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A Parametric Generalization of the Baskakov-Schurer-Szász-Stancu Approximation Operators

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Abstract: In this paper, we introduce and investigate a new class of the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators, which considerably extends the well-known class of the classical Baskakov-Schurer-Szász-Stancu approximation operators. For this new class of approximation operators, we present a Korovkin type theorem and a Grüss-Voronovskaya type theorem, and also study the rate of its convergence. Moreover, we derive several results which are related to the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators in the weighted spaces. Finally, we prove some shape-preserving properties for the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators and, as a special case, we deduce the corresponding shape-preserving properties for the classical Baskakov-Schurer-Szász-Stancu approximation operators.

Keywords: approximation operators; parametric generalization; Baskakov-Schurer-Szász-Stancu operators; Korovkin type theorem; Voronovskaya type theorem; rate of convergence; Grüss-Voronovskaya type theorem; shape-preserving properties

MSC: Primary 40C15, 40G10, 41A36; Secondary 40A35

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1. Introduction

One of the most powerful theorems in the approximation theory is known as the *Weierstrass Approximation Theorem*, which states that any continuous function $f(x)$ defined on the closed interval $[a,b]$ can be approximated by an algebraic polynomial $P(x)$ with real coefficients for each $x \in [a,b]$.

The idea of finding concrete algebraic functions for better approximation has been studied extensively, and a number of polynomial operators have been used directly. The first results are given for the Bernstein operators, which were generalized by Szász [1] as follows:

$$S_n(f,x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f\left(\frac{j}{n}\right)$$

for $x \in [0,\infty)$. Baskakov [2] defined the following sequence of linear operators:

$$L_s(f,x) = \frac{1}{(1+x)^s} \sum_{r=0}^{\infty} \binom{s+r-1}{r} \frac{x^r}{(1+x)^r} f\left(\frac{x}{s}\right)$$

for $s \in \mathbb{N}$ and $x \in [0, \infty)$, $\mathbb{N}$ being the set of positive integers. Subsequently, Schurer [3] generalized the Bernstein operators in the following form:

$$L_{m,p}(f,x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p+k} f\left(\frac{k}{m}\right).$$

Stancu [4] defined the following sequence of operators:

$$S_s^{\alpha,\beta}(f,x) = \sum_{r=0}^{s} \binom{s}{r} x^r (1-x)^{s-r} f\left(\frac{r+\alpha}{s+\beta}\right)$$

for $0 \leqq \alpha \leqq \beta$. More recently, the following form of the Baskakov-Schurer-Szász-Stancu operators was introduced by Sofyalioglu and Kanat [5]:

$$M_{s,p}^{\alpha,\beta}(f;x) := (s+p) \sum_{r=0}^{\infty} \binom{s+p+r-1}{r} \frac{x^r}{(1+x)^{s+p+r}} \cdot \int_0^{\infty} s^{-(s+p)t} \frac{[(s+p)t]^r}{r!} f\left(\frac{(s+p)t+\alpha}{s+p+\beta}\right) \mathrm{d}t,$$

where s is a positive integer, p is a non-negative integer, and $0 \leqq \alpha \leqq \beta$.

2. The New Generalized Baskakov-Schurer-Szász-Stancu Operators

In this paper, we are interested in investigating a more generalized new class of operators, namely, the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators. We define these operators as follows:

$$M_{n,p}^{\gamma,\alpha,\beta}(f;x) = (n+p) \sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x) \int_0^{\infty} f\left(\frac{(n+p)t+\alpha}{n+p+\beta}\right) s_{n,p}^k(t)\, \mathrm{d}t \tag{1}$$

with

$$b_{n,p}^{k,\gamma}(x) = \left[\frac{\gamma x}{1+x}\binom{n+p+k-1}{k} - (1-\gamma)(1+x)\binom{n+p+k-3}{k-2} + (1-\gamma)x\binom{n+p+k-1}{k}\right] \frac{x^{k-1}}{(1+x)^{n+p+k-1}},$$

where $n, p, k \in \mathbb{N}$, $0 \leqq \alpha \leqq \beta$, $\gamma \in \mathbb{R}$ and $x \in [0, \infty)$.

Remark 1. *It is clearly seen that* $M_{n,p}^{1,\alpha,\beta}(f;x) = M_{n,p}^{\alpha,\beta}(f;x)$.

The aims of this paper are to first study the Korovkin type theorem, the Grüss-Voronovskaya type theorem, and the rate of the convergence for the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators. We then present some results related to the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators in the weighted spaces. Finally, in the last section, we give some preserving properties of the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators such as convexity.

3. Preliminary Results

By simple applications of the principle of mathematical induction, one can obtain Lemmas 1 and 2 below:

Lemma 1. *For all* $\ell \geqq 0$,

$$\int_0^\infty t^\ell s_{n,p}^k(t)\,\mathrm{d}t = \frac{\ell!\binom{k+\ell}{\ell}}{(n+p)^{\ell+1}}.$$

Proof. We proceed to the proof by the principle of mathematical induction on N given by

$$N := \ell + k \geqq 0.$$

First of all, for $N = 0$ ($\ell = k = 0$), we have

$$\int_0^\infty s_{n,p}^0(t)\,\mathrm{d}t = \int_0^\infty e^{-(n+p)t}\,\mathrm{d}t = \frac{1}{n+p},$$

as claimed in Lemma 1.

We now assume that the claimed result holds true for some N given by

$$N := \ell + k \geqq 0.$$

We then prove the claimed result for

$$N + 1 := (\ell + 1) + k.$$

Indeed, by partial integration, we have

$$\begin{aligned}\int_0^\infty t^{\ell+1} s_{n,p}^k(t)\,\mathrm{d}t &= \int_0^\infty t^{\ell+k+1} e^{-(n+p)t}\,\frac{(n+p)^k}{k!}\,\mathrm{d}t \\ &= -\int_0^\infty t^{\ell+k+1}\,\frac{\mathrm{d}}{\mathrm{d}t}\left\{e^{-(n+p)t}\,\frac{(n+p)^{k-1}}{k!}\right\}\mathrm{d}t \\ &= \frac{\ell+k+1}{n+p}\int_0^\infty t^{\ell+k}\,e^{-(n+p)t}\,\frac{(n+p)^k}{k!}\,\mathrm{d}t.\end{aligned}$$

Thus, by the induction hypothesis, we have

$$\int_0^\infty t^{\ell+1}\,s_{n,p}^k(t)\,\mathrm{d}t = \frac{(\ell+1)!\,\binom{k+\ell+1}{\ell+1}}{(n+p)^{\ell+2}},$$

which shows that the claimed result also holds true for $N + 1 = \ell + k + 1$. This evidently completes our proof of Lemma 1 by the principle of mathematical induction. □

Lemma 2. *For all* $m \geqq 0$,

$$\sum_{k=0}^\infty b_{n,p}^{k,\gamma}(x)\,\frac{\prod_{j=1}^m (k+j)}{(n+p)^{m+1}} = \frac{f_m - g_m + h_m}{(n+p)^{m+1}},$$

where f_m, g_m *and* h_m *are defined as follows*:

$$f_m = \sum_{k\geqq 0} \gamma \binom{n+p+k-1}{k} \prod_{j=1}^m (k+j)\,\frac{x^k}{(1+x)^{n+p+k}},$$

$$g_m = \sum_{k\geqq 2} (1-\gamma) \binom{n+p+k-3}{k-2} \prod_{j=1}^m (k+j)\,\frac{x^{k-1}}{(1+x)^{n+p+k-2}}$$

and

$$h_m = \sum_{k \geqq 0} (1-\gamma) \binom{n+p+k-1}{k} \prod_{j=1}^{m} (k+j) \frac{x^k}{(1+x)^{n+p+k-1}},$$

and they satisfy the following recurrence relations:

$$f_{m+1} = (m+1+x(n+p))f_m + x(1+x)\ \frac{\mathrm{d}}{\mathrm{d}x}\{f_m\},$$

$$g_{m+1} = (m+2+x(n+p-1))g_m + x(1+x)\ \frac{\mathrm{d}}{\mathrm{d}x}\{g_m\}$$

and

$$h_{m+1} = (m+1+x(n+p-1))h_m + x(1+x)\ \frac{\mathrm{d}}{\mathrm{d}x}\{h_m\},$$

with $f_0 = \gamma$, $g_0 = x(1-\gamma)$ *and* $h_0 = (1+x)(1-\gamma)$.

