma/\delta$. If $p \in \mathcal{H}(1,\delta,\gamma)$, $p(0) := 0$, and*

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} \prec Mz, \quad z \in \mathbb{D}, \tag{15}$$

then

$$p(z) \prec \left(1 - \frac{1}{\delta M + \gamma}\right)Mz, \quad z \in \mathbb{D}.$$

Remark 3. *For the Briot–Bouquet differential subordination, the best dominant was found in [15] (see also [4] (Theorem 3.2j, p. 97)). We will provide it below for the case considered in Corollary 4. Let (see [4] (p. 46))*

$$R_{\gamma,1}(z) := \gamma\frac{1+z}{1-z} + \frac{2z}{1-z^2}, \quad z \in \mathbb{D}.$$

Thus $R_{\gamma,1}(0) = \gamma$ and $R_{\gamma,1}(\mathbb{D})$ is the complex plane with the half-lines $\operatorname{Re} w = 0$ and $|\operatorname{Im} w| \geq \sqrt{1+2\gamma}$ as its two slits. Let

$$q(z) := \left(1 - \frac{1}{\delta M + \gamma}\right)Mz, \quad z \in \mathbb{D}.$$

We have

$$\delta q(0) + \gamma = R_{\gamma,1}(0) = \gamma$$

and since $\delta M < \gamma$, it follows also

$$\operatorname{Re}(\delta q(z) + \gamma) > -\delta\left(1 - \frac{1}{\delta M + \gamma}\right)M + \gamma$$

$$= \gamma - \delta M + \frac{\delta M}{\delta M + \gamma} > 0, \quad z \in \mathbb{D}.$$

Hence

$$\delta q + \gamma \prec R_{\gamma,1}.$$

Now applying Theorem 3.2j of [4] we state that the function

$$\widetilde{q}(z) := z^{\gamma} \exp(\delta M z) \left(\delta \int_0^z t^{\gamma-1} \exp(\delta M t) dt \right)^{-1} - \frac{\gamma}{\delta}, \quad z \in \mathbb{D},$$

is a univalent solution of the differential equation

$$q(z) + \frac{zq'(z)}{\delta q(z) + \gamma} = Mz, \quad z \in \mathbb{D}.$$

Consequently, if $p \in \mathcal{H}(1, \delta, \gamma)$, $p(0) := 0$, *satisfies* (15), *then*

$$p(z) \prec \widetilde{q}(z) \prec \left(1 - \frac{1}{\delta M + \gamma}\right) Mz, \quad z \in \mathbb{D}$$

and $\widetilde{q}$ *is the best dominant of* (15).

For $M = 1$, which holds when $\gamma > \delta$, we have the following.

Corollary 5. *Let* $0 < \delta < \gamma$. *If* $p \in \mathcal{H}(1, \delta, \gamma)$, $p(0) := 0$, *and*

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} \prec z, \quad z \in \mathbb{D},$$

then

$$p(z) \prec \left(1 - \frac{1}{\delta + \gamma}\right) z, \quad z \in \mathbb{D}.$$

The case $\delta = 0$ in Theorem 3 reduces to Corollary 2.7 in [3]. To be self-contained, we will provide more detailed proof than in [3], where it has been shown that q given in (17) is the best dominant.

Corollary 6. *Let* $\beta \in (0, 1]$, $\gamma > 0$, *and* $M > 0$. *If* $p \in \mathcal{H}(\beta, 0, \gamma)$, $p(0) := 0$, *and*

$$\frac{p(z)\left(p(z) + \dfrac{zp'(z)}{\gamma}\right)}{p(z) + (1-\beta)\dfrac{zp'(z)}{\gamma}} \prec Mz, \quad z \in \mathbb{D}, \tag{16}$$

then

$$p(z) \prec q(z) := \left(1 - \frac{\beta}{\gamma + 1}\right) Mz, \quad z \in \mathbb{D}. \tag{17}$$

Moreover, the function q *is the best dominant of* (16).

