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An Application of the Principle of Differential Subordination to Analytic Functions Involving Atangana–Baleanu Fractional Integral of Bessel Functions

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Abstract: The aim of this paper is to establish certain subordination results for analytic functions involving Atangana–Baleanu fractional integral of Bessel functions. Studying subordination properties by using various types of operators is a technique that is widely used.

1. Introduction and Preliminary Results

In recent years, in the field of fractional calculus, many definitions of fractional integral operators have been derived. These operators have been proven to be particularly useful in many areas of applicability by modeling various phenomena and processes. In the context of fractional calculus study, an important issue is to generalize the concept of nth derivatives and nth integrals. Originally conceived for natural numbers n, the study was extended to the concept of λnt derivatives and λnt integrals. These are often considered both as differintegrals for more general types of λ. An important and interesting step in this study is to consider fractional extensions for the complex plane. Most of the concepts of fractional λnt differintegrals are applied equally well for $\lambda \in \mathbb{C}$ as for $\lambda \in \mathbb{R}$. For example, the most well-known definition used, the Riemann–Liouville, denoted RL, is as follows:

$$
{}^{RL}_c I^\nu_z f(z) = \frac{1}{\Gamma(\nu)} \int_c^z (z-w)^{\nu-1} f(w) dw, \ \ \mathrm{Re}\,(\nu) > 0. \tag{1}
$$

Here, (1) is the definition of fractional integral [1].

In the field of fractional calculus on the complex plane, a reference problem is the topic of branch points and branch cuts. In the integrant of the relation (1) appears the singular function $(z-w)^{\nu-1}$, which has a branch point at $w = z$, the point at the end of the integration curve. Therefore, choices of branch function have to be considered so as to produce a suitable form for the relation (1). This topic is developed in [1], §22.

Certain differential and superordination results implying integral operators were studied recently in [2]. Using fractional integral operators, the authors of [3,4] obtained properties and inequalities regarding a subclass of analytic functions. We also recall here some results obtained by applying fractional integral on different hypergeometric functions seen in papers [5–7]. Another interesting study of confluent (or Kummer) hypergeometric function was made in [8], which extended the study made in [9]. The univalence of confluent Kummer function was also studied in [10]. Inspired by the study from [8], we present here a new fractional integral operator connecting two other important operators, namely the Atangana–Baleanu integral operator and Riemann–Liouville.

The Atangana–Baleanu definition of fractional calculus was first introduced in [11] and has been studied in works such as [12–14] and others. The definition of Atangana–

Baleanu integral operator can be extended to differently complex values of differentiation order ν by using analytic continuation.

Many interesting results were obtained recently as applications of fractional integral operator in the complex plane to fluid mechanics. We recall here the idea of Tsallis entropy in the complex plane. Motivated by this important topic of fractional integral operators, we intended to extend the Atangana–Baleanu integral operator to other results in Geometric Function Theory, namely differential subordinations results. The Atangana–Baleanu integral operator has been proven useful by its many applications. Taking note of these results will be useful to introduce in future works a new symmetric differential operator and its integral.

Definition 1. *From [15], let c be a fixed complex number, f be a complex function that is analytic on an open star-domain D centered at c, and let $B(\nu)$ be a multiplier function that is also analytic. The extended Atangana–Baleanu integral, denoted by $_c^{AB}I_z^\nu f(z)$, is defined for any $\nu \in \mathbb{C}$, with $Re\nu > 0$ and any $z \in D\backslash\{c\}$ by:*

$$_c^{AB}I_z^\nu f(z) = \frac{1-\nu}{B(\nu)}f(z) + \frac{\nu}{B(\nu)}{}_c^{RL}I_z^\nu f(z). \tag{2}$$

Proposition 1. *From [15], the extended Atangana–Baleanu integral operator given in Definition 1 is:*

1. *a function that is an analytic one of argument $z \in D\backslash\{c\}$ and also of $\nu \in \mathbb{C}$, provided f and B are analytic functions and B is nonzero.*
2. *similar to the original formula when $0 < \nu < 1$ and $c < z$ in $\mathbb{R}$.*
    Hence, the above extended integral operator yields an analytic continuation of the original Atangana–Baleanu integral operator to complex values z and $\nu \in \mathbb{C}$.