Proof. By using similar arguments as in the proof of Lemma 1, we can establish the result asserted by Lemma 2. We choose to skip the details involved. □

By means of Lemmas 1 and 2, and, by using the principle of mathematical induction on m, we are led to the following result.

Proposition 1. *For all* $m \geqq 0$,

$$\begin{aligned} f_m &= \sum_{k \geqq 0} \gamma \binom{n+p+k-1}{k} \prod_{j=1}^{m} (k+j) \frac{x^k}{(1+x)^{n+p+k}} \\ &= m!\ \gamma \sum_{j=0}^{m} \binom{m}{j} \binom{n+p-1+j}{j} x^j, \end{aligned}$$

$$\begin{aligned} g_m &= \sum_{k \geqq 2} (1-\gamma) \binom{n+p+k-3}{k-2} \prod_{j=1}^{m} (k+j) \frac{x^{k-1}}{(1+x)^{n+p+k-2}} \\ &= (1-\gamma) m!\ \sum_{j=0}^{m} \binom{m+2}{j+2} \binom{n+p-1+j}{j} x^{j+1} \end{aligned}$$

and

$$\begin{aligned} h_m &= \sum_{k \geqq 0} (1-\gamma) \binom{n+p+k-1}{k} \prod_{j=1}^{m} (k+j) \frac{x^k}{(1+x)^{n+p+k-1}} \\ &= (1-\gamma)(1+x) m!\ \sum_{j=0}^{m} \binom{m}{j} \binom{n+p-1+j}{j} x^j. \end{aligned}$$

Furthermore, for all $m \geqq 0$,

$$\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\ \frac{\prod_{j=1}^{m} (k+j)}{(n+p)^{m+1}} = \frac{m!\ \sum_{j=0}^{m} \left[\binom{m}{j} + (1-\gamma) \left(\binom{m}{j} - \binom{m+2}{j+2} \right) x \right] \binom{n+p-1+j}{j} x^j}{(n+p)^{m+1}}.$$

By (1), we have

$$M_{n,p}^{\gamma,\alpha,\beta}(t^\ell;x) = (n+p)\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^\infty \frac{((n+p)t+\alpha)^\ell}{(n+p+\beta)^\ell} s_{n,p}^k(t)\,\mathrm{d}t$$
$$= \sum_{j=0}^{\ell} \frac{(n+p)^{j+1}\alpha^{\ell-j}\binom{\ell}{j}}{(n+p+\beta)^\ell} \sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^\infty t^j s_{n,p}^k(t)\,\mathrm{d}t.$$

Thus, by Lemma 1, we obtain

$$M_{n,p}^{\gamma,\alpha,\beta}(t^\ell;x) = \sum_{j=0}^{\ell} \frac{\alpha^{\ell-j}}{(n+p+\beta)^\ell}\binom{\ell}{j}\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\binom{k+j}{j} j!,$$

which, by Lemma 2, implies that

$$M_{n,p}^{\gamma,\alpha,\beta}(t^\ell;x) = \sum_{j=0}^{\ell} \frac{\alpha^{\ell-j}}{(n+p+\beta)^\ell}\binom{\ell}{j}(f_j - g_j + h_j).$$

Hence, by applying the above Proposition, we can prove the following result.

Theorem 1. *For all $\ell \geqq 0$,*

$$M_{n,p}^{\gamma,\alpha,\beta}(e_\ell;x) = \frac{\sum\limits_{j=0}^{\ell} \alpha^{\ell-j} j!\binom{\ell}{j}\sum\limits_{i=0}^{j}\left(\binom{j}{i} + (1-\gamma)\left[\binom{j}{i} - \binom{j+2}{i+2}\right]x\right)\binom{n+p-1+i}{i}x^i}{(n+p+\beta)^\ell}.$$

For instance, Theorem 1 for $\ell = 0,1,2$ gives the following moments:

1. $M_{n,p}^{\gamma,\alpha,\beta}(e_0;x) = 1,$
2. $M_{n,p}^{\gamma,\alpha,\beta}(e_1;x) = \dfrac{\alpha+1}{n+p+\beta} + \dfrac{n+p+2\gamma-2}{n+p+\beta}\,x,$
3. $M_{n,p}^{\gamma,\alpha,\beta}(e_2;x) = \dfrac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \dfrac{2[(n+p+2\gamma)\alpha+2n+2p-2\alpha+5\gamma-5]}{(n+p+\beta)^2}\,x$
$+\dfrac{(n+p)(n+4\gamma+p-3)}{(n+p+\beta)^2}\,x^2.$

In what follows, we will prove the Korovkin type theorem for the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators. In the last several years, this subject is widely studied, and it is treated, among others, in the following references (see, for example, Refs. [6–20]). Some other related recent developments on this subject can be found in [21–24].

Theorem 2. *Let $(M_{n,p}^{\gamma,\alpha,\beta})$ be a sequence of positive linear operators defined on $C[0,R]$ for any finite R such that, for every $i \in \{0,1,2\}$,*

$$\lim_{n\to\infty} \left\| M_{n,p}^{\gamma,\alpha,\beta}(e_i;x) - e_i \right\| = 0, \tag{2}$$

where $e_i = x^i$. Then, for every $f \in C[0,R]$,

$$\lim_{n\to\infty} \left\| M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f \right\| = 0. \tag{3}$$

Proof. From Theorem 1, we have

$$\left\| M_{n,p}^{\gamma,\alpha,\beta}(e_0;x) - e_0 \right\| = 1 - 1 = 0,$$

$$\left\|M_{n,p}^{\gamma,\alpha,\beta}(e_1;x)-e_1\right\|=\left\|\frac{\alpha+1}{n+p+\beta}+\frac{n+p+2\gamma-2}{n+p+\beta}x-x\right\|=0$$

and

$$\begin{aligned}&\left\|M_{n,p}^{\gamma,\alpha,\beta}(e_2;x)-e_2\right\|\\&=\left\|\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2}+\frac{2((n+p+2\gamma)\alpha+2n+2p-2\alpha+5\gamma-5)}{(n+p+\beta)^2}x\right.\\&\left.+\frac{(n+p)(n+4\gamma+p-3)}{(n+p+\beta)^2}x^2-x^2\right\|=0.\end{aligned}$$

By means of the basic form of the Korovkin type theorem (see, for example, Ref. [25]), we complete the proof of Theorem 2. □