Proof. We will show that q is the best dominant of (16). We will find the univalent solution q of the differential equation

$$\frac{q(z)\left(q(z) + \dfrac{zq'(z)}{\gamma}\right)}{q(z) + (1-\beta)\dfrac{zq'(z)}{\gamma}} = Mz, \quad z \in \mathbb{D}, \tag{18}$$

such that $q(0) := 0$. We apply the technique of power series to find the analytic solution of (18) of the form

$$q(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{19}$$

Let $\alpha := 1/\gamma$. Since q is required to be univalent, we have

$$a_1 = q'(0) \neq 0. \tag{20}$$

From (18) we equivalently obtain

$$(q(z))^2 + \alpha z q(z) q'(z) = M\Big[zq(z) + \alpha(1-\beta)z^2 q'(z)\Big], \quad z \in \mathbb{D}.$$

Putting the series from (19) into the above equality we get

$$\begin{aligned} & a_1 z^2 + 2a_1 a_2 z^3 + (2a_1 a_3 + a_2^2)z^4 + (2a_1 a_4 + 2a_2 a_3)z^5 + \dots \\ & +\alpha\Big(a_1^2 z^2 + 3a_1 a_2 z^3 + (4a_1 a_3 + 2a_2^2)z^4 + (5a_1 a_4 + 5a_2 a_3)z^5 + \dots\Big) \\ & = M\Big[a_1 z^2 + a_2 z^3 + a_3 z^4 + a_4 z^5 + \dots \\ & +\alpha(1-\beta)\Big(a_1 z^2 + 2a_2 z^3 + 3a_3 z^4 + 4a_4 z^5 + \dots\Big)\Big], \quad z \in \mathbb{D}. \end{aligned}$$

Comparing the early coefficients, we get

$$\begin{aligned} & a_1^2(1+\alpha) = Ma_1(1+\alpha(1-\beta)), \\ & a_1 a_2(2+3\alpha) = Ma_2(1+2\alpha(1-\beta)), \\ & a_1 a_3(2+4\alpha) + a_2^2(1+2\alpha) = Ma_3(1+3\alpha(1-\beta)), \\ & a_1 a_4(2+5\alpha) + a_2 a_3(2+5\alpha) = Ma_4(1+4\alpha(1-\beta)), \end{aligned} \tag{21}$$

and generally, for $n = 2k-1,\ k \geq 2$,

$$\begin{aligned} & (2+(2k-1)\alpha)(a_1 a_{2k-2} + a_2 a_{2k-3} + \cdots + a_{k-1} a_k) \\ & = M(1+(2k-2)\alpha(1-\beta))a_{2k-2}, \end{aligned} \tag{22}$$

and for $n = 2k,\ k \geq 2$,

$$\begin{aligned} & 2(1+k\alpha)\left(a_1 a_{2k-1} + a_2 a_{2k-2} + \cdots + a_{k-1} a_{k+1} + \frac{1}{2} a_k^2\right) \\ & = M(1+(2k-1)\alpha(1-\beta))a_{2k-1}. \end{aligned} \tag{23}$$

Taking (20) into account, from the first equation in (21) it follows that

$$a_1 = \frac{M(1+\alpha(1-\beta))}{1+\alpha}. \tag{24}$$

This and the second equation in (21) gives $a_2 = 0$. Substituting $a_2 = 0$ into the third equation in (21), because of (20), we see that $a_3 = 0$. This way, using mathematical induction, we can prove that

$$a_2 = a_3 = \cdots = a_{2k-3} = 0, \tag{25}$$

and that the Formula (22) reduces to

$$(2+(2k-1)\alpha)a_1 a_{2k-2} = M(1+(2k-2)\alpha(1-\beta))a_{2k-2},$$

which in view of (24) yields $a_{2k-2} = 0$. So, using (25), Equation (23) reduces to

$$2(1 + k\alpha)a_1 a_{2k-1} = M(1 + (2k - 1)\alpha(1 - \beta))a_{2k-1},$$

which in view of (24) yields $a_{2k-1} = 0$. Thus, we proved that $a_n = 0$ for all $n \geq 2$. In this way, by (19) and (24) it follows that

$$q(z) = \frac{M(1 + \alpha(1 - \beta))}{1 + \alpha} z, \quad z \in \mathbb{D},$$

is the unique analytic univalent solution of (18). This ends the proof of the lemma. □

3. Conclusions

Research on the differential subordinations of the harmonic mean began recently with two papers [1,2]. In these papers, general theorems for the differential subordinations of the harmonic mean, in which any convex function is the dominant, were proved. Detailed studies of such subordinations, in which the dominant is a specific convex function, offer a number of new and non-trivial problems. One of them is to determine the best dominant or one that is close to it. It also means an improvement for a specific convex function of the above-mentioned general results. This issue is difficult, and at the same time, interesting for study. Such research was undertaken only in [3]. In this paper, a situation is considered in which the dominant is a linear function, and the scheme of the differential subordination of the harmonic mean is constructed in such a way as to be a generalization of the Briot–Bouquet differential subordination. The main result of this paper is contained in Theorem 2, in which the constant on the right side of the inequality (7) is determined, which increases the initial constant M. The result of Theorem 3 is equivalent to this. As noted in Remark 1, the obtained linear function is not the best dominant. This problem is therefore still open.

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