Definition 2. *From [16,17], consider $\delta, b, c \in \mathbb{C}$ and the second-order linear homogenous differential equation*

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - \delta^2 + (1-b)\delta \right] w(z) = 0 \tag{3}$$

which is a natural extension of Bessel's equation. The solution $w(z)$ has a series representation

$$w_{\delta,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!\Gamma\left(\delta + k + \frac{b+1}{2}\right)} \cdot \left(\frac{z}{2}\right)^{2k+\delta} \tag{4}$$

and we called the generalized Bessel function of the first kind of order δ.

Recall the well-known symbols and results.

Consider $\mathcal{H}$ the class of analytic functions in

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ containing functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots.$$

Let $\mathcal{A}(p, n)$ be the class of functions normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}) \tag{5}$$

which are analytic in U. In particular, we have

$$\mathcal{A}(p, 1) := \mathcal{A}_p \quad \text{and} \quad \mathcal{A}(1, 1) := \mathcal{A} = \mathcal{A}_1.$$

Consider the class

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots\}$$

with $\mathcal{A}_1 := \mathcal{A}$.

For two analytic functions f and g, we recall here the principle of subordination. We say that the function f is subordinate to g, written as

$$f \prec g \quad \text{and} \quad f(z) \prec g(z), \quad (z \in U) \tag{6}$$

if there exists a Schwarz function w in U that is analytic in U such that $f(z) = g(w(z)), z \in U$.

If the function g is univalent in U, then we can rewrite (6) equivalently

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Consider $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$.

If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \tag{7}$$

then p is a solution of the differential superordination (7). If f is subordinate to F, then F is superordinate to f.

An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (7). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (7) is said to be the best subordinant. In [18], Miller and Mocanu give conditions on h, q, and ϕ such that:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \ \Rightarrow \ q(z) \prec p(z).$$

Applying the results of Miller and Mocanu [18], Bulboacă [19] gives classes of first-order differential superordinations preserving integral operators [20]. Srivastava and Lashin [21] studied star-like functions of complex order and also convex functions of complex order using the Briot–Bouquet differential subordination technique.

Definition 3. *[18] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} - E(f)$, where*

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Theorem 1. *[22] Let the function q be univalent in the open unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that
(1) $Q(z)$ is starlike univalent in Δ and
(2) $\operatorname{Re}\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If p is an analytic function in U with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

2. Main Results

By making use of Definitions 1 and 2, we introduce the following integral operator.

Definition 4. *Consider $\delta, b, c, \lambda \in \mathbb{C}$ with $\Re\lambda > 0$ and*

$$^{AB}I_z^\lambda w_{\delta,b,c}(z) = \frac{1-\lambda}{B(\lambda)} w_{\delta,b,c} + \frac{\lambda}{B(\lambda)} {}^{RL}I_z^\lambda w_{\delta,b,c}(z) \tag{8}$$

where $^{AB}I_z^\lambda w_{\delta,b,c}(z) = {}_0^{AB} I_z^\lambda w_{\delta,b,c}(z), {}^{RL}I_z^\lambda w_{\delta,b,c}(z) = {}_0^{RL} I_z^\lambda w_{\delta,b,c}(z)$ and $B(\lambda)$ is a normalization function $B(0) = B(1) = 1$.

Theorem 2. *Let $\alpha, \beta, \gamma, \xi, \mu \in \mathbb{C}$, $\mu \neq 0$, $\xi \neq 0$, $\mathrm{Re}\,\lambda > 0$ and q be univalent in the open unit disc U such that $q(z) \neq 0$.*

Suppose that $\dfrac{zq'(z)}{q(z)}$ is star-like univalent in U. Let

$$\mathrm{Re}\left\{ \frac{\beta}{\xi}q(z) + \frac{2\gamma}{\xi}(q(z))^2 + \frac{\alpha}{\xi}\frac{q(z)}{q'(z)} \right\} > 0 \tag{9}$$

and

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z) := \alpha + \beta\left[\frac{^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right]^\mu + \tag{10}$$

$$+\gamma\left[\frac{^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right]^{2\mu} + \xi\mu\left[\frac{z \cdot \left(^{AB}I_z^\lambda w_{\delta,b,c}(z)\right)'}{^{AB}I_z^\lambda w_{\delta,b,c}(z)} - 1\right].$$

If q satisfies the following subordination

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z) \prec \alpha + \beta q(z) + \gamma(q(z))^2 + \xi\frac{zq'(z)}{q(z)} \tag{11}$$

then

$$\left(\frac{^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec q(z), \quad z \in U, z \neq 0 \tag{12}$$

and q is the best dominant.