Lemma 3. *For all $\ell \geqq 0$,*

$$M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^\ell;x\right)=\sum_{j=0}^{\ell}\binom{\ell}{j}(-x)^{\ell-j}M_{n,p}^{\gamma,\alpha,\beta}(e_j;x).$$

Proof. Lemma 3 follows immediately from (1). □

Example 1. *By Theorem 1 and Lemma 3 for $\ell = 0, 1, 2$, we obtain*

$$M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^0;x\right)=1,$$

$$M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^1;x\right)=\frac{\alpha+1}{n+p+\beta}+\frac{2(\gamma-1)-\beta}{n+p+\beta}x$$

and

$$\begin{aligned}M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^2;x\right)=&\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2}+\frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}x\\&+\frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}x^2.\end{aligned}$$

Moreover, if we consider Lemma 3 for $\ell \leqq 6$ and $n \to \infty$, we obtain

$$\lim_{n\mapsto\infty} nM_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^1;x\right)=\alpha+1+(2\gamma-2-\beta)x,$$

$$\lim_{n\mapsto\infty} nM_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^2;x\right)=2x+x^2,$$

$$\lim_{n\mapsto\infty} n^2\, M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^3;x\right)=6(\alpha+2)x+3(\alpha-2\beta+4\gamma-1)x^2+(6\gamma-3\beta-4)x^3,$$

$$\lim_{n\mapsto\infty} n^2\, M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^4;x\right)=12x^2+12x^3+3x^4,$$

$$\begin{aligned}\lim_{n\mapsto\infty} n^3 M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^5;x\right)=&\,60(\alpha+3)x^2+60(\alpha-\beta+2\gamma+2)x^3\\&+5(3\alpha-12\beta+24\gamma-1)x^4+5(6\gamma-3\beta-2)x^5\end{aligned}$$

and

$$\lim_{n\mapsto\infty} n^3\, M_{n,p}^{\gamma,\alpha,\beta}\left((y-x)^6;x\right)=120x^3+180x^4+90x^5+15x^6.$$

4. Direct Estimates

With $B[0,\infty)$, $C[0,\infty)$, and $C_B([0,\infty))$, we will denote the space of all bounded functions, the space of all continuous functions, and the space of all continuous and bounded functions defined in the interval $[0,\infty)$, respectively, endowed with the norm given by

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

The modulus of continuity of the function $f \in C[0,\infty)$ is defined by

$$\omega(f;\delta) := \sup\{|f(x) - f(y)| : \ x, y \in [0,\infty) \quad \text{and} \quad |x-y| \leqq \delta\}.$$

It is known that, for any value of the $|x-y|$, we have

$$|f(x) - f(y)| \leqq \omega(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right).$$

Theorem 3. *Let $f \in C_B[0,\infty)$. Then, the following inequality for the operators (1) holds true:*

$$\begin{aligned}
&\|M_{n,p}^{\gamma,\alpha,\beta} f - f\| \\
&\leqq \omega(f;\sqrt{n})\left(1 + \frac{n+p}{\sqrt{n}}\left[1 - \frac{(1-\alpha)\big((n+p+1)x+1\big)}{(1+x)^{n+p}}\right]^{\frac{1}{2}}\right. \\
&\cdot\left[\frac{2\alpha+1+(8\alpha\gamma-8\alpha-2\beta+14\gamma-14)x-8\beta(\gamma-1)x^2}{4(n+p)(n+p+\beta)^2}\right. \\
&+\frac{4\alpha^2+(-8\alpha\beta+4\alpha+3)x+(4\beta^2-4\beta+2)x^2}{4(n+p+\beta)^2} \\
&\left.\left.+2x(4x\beta-x-4\alpha)(n+p)^2+\frac{x^2(n+p)^2}{(n+p+\beta)^2}\right]^{\frac{1}{2}}\right).
\end{aligned}$$

Proof. We know that operators $M_{n,p}^{\gamma,\alpha,\beta}$ are linear and positive. Let $f \in C_B[0,\infty)$. In view of the modulus of continuity, we have

$$\begin{aligned}
\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| &\leqq (n+p)\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^{\infty}\left|f\left(\frac{(n+p)t+\alpha}{n+p+\beta}\right) - f(x)\right| s_{n,p}^k(t)\,\mathrm{d}t \\
&\leqq \omega(f;\delta)\left(1 + \frac{1}{\delta}(n+p)\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^{\infty}\left|\frac{(n+p)t+\alpha}{n+p+\beta} - x\right| s_{n,p}^k(t)\,\mathrm{d}t\right). \qquad (4)
\end{aligned}$$

Let us set

$$B := \sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^{\infty}\left|\frac{(n+p)t+\alpha}{n+p+\beta} - x\right| s_{n,p}^k(t)\,\mathrm{d}t.$$

Then, by the Cauchy-Schwarz inequality, we get

$$B \leqq \left[\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\right]^{\frac{1}{2}} \cdot \left[\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^{\infty}\left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^2 \cdot \left[s_{n,p}^k(t)\right]^2\mathrm{d}t\right]^{\frac{1}{2}}. \qquad (5)$$

By direct calculations, we see that

$$\sum_{k\geqq 0} b_{n,p}^{k,\gamma}(x) = 1 - \frac{(1-\gamma)[(n+p+1)x+1]}{(1+x)^{n+p}}, \qquad (6)$$

and also that

$$A_0 = \int_0^\infty 1 \cdot \left[s_{n,p}^k(t)\right]^2 \,\mathrm{d}t = \frac{\binom{2k}{k}}{2^{2k+1}(n+p)},$$

$$A_1 = \int_0^\infty t \cdot \left[s_{n,p}^k(t)\right]^2 \,\mathrm{d}t = \frac{(2k+1)\binom{2k}{k}}{2^{2k+2}(n+p)^2}$$

and

$$A_2 = \int_0^\infty t^2 \cdot \left[s_{n,p}^k(t)\right]^2 \,\mathrm{d}t = \frac{(k+1)(2k+1)\binom{2k}{k}}{2^{2k+2}(n+p)^3}.$$

These last three equalities lead us to the following consequence:

$$\begin{aligned}
&\int_0^\infty \left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^2 \cdot \left[s_{n,p}^k(t)\right]^2 \,\mathrm{d}t \\
&\quad = \frac{(n+p)^2}{(n+p+\beta)^2} A_2 + 2\left(\frac{\alpha}{n+p+\beta} - x\right)\frac{n+p}{n+p+\beta} A_1 + \left(\frac{\alpha}{n+p+\beta} - x\right)^2 A_0.
\end{aligned}$$

Hence, in view of the positivity of $b_{n,p}^{k,\alpha}(x)$, if we use the following expression:

$$\int_0^\infty \left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^2 \cdot \left[s_{n,p}^k(t)\right]^2 \,\mathrm{d}t$$

together with the fact the

$$\binom{2k}{k} \leqq 2^{2k},$$

we obtain

$$\sum_{k=0}^\infty b_{n,p}^{k,\alpha}(x) \int_0^\infty \left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^2 \cdot \left[s_{n,p}^k(t)\right]^2 \,\mathrm{d}t \leqq U,$$

where

$$\begin{aligned}
U = {} & \frac{2\alpha + 1 + (8\alpha\gamma - 8\alpha - 2\beta + 14\gamma - 14)x - 8\beta(\gamma-1)x^2}{4(n+p)(n+p+\beta)^2} \\
& + \frac{4\alpha^2 + (-8\alpha\beta + 4\alpha + 3)x + (4\beta^2 - 4\beta + 2)x^2}{4(n+p+\beta)^2} \\
& + 2x(4x\beta - x - 4\alpha)(n+p)^2 + \frac{x^2(n+p)^2}{(n+p+\beta)^2}
\end{aligned}$$

From (6) and (5), we find that

$$B \leqq \sqrt{\left(1 - \frac{(1-\alpha)((n+p+1)x+1}{(1+x)^{n+p}}\right)U}.$$

Putting $\delta = \sqrt{n}$, we get the result asserted by Theorem 3. □

In what follows, we will give an upper bound for the sequence of the parametric generalization of the Baskakov-Schurer-Szász operators.

Theorem 4. *For any* $f \in C_B[0,\infty)$,

$$\left\| M_{n,p}^{\gamma,\alpha,\beta}(f;x) \right\| \leqq \|f\|_C.$$

Proof. From the definition of the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators in (1), we have

$$\begin{aligned} \left| M_{n,p}^{\gamma,\alpha,\beta}(f;x) \right| &\leqq \sup_{t \in \mathbb{R}^+} |f(t)| \cdot (n+p) \sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x) \int_0^{\infty} s_{n,p}^{k}(t)\, \mathrm{d}t \\ &= \sup_{t \in \mathbb{R}^+} |f(t)| \cdot M_{n,p}^{\gamma,\alpha,\beta}(e_0;x) = \|f\|_C, \end{aligned}$$

as asserted by Theorem 4. □

For $f \in C[0,\infty)$ and $\delta > 0$, the second-order modulus of smoothness of f is defined as follows:

$$w_2(f,\sqrt{\delta}) := \sup_{0<h\leqq\sqrt{\delta}} \ \sup_{x,x+h\in[0,\infty)} \{|f(x+2h) - 2f(x) + f(x-h)|\}.$$

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf \{ \|f-g\|_{C[0,\infty)} + \delta \|g''\|_{C[0,\infty)} : g \in W^2 \},$$

where $\delta > 0$ and

$$W^2 = \{g \in C[0,\infty) : g', g'' \in C[0,\infty)\}.$$

It is known that there exists a positive constant $\mathcal{C} > 0$ such that (see [26] (Theorem 3.1.2)),

$$K_2(f,\delta) \leqq \mathcal{C}\, w_2(f,\sqrt{\delta}) \qquad (\delta > 0).$$

Theorem 5. *Let* $f \in C[0,A]$ *for any finite real number* A*. Then,*

$$\left| M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x) \right| \leqq \frac{2}{A}\|f\| b^2 + \frac{3}{4}(A + b^2 + 2)\omega_2(f;b),$$

where

$$b = \sqrt[4]{M_{n,p}^{\gamma,\alpha,\beta}\big((s-x)^2;x\big)}.$$

Proof. Let f_S be the Steklov function of the second order for the function $f(x)$. Knowing that

$$M_{n,p}^{\gamma,\alpha,\beta}(e_0;x) = 1,$$

which follows from Theorem 1, we have

$$\begin{aligned} \left| M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x) \right| &\leqq \left| M_{n,p}^{\gamma,\alpha,\beta}(f - f_S;x) \right| + \left| M_{n,p}^{\gamma,\alpha,\beta}(f_S;x) - f_S(x) \right| + |f_S(x) - f(x)| \\ &\leqq 2\|f_S - f\| + \left| M_{n,p}^{\gamma,\alpha,\beta}(f_S;x) - f_S(x) \right|. \end{aligned} \tag{7}$$

Now, from the Lemmas in [27], we find that

$$\left| M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x) \right| \leqq \frac{3}{2}\omega_2(f;b) + \left| M_{n,p}^{\gamma,\alpha,\beta}(f_S;x) - f_S(x) \right|. \tag{8}$$

Knowing that $f_S \in C^2[0,A]$, and from the Lemmas in [17], we obtain

$$\left| M_{n,p}^{\gamma,\alpha,\beta}(f_S;x) - f_S(x) \right| \leqq \|f_S'\| \sqrt{M_{n,p}^{\gamma,\alpha,\beta}\big((s-x)^2;x\big)} + \frac{1}{2}\|f_S''\| M_{n,p}^{\gamma,\alpha,\beta}\big((s-x)^2;x\big).$$

The following inequality is valid (see [27]):

$$\|f_S''\| \leqq \frac{3}{2b^2}\omega_2(f;b). \tag{9}$$

In light of (8) and (9), (7) takes the following form:

$$\left|M_{n,p}^{\gamma,\alpha,\beta}(f_S;x) - f_S(x)\right| \leqq \|f_S'\| \sqrt{M_{n,p}^{\gamma,\alpha,\beta}((s-x)^2;x)} + \frac{3}{4b^2}\omega_2(f;b)M_{n,p}^{\gamma,\alpha,\beta}((s-x)^2;x).$$

From the relation (9) and the Landau inequality (see [28]), we get

$$\|f_S'\| \leqq \frac{2}{A}\|f\| + \frac{3A}{4b^2}\omega_2(f;b). \tag{10}$$

Using relations (9) and (10), and upon setting

$$b = \sqrt[4]{M_{n,p}^{\gamma,\alpha,\beta}((s-x)^2;x)},$$

we obtain

$$\left|M_{n,p}^{\gamma,\alpha,\beta}(f_S;x) - f_S(x)\right| \leqq \frac{2}{A}\|f\|b^2 + \frac{3}{4}(A+b^2)\omega_2(f;b).$$

Now, from relation (8), we complete the proof of Theorem 5. □

Let

$$C_B^2[0,\infty) = \{f \in C_B[0,\infty) : f', f'' \in C_B[0,\infty)\}$$

with the norm given by

$$\|f\|_{C_B^2} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$$

and the Peetre's K-functional given by (see [29])

$$K(f;\delta) = \{\|f-g\|_\infty + \delta\|g\|_{C_B^2}\}.$$

Theorem 6. *Let $f \in C_B[0,\infty)$. Then, the following inequality holds true:*

$$\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| \leqq A(n,p,\alpha,\beta,\gamma,x)\|f\|_{C_B^2},$$

for every $x \geqq 0$, where

$$\begin{aligned} A(n,p,\alpha,\beta,\gamma,x) = & \left[\frac{\alpha+1}{n+p+\beta} + \frac{2(\gamma-1)-\beta}{n+p+\beta}x\right] \\ & + \left[\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}x \right. \\ & \left. + \frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}x^2\right]. \end{aligned}$$