Proof. Consider the function

$$p(z) := \left(\frac{^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu, \quad z \in U, z \neq 0.$$

By a straightforward computation, one obtains

$$\frac{zp'(z)}{p(z)} = \mu\left[\frac{z(^{AB}I_z^\lambda w_{\delta,b,c}(z))'}{^{AB}I_z^\lambda w_{\delta,b,c}(z)} - 1\right].$$

By setting

$$\theta(w) : = \alpha + \beta w + \gamma w^2 \text{ and}$$
$$\phi(w) : = \frac{\xi}{w}$$

it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$, and $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Considering

$$Q(z) = zq'(z)\phi(q(z)) = \xi \frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z)) + Q(z) = \alpha + \beta q(z) + \gamma(q(z))^2 + \xi \frac{zq'(z)}{q(z)}$$

we get that $Q(z)$ is a star-like univalent in U and

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\beta}{\xi}q(z) + \frac{2\gamma}{\xi}(q(z))^2 + \frac{\alpha}{\xi}\frac{q(z)}{q'(z)}\right\} > 0.$$

The assertion (12) of Theorem 2 follows by applying Theorem 1. □

For the choices $q(z) = \dfrac{1 + Az}{1 + Bz}$, $-1 \le B < A \le 1$, and $q(z) = \left(\dfrac{1+z}{1-z}\right)^\eta$, $0 < \eta \le 1$ in Theorem 2, we obtain the next two corollaries.

Corollary 1. *Let* $\alpha, \beta, \gamma, \xi, \mu \in \mathbb{C}$, $\mu \ne 0$, $\xi \ne 0$, $\mathrm{Re}\,\lambda > 0$, $-1 \le B < A \le 1$, *and*

$$\mathrm{Re}\left\{\frac{\beta}{\xi}\frac{1 + Az}{1 + Bz} + \frac{2\gamma}{\xi}\left(\frac{1 + Az}{1 + Bz}\right)^2 + \frac{\alpha}{\xi}\frac{(1 + Az)(1 + Bz)}{A - B}\right\} > 0. \tag{13}$$

If

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z) \prec \alpha + \beta\frac{1 + Az}{1 + Bz} + \gamma\left(\frac{1 + Az}{1 + Bz}\right)^2 + \xi\frac{(A - B)z}{(1 + Az)(1 + Bz)} \tag{14}$$

where $\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z)$ *is defined in* (10), *then*

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec \frac{1 + Az}{1 + Bz} \tag{15}$$

and $\dfrac{1 + Az}{1 + Bz}$ *is the best dominant.*

Corollary 2. *Let* $\alpha, \beta, \gamma, \xi, \mu \in \mathbb{C}$, $\mu \ne 0$, $\xi \ne 0$, $\mathrm{Re}\,\lambda > 0$, $0 < \eta \le 1$, *and*

$$\mathrm{Re}\left\{\frac{\beta}{\xi}\left(\frac{1+z}{1-z}\right)^\eta + \frac{2\gamma}{\xi}\left(\frac{1+z}{1-z}\right)^{2\eta} + \frac{\alpha}{2\xi\eta}(1 - z^2)\right\} > 0. \tag{16}$$

If

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z) \prec \alpha + \beta\left(\frac{1+z}{1-z}\right)^\eta + \gamma\left(\frac{1+z}{1-z}\right)^{2\eta} + \frac{2\xi\eta z}{1 - z^2} \tag{17}$$

where $\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z)$ *is defined in* (10), *then*

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec \left(\frac{1+z}{1-z}\right)^\eta \tag{18}$$

and $\left(\dfrac{1+z}{1-z}\right)^\eta$ *is the best dominant.*

For the special case $q(z) = e^{\eta A z}$, with $|\eta A| < \pi$, Theorem 2 readily yields another corollary.