Proof. By using the Taylor formula and the linearity of the operators $M_{n,p}^{\gamma,\alpha,\beta}(f;x)$, we obtain

$$M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x) = M_{n,p}^{\gamma,\alpha,\beta}((t-x);x)f'(x) + \frac{1}{2}M_{n,p}^{\gamma,\alpha,\beta}((t-x)^2;x)f''(\varphi),$$

where $\varphi \in (x, t)$. In addition, from the above Example, we have

$$\begin{aligned}
\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| &= \|f'\| \cdot \left[\frac{\alpha+1}{n+p+\beta} + \frac{2(\gamma-1)-\beta}{n+p+\beta}x\right] \\
&\quad + \frac{\|f''\|}{2}\left[\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}x\right. \\
&\quad \left. + \frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}x^2\right] \\
&\leqq A(n,p,\alpha,\beta,\gamma,x)\|f\|_{C_B^2},
\end{aligned}$$

where

$$\begin{aligned}
A(n,p,\alpha,\beta,\gamma,x) &= \left[\frac{\alpha+1}{n+p+\beta} + \frac{2(\gamma-1)-\beta}{n+p+\beta}x\right] \\
&\quad + \left[\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}x\right. \\
&\quad \left. + \frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}x^2\right],
\end{aligned}$$

which proves Theorem 6. □

Theorem 7. *Let $f \in C[0,\infty)$. Then,*

$$\begin{aligned}
\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| \leqq 2\mathcal{M}\Bigg[\omega_2\left(f;\sqrt{\frac{1}{2}A(n,p,\alpha,\beta,\gamma,x)}\right) \\
+ \min\left\{1, \frac{1}{2}A(n,p,\alpha,\beta,\gamma,x)\right\}\|f\|_\infty\Bigg],
\end{aligned}$$

where $\mathcal{M}$ is a positive constant and $A(n,p,\alpha,\beta,\gamma,x)$ is defined as in Theorem 6.

Proof. From the linearity of the operator $M_{n,p}^{\gamma,\alpha,\beta}(f;x)$ and the following relation:

$$f(t) - f(x) = f(t) - g(t) + g(t) - g(x) + g(x) - g(t),$$

we obtain

$$\begin{aligned}
\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| &\leqq \left|M_{n,p}^{\gamma,\alpha,\beta}(f-g;x) - f(x)\right| \\
&\quad + \left|M_{n,p}^{\gamma,\alpha,\beta}(g;x) - g(x)\right| + |f(x) - g(x)|.
\end{aligned}$$

Now, from Theorems 4 and 6, and, by considering that $g \in C_B^2$, we get

$$\begin{aligned}
\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| &\leqq 2\|f-g\| + A(n,p,\alpha,\beta,\gamma,x)\|g\|_{C_B^2} \\
&= 2K\left(f; \frac{1}{2}A(n,p,\alpha,\beta,\gamma,x)\right).
\end{aligned}$$

It is known that

$$K(f;\delta) \leqq \mathcal{C}\left[\omega_2(f;\sqrt{\delta}) + \min\{1,\delta\}\|f\|_\infty\right],$$

where $\mathcal{C}$ is a positive constant, holds true for every $\delta > 0$ (see [26]). From the last two relations, we get the result asserted by Theorem 7. □

We will give the Voronovskaya type theorem for the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators.

Theorem 8. *For $f \in C_B[0,\infty)$, the following limit relation*:

$$\lim_{n\to\infty} n\left[M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right] = f'(x)[1+\alpha+(2\gamma-2-\beta)x] + \frac{f''(x)}{2}\left(2x+x^2\right),$$

holds true for every $x \in [0,\mathfrak{M}]$ and any finite $\mathfrak{M}$.

Proof. By Taylor's expansion theorem of the function f in $C_B[0,\infty)$, we obtain:

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + (t-x)^2\psi_x(t),$$

where

$$\psi_x(t) = \begin{cases} \dfrac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2 f''(x)}{(t-x)^2} & (x \neq t) \\ 0 & (x = t) \end{cases}$$

and the function $\psi_x(\cdot)$ is the Peano form of the remainder, $\psi_x(\cdot) \in C_B[0,\infty)$ and $\psi_x(t) \to 0$ as $t \to x$. Applying the operator $M_{n,p}^{\gamma,\alpha,\beta}$ on both sides of the above relation, we find that

$$nM_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x) = f'(x)nM_{n,p}^{\gamma,\alpha,\beta}((t-x);x) + \frac{f''(x)}{2}nM_{n,p}^{\gamma,\alpha,\beta}((t-x)^2;x) + nM_{n,p}^{\gamma,\alpha,\beta}((t-x)^2\psi_x(t);x).$$

In addition, from the above Example, we get

$$\lim_{n\to\infty} n\left[M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right] = f'(x)[1+\alpha+(2\gamma-2-\beta)x] + \frac{f''(x)}{2}\left(2x+x^2\right) + \lim_{n\to\infty} nM_{n,p}^{\gamma,\alpha,\beta}((t-x)^2\psi_x(s);x),$$

which, after applying the Cauchy-Schwarz inequality, yields

$$nM_{n,p}^{\gamma,\alpha,\beta}((t-x)^2\psi_x(s);x) \leqq \left\{n^2 M_{n,p}^{\gamma,\alpha,\beta}((t-x)^4;x)\right\}^{\frac{1}{2}}\left\{M_{n,p}^{\gamma,\alpha,\beta}(\psi_x^2(t);x)\right\}^{\frac{1}{2}}.$$

We now observe that $\psi_x^2(t) \to 0$ as $t \to x$ and $\psi_x^2(\cdot) \in C_B[0,\infty)$. Thus, from Theorem 2, it follows that

$$\left\{M_{n,p}^{\gamma,\alpha,\beta}(\psi_x^2(t);x)\right\}^{\frac{1}{2}} \to 0$$

as $n \to \infty$. Then, by using the last relations for every $x \in [0,M]$, we get

$$\lim_{n\to\infty} n\left[M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right] = f'(x)[1+\alpha+(2\gamma-2-\beta)x] + \frac{f''(x)}{2}\left(2x+x^2\right).$$

This completes the proof of Theorem 8. □

In what follows, we will give the Grüss-Voronovskaya type theorem (see [30]) for the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators.

Theorem 9. *Let $f', f'', g', g'' \in C_B[0,\infty)$. Then,*

$$\lim_{n\to\infty} n\left|M_{n,p}^{\gamma,\alpha,\beta}(fg;x) - M_{n,p}^{\gamma,\alpha,\beta}(f,x)M_{n,p}^{\gamma,\alpha,\beta}(g;x)\right| = (2x+x^2)f'(x)g'(x),$$

for each $x \in [0, \mathfrak{M}]$, where $\mathfrak{M}$ is finite.

Proof. After some calculations, we obtain

$$\begin{aligned}
& n\Big[M_{n,p}^{\gamma,\alpha,\beta}(fg;x) - M_{n,p}^{\gamma,\alpha,\beta}(f;x)M_{n,p}^{\gamma,\alpha,\beta}(g,x)\Big] \\
& = \left[n\Big(M_{n,p}^{\gamma,\alpha,\beta}(fg;x) - fg)\Big) - [1+\alpha+(2\gamma-2-\beta)x](fg)'(x) - (2x+x^2)\frac{(fg)''(x)}{2}\right] \\
& \quad - g(x)\left[n\Big(M_{n,p}^{\gamma,\alpha,\beta}(f,x) - f(x)\Big) - [1+\alpha+(2\gamma-2-\beta)x]f'(x) - (2x+x^2)\frac{f''(x)}{2}\right] \\
& \quad - M_{n,p}^{\gamma,\alpha,\beta}(f;x)\Bigg[n\Big(M_{n,p}^{\gamma,\alpha,\beta}(g;x) - g(x)\Big) - [1+\alpha+(2\gamma-2-\beta)x]g'(x) \\
& \qquad\qquad - (2x+x^2)\,\frac{g''(x)}{2}\Bigg] \\
& \quad + \Big(2x+x^2\Big)f'(x)g'(x) + \Big(2x+x^2\Big)\,\frac{g''(x)}{2}[f(x) - M_{n,p}^{\gamma,\alpha,\beta}(f;x)] \\
& \quad + [1+\alpha+(2\gamma-2-\beta)x]g'(x)[f(x) - M_{n,p}^{\gamma,\alpha,\beta}(f;x)].
\end{aligned}$$

The proof of Theorem 9 now follows from Theorem 8 and the above Example. □

The following results give light to the speed of the change between the difference of the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators and their derivatives, measured in terms of the modulus of continuity.