Corollary 3. *Let* $A, \alpha, \beta, \gamma, \xi, \mu \in \mathbb{C}$, $\mu \neq 0$, $\xi \neq 0$, $|\eta A| < \pi$, $\operatorname{Re} \lambda > 0$, *and*

$$\operatorname{Re}\left\{\frac{\alpha}{\xi \eta A} + \frac{\beta}{\xi} e^{\eta A z} + \frac{2\gamma}{\xi} e^{2\eta A z}\right\} > 0. \tag{19}$$

If

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z) \prec \alpha + \beta e^{\eta A z} + \gamma e^{2\eta A z} + \xi A \eta z \tag{20}$$

where $\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z)$ *is defined in* (10), *then*

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec e^{\eta A z} \tag{21}$$

and $e^{\eta A z}$ *is the best dominant.*

Corollary 4. *Let* $\alpha, \beta, \gamma, \xi, \mu, \eta \in \mathbb{C}, \mu \neq 0, \eta \neq 0, \xi \neq 0, \operatorname{Re} \lambda > 0, -1 \leq B < A \leq 1$, *and*

$$\operatorname{Re}\left\{\frac{\beta}{\xi}(1 + Bz)^{\frac{\eta(A-B)}{B}} + \frac{2\gamma}{\xi}(1 + Bz)^{\frac{2\eta(A-B)}{B}} + \frac{\alpha(1 + Bz)}{\xi \eta(A - B)}\right\} > 0.$$

If

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z) \prec \alpha + \beta(1 + Bz)^{\frac{\eta(A-B)}{B}} + \gamma(1 + Bz)^{\frac{2\eta(A-B)}{B}} + \xi \frac{z\eta(A - B)}{1 + Bz} \tag{22}$$

where $\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \gamma, \xi; z)$ *is defined in* (10), *then*

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec (1 + Bz)^{\frac{\eta(A-B)}{B}} \tag{23}$$

and $(1 + Bz)^{\frac{\eta(A-B)}{B}}$ *is the best dominant.*

We remark that $q(z) = (1 + Bz)^{\frac{\eta(A-B)}{B}}$ is univalent if and only if either

$$\left|\frac{\eta(A - B)}{B} - 1\right| \leq 1 \text{ or } \left|\frac{\eta(A - B)}{B} + 1\right| \leq 1.$$

Theorem 3. *Let the function* q *be univalent in the unit disc* U *such that* $q(z) \neq 0$ *and* $\alpha, \beta, \xi, \mu \in \mathbb{C}$, $\mu \neq 0, \xi \neq 0, \operatorname{Re} \lambda > 0$.

Suppose that $\frac{zq'(z)}{q(z)}$ *is a star-like univalent in* U. *Consider the inequality*

$$\operatorname{Re}\left\{1 + \frac{\beta}{\xi} + \frac{zq''(z)}{q'(z)}\right\} > 0 \tag{24}$$

and

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \xi; z) := \alpha + (\beta - \mu\xi)\left[\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right]^\mu + \tag{25}$$

$$+ \xi \mu \left({}^{AB}I_z^\lambda w_{\delta,b,c}(z)\right)' \left[\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right]^{\mu-1}.$$

If the following subordination holds for q

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \xi; z) \prec \alpha + \beta q(z) + \xi z q'(z) \tag{26}$$

then

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z} \right)^\mu \prec q(z), \quad z \in U, \, z \neq 0 \tag{27}$$

and q is the best dominant.

Proof. Define the analytic function p as follows

$$p(z) := \left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z} \right)^\mu, \quad z \in U, \, z \neq 0.$$

Differentiating with respect to z, one obtains

$$p'(z) = \mu \left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z} \right)^{\mu-1} \left[\frac{\left({}^{AB}I_z^\lambda w_{\delta,b,c}(z)\right)'}{z} - \frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z^2} \right]. \tag{28}$$

Considering $\phi(w) := \xi$ and $\theta(w) := \alpha + \beta w$, we can observe that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$, and also $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Setting

$$Q(z) = zq'(z)\phi(q(z)) = \xi z q'(z)$$

and

$$h(z) := \theta(q(z)) + Q(z) = \alpha + \beta q(z) + \xi z q'(z)$$

we obtain that $Q(z)$ is starlike univalent in U and

$$\mathrm{Re}\left\{ \frac{zh'(z)}{Q(z)} \right\} = \mathrm{Re}\left\{ 1 + \frac{\beta}{\xi} + \frac{zq''(z)}{q'(z)} \right\} > 0.$$

Thus, in view of Theorem 1, the assertion (27) of Theorem 3 holds. □

By a direct application of Theorem 3, we have the following results:

Corollary 5. *Consider $\alpha, \beta, \xi, \mu \in \mathbb{C}$, $\mu \neq 0$, $\xi \neq 0$, $\mathrm{Re}\,\lambda > 0$, $-1 \leq B < A \leq 1$, and*

$$\mathrm{Re}\left\{ 1 + \frac{\beta}{\xi} - \frac{2Bz}{1+Bz} \right\} > 0. \tag{29}$$

If

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \xi; z) \prec \alpha + \beta \frac{1+Az}{1+Bz} + \xi \frac{(A-B)z}{(1+Bz)^2} \tag{30}$$

where $\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \xi; z)$ is defined in (25), then

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z} \right)^\mu \prec \frac{1+Az}{1+Bz} \tag{31}$$

and $\dfrac{1+Az}{1+Bz}$ is the best dominant.

Corollary 6. *Consider $\alpha, \beta, \xi, \mu \in \mathbb{C}$, $\mu \neq 0$, $\xi \neq 0$, $\mathrm{Re}\,\lambda > 0$, $0 < \eta \leq 1$ and*

$$\mathrm{Re}\left\{ 1 + \frac{\beta}{\xi} + \frac{2z(\eta+z)}{1-z^2} \right\} > 0. \tag{32}$$

If

$$\psi_\lambda^{\delta,b,c}(\mu, \alpha, \beta, \xi; z) \prec \alpha + \beta \left(\frac{1+z}{1-z} \right)^\eta + \xi z \frac{(1+z)^{\eta-1}}{(1-z)^{\eta+1}} \tag{33}$$

where $\psi_\lambda^{\delta,b,c}(\mu,\alpha,\beta,\xi;z)$ is defined in (25), then

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec \left(\frac{1+z}{1-z}\right)^\eta \tag{34}$$

and $\left(\dfrac{1+z}{1-z}\right)^\eta$ is the best dominant.

Corollary 7. *Let* $A,\alpha,\beta,\xi,\mu \in \mathbb{C}$, $\mu \neq 0$, $\xi \neq 0$, $|\eta A| < \pi$, Re $\lambda > 0$, *and*

$$\operatorname{Re}\left(1 + \frac{\beta}{\xi} + z\eta A\right) > 0. \tag{35}$$

If

$$\psi_\lambda^{\delta,b,c}(\mu,\alpha,\beta,\xi;z) \prec \alpha + e^{\eta A z}(\beta + \xi A \eta z) \tag{36}$$

where $\psi_\lambda^{\delta,b,c}(\mu,\alpha,\beta,\xi;z)$ is defined in (25), then

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec e^{\eta A z} \tag{37}$$

and $e^{\eta A z}$ is the best dominant.

Corollary 8. *Let* $\alpha,\beta,\xi,\mu,\eta \in \mathbb{C}$, $\mu \neq 0$, $\eta \neq 0$, $\xi \neq 0$, Re $\lambda > 0$, $-1 \leq B < A \leq 1$, *and*

$$\operatorname{Re}\left\{1 + \frac{\beta}{\xi}\frac{z[\eta(A-B)-B]}{(1+Bz)}\right\} > 0.$$

If

$$\psi_\lambda^{\delta,b,c}(\mu,\alpha,\beta,\xi;z) \prec \alpha + \beta(1+Bz)^{\frac{\eta(A-B)}{B}} + \xi z\eta(A-B)(1+Bz)^{\frac{\eta(A-B)-B}{B}} \tag{38}$$

where $\psi_\lambda^{\delta,b,c}(\mu,\alpha,\beta,\xi;z)$ is defined in (25), then

$$\left(\frac{{}^{AB}I_z^\lambda w_{\delta,b,c}(z)}{z}\right)^\mu \prec (1+Bz)^{\frac{\eta(A-B)}{B}} \tag{39}$$

and $(1+Bz)^{\frac{\eta(A-B)}{B}}$ is the best dominant.

3. Discussion

Motivated by the results obtained by the first author related to the study of applying fractional integral operator on hypergeometric functions, we considered the study of a certain integral operator involving boththe Atangana–Baleanu integral operator and the Riemann–Liouville one. In the present paper, we provided new differential subordinations results based on an fractional integral operator. Moreover, using specific well-known univalent functions, we established certain statements that resulted in specific corollaries that provide the best dominants.

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