Theorem 10. *Let $\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'' \in C[0,\infty)$. Then,*

$$\begin{aligned}
& \Bigg|(n+p+\beta)\,[M_{n,p}^{\gamma,\alpha,\beta}(\mathfrak{h};x) - \mathfrak{h}(x)] - \mathfrak{h}'(x)[\alpha+1+(2(\gamma-1)-\beta)x] \\
& \qquad - \frac{\mathfrak{h}''(x)}{2}\left(\frac{\alpha^2+2\alpha+2}{n+p+\beta} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{n+p+\beta}x\right. \\
& \qquad \left. + (\beta^2-4\beta\gamma+n+p+4\beta)x^2\right)\Bigg| = O(1)\cdot\omega\left(\mathfrak{h}''; \frac{1}{\sqrt{n}}\right) \qquad (n\to\infty)
\end{aligned}$$

for every $x \in [0, M]$ for any finite M.

Proof. From the Taylor's theorem, we have

$$\mathfrak{h}(u) = \mathfrak{h}(x) + \mathfrak{h}'(x)(u-x) + \frac{\mathfrak{h}''(x)}{2}(u-x)^2 + R(u,x),$$

where

$$R(u,x) = \frac{\mathfrak{h}''(\theta) - \mathfrak{h}''(x)}{2}(u-x)^2$$

for $\theta \in (u,x)$. We thus find that

$$\begin{aligned}
& \left|M_{n,p}^{\gamma,\alpha,\beta}(\mathfrak{h};x) - \mathfrak{h}(x) - \mathfrak{h}'(x)M_{n,p}^{\gamma,\alpha,\beta}(u-x;x) - \frac{\mathfrak{h}''(x)}{2}M_{n,p}^{\gamma,\alpha,\beta}\left((u-x)^2;x\right)\right| \\
& \qquad \leqq M_{n,p}^{\gamma,\alpha,\beta}(|R(u,x)|;x).
\end{aligned}$$

From this last relation, we get

$$\Big|(n+p+\beta)\big[M_{n,p}^{\gamma,\alpha,\beta}(\mathfrak{h};x)-\mathfrak{h}(x)\big]-\mathfrak{h}'(x)\big[\alpha+1+\big(2(\gamma-1)-\beta\big)x\big]$$
$$-\frac{\mathfrak{h}''(x)}{2}\Bigg[\frac{\alpha^2+2\alpha+2}{n+p+\beta}+\frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{n+p+\beta}x$$
$$+(\beta^2-4\beta\gamma+n+p+4\beta)x^2\Bigg]\Bigg|\leqq(n+p+\beta)M_{n,p}^{\gamma,\alpha,\beta}(|R(u,x)|;x).$$

By the properties of the modulus of continuity, we have

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\leqq\frac{1}{2!}\left(1+\frac{|\theta-x|}{\delta}\right)\omega(\mathfrak{h}'';\delta).$$

On the other hand, it is easily seen that

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\leqq\begin{cases}\omega(\mathfrak{h}'';\delta) & (|u-x|\leqq\delta)\\[2mm] \dfrac{(t-x)^4}{\delta^4}\omega(\mathfrak{h}'';\delta) & (|u-x|\geqq\delta).\end{cases}$$

For $0<\delta<1$, we obtain that

$$\left|\frac{\mathfrak{h}''(\theta)-\mathfrak{h}''(x)}{2!}\right|\leqq\omega(\mathfrak{h}'';\delta)\left(1+\frac{(u-x)^4}{\delta^4}\right),$$

which yields

$$|R(u,x)|\leqq\omega(\mathfrak{h}'';\delta)\left(1+\frac{(u-x)^4}{\delta^4}\right)(u-x)^2=\omega(\mathfrak{h}'';\delta)\left((u-x)^2+\frac{(u-x)^6}{\delta^4}\right).$$

By the linearity of $M_{n,p}^{\gamma,\alpha,\beta}$ and the above relation, we obtain

$$M_{n,p}^{\gamma,\alpha,\beta}(|R(u,x)|;x)\leqq\omega(\mathfrak{h}'';\delta)\left(M_{n,p}^{\gamma,\alpha,\beta}((u-x)^2;x)+\frac{1}{\delta^4}M_{n,p}^{\gamma,\alpha,\beta}((u-x)^6;x)\right).$$

Now, in view of the above Example, for every $x\in[0,\mathfrak{M}]$, we have

$$M_{n,p}^{\gamma,\alpha,\beta}(|R(u,x)|;x)\leqq\omega(\mathfrak{h}'';\delta)\left[O\left(\frac{1}{n}\right)+\frac{1}{\delta^4}\,O\left(\frac{1}{n^3}\right)\right]=O\left(\frac{1}{n}\right)\omega(\mathfrak{h}'';\delta).$$

Thus, for

$$\delta=\frac{1}{\sqrt{n}},$$

we complete the proof of Theorem 10. □

The next result gives an estimation of the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators in the special Lipschitz-type space $\mathrm{Lip}_M^*\alpha$([1]), defined as follows:

$$\mathrm{Lip}_M^*(\alpha):=\left\{f\in C_B[0,\infty):|f(s)-f(x)|\leqq\mathcal{M}\frac{|s-x|^\alpha}{(x+s)^{\frac{\alpha}{2}}},\ x\in(0,\infty)\ \text{ and }\ s\in(0,\infty)\right\},$$

where $\mathcal{M}$ is a positive constant and $\alpha\in(0,1]$.

Theorem 11. *Let $f \in \mathrm{Lip}_M^*(\alpha)$. Then, for all $x, t \in (0, \infty)$, $n \in \mathbb{N}$ and $\alpha \in (0, 1]$,*

$$\begin{aligned}&\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| \\ &\leqq \frac{\mathcal{M}}{(x+t)^{\frac{\alpha}{2}}}\left(\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}x \right. \\ &\quad \left. + \frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}x^2\right)^{\frac{\alpha}{2}},\end{aligned}$$

where $\mathcal{M}$ is a positive constant.

Proof. Let $f \in \mathrm{Lip}_M^*(\alpha)$ and $\alpha \in (0, 1]$. We will distinguish between the following two cases.

I. For $\alpha = 1$, we have

$$\begin{aligned}&\left|M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right| \\ &\leqq \left|M_{n,p}^{\gamma,\alpha,\beta}(|f(t)-f(x)|;x)\right| \\ &\leqq \mathcal{M}\cdot M_{n,p}^{\gamma,\alpha,\beta}\left(\frac{|t-x|}{(x+t)^{\frac{1}{2}}};x\right) \\ &\leqq \frac{M}{(x+t)^{\frac{1}{2}}}M_{n,p}^{\gamma,\alpha,\beta}(|t-x|;x)\end{aligned}$$

for a positive constant $\mathcal{M}$.

If we apply the Cauchy-Schwarz inequality in the last expression, we get

$$\begin{aligned}&\left|M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right| \\ &\leqq \frac{\mathcal{M}}{(x+t)^{\frac{1}{2}}}\, M_{n,p}^{\gamma,\alpha,\beta}(|t-x|;x) \\ &\leqq \frac{\mathcal{M}}{(x+t)^{\frac{1}{2}}}\,\sqrt{M_{n,p}^{\gamma,\alpha,\beta}((t-x)^2;x)} \\ &= \frac{\mathcal{M}}{(x+t)^{\frac{1}{2}}}\,\left(\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}\,x \right. \\ &\quad \left. + \frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}\,x^2\right)^{\frac{1}{2}}.\end{aligned}$$

II. For $\alpha \in (0, 1)$, we have

$$\begin{aligned}&\left|M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right| \\ &\leqq \left|M_{n,p}^{\gamma,\alpha,\beta}(|f(t)-f(x)|;x)\right| \\ &\leqq \mathcal{M}\cdot M_{n,p}^{\gamma,\alpha,\beta}\left(\frac{|t-x|^{\alpha}}{(x+t)^{\frac{\alpha}{2}}};x\right) \\ &\leqq \frac{M}{(x+t)^{\frac{\alpha}{2}}}\, M_{n,p}^{\gamma,\alpha,\beta}(|t-x|^{\alpha};x).\end{aligned}$$

If we apply the Hölder inequality on the last relation under the conditions that

$$p = \frac{1}{\alpha}, q = \frac{1}{1-\alpha},$$

we obtain

$$\left|M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right| \leqq \frac{\mathcal{M}}{(x+t)^{\frac{\alpha}{2}}}\left[M_{n,p}^{\gamma,\alpha,\beta}(|t-x|;x)\right]^{\alpha}$$

for a positive constant $\mathcal{M}$. Thus, after applying the Cauchy-Schwarz inequality, obtain the following estimate:

$$\begin{aligned}
&\left|M_{n,p}^{\gamma,\alpha,\beta}(f(t);x) - f(x)\right| \\
&\quad \leqq \frac{\mathcal{M}}{(x+t)^{\frac{\alpha}{2}}}\left[\sqrt{M_{n,p}^{\gamma,\alpha,\beta}((t-x)^2;x)}\right]^{\alpha} \\
&\quad = \frac{\mathcal{M}}{(x+t)^{\frac{\alpha}{2}}}\left\{\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2(n+p-\alpha\beta+2\alpha\gamma-2\alpha-\beta+5\gamma-5)}{(n+p+\beta)^2}\,x \right. \\
&\qquad \left. + \frac{\beta^2-4\beta\gamma+n+p+4\beta}{(n+p+\beta)}\,x^2\right\}^{\frac{\alpha}{2}},
\end{aligned}$$

which completes the proof of Theorem 11. □

The Ditzian-Totik uniform modulus of smoothness of the first and the second orders are defined as follows (see [26]):

$$\omega_{\gamma}(f;\delta) := \sup_{0<|h|\leqq\delta} \ \sup_{x,x+h\gamma(x)\in[0,\infty)} \left\{\left|f(x+h\gamma(x)) - f(x)\right|\right\}$$

and

$$\omega_2^{\phi}(f;\delta) := \sup_{0<|h|\leqq\delta} \ \sup_{x,x\pm h\phi(x)\in[0,\infty)} \left\{\left|f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))\right|\right\},$$

respectively, where ϕ is an admissible step-weight function on $[a,b]$, that is,

$$\phi(x) = [(x-a)(b-x)]^{1/2}$$

if $x \in [a,b]$. The corresponding K-functional is defined as follows:

$$K_{2,\phi(x)}(f,\delta) = \inf_{g\in W^2(\phi)}\left\{\|f-g\|_{C[0,\infty)} + \delta\|\phi^2 g''\|_{C[0,\infty)}\right\},$$

where $\delta > 0$,

$$W^2(\phi) = \{g \in C_B[0,\infty) : g' \in AC[0,\infty),\ \phi^2 g'' \in C_B[0,\infty)\} \text{ and } g' \in AC[0,\infty)$$

means that g' is absolutely continuous on $[0,\infty)$. It is known that there exists an absolute constant $\mathfrak{C} > 0$ such that (see [26])

$$C^{-1}\omega_2^{\phi}(f;\sqrt{\delta}) \leqq K_{2,\phi(x)}(f,\delta) \leqq \mathfrak{C}\,\omega_2^{\phi}(f;\sqrt{\delta}). \tag{11}$$

Theorem 12. *Let $\Phi = \sqrt{x(1-x)}$ $(x \in [0,1])$ be a step-weight function of the Ditzian-Totik modulus of smoothness. Then, for any $f \in C_B[0,1]$ and $x \in [0,1]$, $n \in \mathbb{N}$ and $2\gamma < \beta + 2$,*

$$\left\| M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x) \right\| \leqq 4K_{2,\Phi(x)}\left(f, \frac{M_{n,p}^{\gamma,\alpha,\beta}\big((s-x)^2;x\big) + \alpha_1(n,p,\alpha,\beta)}{4\Phi^2(x)} \right) + \omega_\gamma\left(f; \frac{\alpha_1(n,p,\alpha,\beta)}{\gamma(x)} \right),$$

where

$$\alpha_1(n,p,\alpha,\beta) = \frac{\alpha+1}{n+p+\beta} + \frac{2(\gamma-1)-\beta}{n+p+\beta}.$$

Proof. Let

$$M_{n,p}^{\gamma,\alpha,\beta,*}(f;x) = M_{n,p}^{\gamma,\alpha,\beta}(f;x) + f(x) - f\big(x + \beta_1(n,p,\alpha,\beta,x)\big),$$

where

$$\beta_1(n,p,\alpha,\beta,x) = \frac{\alpha+1}{n+p+\beta} + \frac{2(\gamma-1)-\beta}{n+p+\beta}x.$$

We then observe that

$$M_{n,p}^{\gamma,\alpha,\beta,*}(1;x) = 1 \qquad \text{and} \qquad M_{n,p}^{\gamma,\alpha,\beta,*}\big((s-x);x\big) = 0.$$

Let $g \in W^2(\phi)$. Then, by using Taylor's expansion, we write

$$g(s) = g(x) + g'(x)(s-x) + \int_x^s (s-u)g''(u)\,\mathrm{d}u \qquad \big(s \in [0,\infty)\big),$$

which implies that

$$M_{n,p}^{\gamma,\alpha,\beta,*}(g;x) - g(x) = M_{n,p}^{\gamma,\alpha,\beta}\left(\int_x^s (s-u)g''(u)\,\mathrm{d}u; x \right) - \int_x^{x+\beta_1(n,p,\alpha,\beta,x)} [x + \beta_1(n,p,\alpha,\beta,x) - u]g''(u)\,\mathrm{d}u.$$

Therefore, we have

$$\begin{aligned}
&\left| M_{n,p}^{\gamma,\alpha,\beta,*}(g;x) - g(x) \right| \\
&\quad \leqq M_{n,p}^{\gamma,\alpha,\beta}\left(\left| \int_x^s (s-u)g''(u)\,\mathrm{d}u \right|; x \right) \\
&\qquad + \int_x^{x+\beta_1(n,p,\alpha,\beta,x)} |x + \beta_1(n,p,\alpha,\beta,x) - u| \cdot |g''(u)|\,\mathrm{d}u \\
&\quad \leqq \left\| \phi^2 g''(x)\, M_{n,p}^{\gamma,\alpha,\beta}\left(\left| \int_x^s \frac{|s-u|}{\phi^2(u)}\,\mathrm{d}u \right|; x \right) + \left\| \phi^2 g''(x) \right\| \right\| \\
&\qquad \cdot \left| \int_x^{x+\beta_1(n,p,\alpha,\beta,x)} \frac{|x + \beta_1(n,p,\alpha,\beta,x) - u|}{\phi^2(u)}\,\mathrm{d}u \right|.
\end{aligned}$$

Let $u = \rho x + (1-\rho)s$ $(\rho \in [0,1])$. Since ϕ^2 is concave on $[0,\infty)$, it follows that $\phi^2(u) \geq \rho\phi^2(x) + (1-\rho)\phi^2(s)$ and hence

$$\frac{|s-u|}{\phi^2(u)} = \frac{\rho|x-s|}{\phi^2(u)} \leqq \frac{\rho|x-s|}{\rho\phi^2(x) + (1-\rho)\phi^2(s)} \leqq \frac{|x-s|}{\phi^2(x)}.$$

We thus obtain

$$\left\|M_{n,p}^{\gamma,\alpha,\beta,*}(g)-g\right\| \leqq \frac{\|\phi^2 g''\|_{C[0,\infty)}}{\phi^2(x)}\left(\left[M_{n,p}^{\gamma,\alpha,\beta}\left((s-x)^2;x\right)\right]+x\beta_1(n,p,\alpha,\beta,x)\right).$$

From the above relations, we obtain

$$\begin{aligned}
&\left\|M_{n,p}^{\gamma,\alpha,\beta,*}(f,x)-f(x)\right\| \\
&\quad\leqq \left\|M_{n,p}^{\gamma,\alpha,\beta,*}(f-g)\right\|+\left\|M_{n,p}^{\gamma,\alpha,\beta,*}(g)-g\right\|+\|f-g\|+\|f(x+\beta_1(n,p,\alpha,\beta,x))-f(x)\| \\
&\quad\leqq 4\|f-g\|+\frac{\|\phi^2 g''\|}{\phi^2(x)}\left[M_{n,p}^{\gamma,\alpha,\beta}((s-x)^2;x)+x\beta_1(n,p,\alpha,\beta,x)\right] \\
&\qquad\quad+\|f(x+\beta_1(n,p,\alpha,\beta,x))-f(x)\|.
\end{aligned}$$

We know that

$$\begin{aligned}
\|f(x+\beta_1(n,p,\alpha,\beta,x))-f(x)\| &\leqq \left\|f\left(x+\gamma(x)\frac{M_{n,p}^{\gamma,\alpha,\beta}\left((s-x);x\right)}{\gamma(x)}\right)-f(x)\right\| \\
&\leqq \omega_\gamma\left(f;\frac{\beta_1(n,p,\alpha,\beta,x)}{\gamma(x)}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\|M_{n,p}^{\gamma,\alpha,\beta}(f,x)-f(x)\right\| \leqq 4K_{2,\Phi(x)}&\left(f,\frac{M_{n,p}^{\gamma,\alpha,\beta}((s-x)^2;x)+x\beta_1(n,p,\alpha,\beta,x)}{4\Phi^2(x)}\right) \\
&+\omega_\gamma\left(f;\frac{\beta_1(n,p,\alpha,\beta,x)}{\gamma(x)}\right). \qquad (12)
\end{aligned}$$

From the conditions given in Theorem 12, the properties of the K-functional, and the modulus of continuity, we get

1.
$$\frac{\alpha+1}{n+p+\beta}+\frac{2(\gamma-1)-\beta}{n+p+\beta}x \leqq \frac{\alpha+1}{n+p+\beta}+\frac{2(\gamma-1)-\beta}{n+p+\beta},$$

2.
$$\begin{aligned}
&K_{2,\Phi(x)}\left(f,\frac{M_{n,p}^{\gamma,\alpha,\beta}\left((s-x)^2;x\right)+x\beta_1(n,p,\alpha,\beta,x)}{4\Phi^2(x)}\right) \\
&\qquad\leqq K_{2,\Phi(x)}\left(f,\frac{M_{n,p}^{\gamma,\alpha,\beta}\left((s-x)^2;x\right)+\alpha_1(n,p,\alpha,\beta)}{4\Phi^2(x)}\right),
\end{aligned}$$

3.
$$\omega_\gamma\left(f;\frac{\beta_1(n,p,\alpha,\beta,x)}{\gamma(x)}\right) \leqq \omega_\gamma\left(f;\frac{\alpha_1(n,p,\alpha,\beta)}{\gamma(x)}\right)$$

for every $x \in [0,1]$. Combining the relation (12) and the other preceding relations, we obtain

$$\begin{aligned}
\left\|M_{n,p}^{\gamma,\alpha,\beta}(f;x)-f(x)\right\| \leqq 4K_{2,\Phi(x)}&\left(f,\frac{M_{n,p}^{\gamma,\alpha,\beta}\left((s-x)^2;x\right)+\alpha_1(n,p,\alpha,\beta)}{4\Phi^2(x)}\right) \\
&+\omega_\gamma\left(f;\frac{\alpha_1(n,p,\alpha,\beta)}{\gamma(x)}\right),
\end{aligned}$$

as asserted by Theorem 12. □

5. Weighted Approximation

Let $\rho(x) = x^2 + 1$ be the weight function and let $\mathcal{M}_f$ be a positive constant. We define the weighted space of functions as follows:

(i) $B_\rho[0, \infty)$ is the space of functions f defined on $[0, \infty)$ and satisfying $|f(x)| \leqq M_f \rho(x)$.
(ii) $C_\rho[0, \infty)$ is the subspace of all continuous functions in $B_\rho[0, \infty)$.
(iii) $C^*_\rho[0, \infty)$ is the subspace of functions $f \in C_\rho[0, \infty)$ for which $\frac{f(x)}{\rho(x)}$ is convergent as $x \to \infty$.

We note that the space $B_\rho[0, \infty)$ is a normed linear space with the norm given by

$$\|f\|_\rho = \sup_{x \geqq 0} \frac{|f(x)|}{\rho(x)}.$$

In order to calculate the rate of convergence, we consider the weighted modulus of continuity $\Omega(f; \delta)$ defined on infinite interval $[0, \infty)$ as

$$\Omega(f; \delta) = \sup_{x \geqq 0;\, 0<|h| \leqq \delta} \frac{|f(x+h) - f(x)|}{(1+h^2)\rho(x)} \qquad \left(\forall\, f \in C^*_\rho[0, \infty)\right).$$

For any $\mu \in [0, \infty)$, the weighted modulus of continuity $\Omega(f; \delta)$ verifies the following inequality:

$$\Omega(f; \mu\delta) \leqq 2(1+\mu)(1+\delta^2)\Omega(f; \delta),$$

and, for every $f \in C^*_\rho[0, \infty)$, we get

$$|f(t) - f(x)| \leqq 2\left(\frac{|t-x|}{\delta} + 1\right)(1+\delta^2)\Omega(f; \delta)(1+x^2)(1+(t-x)^2).$$

Theorem 13. *Let $\rho(x)$ be a weight function on $[0, \infty)$. Then, for each function $f \in C^*_\rho[0, \infty)$,*

$$\lim_{n \to \infty} \left\| M^{\gamma,\alpha,\beta}_{n,p}(f; x) - f(x) \right\|_\rho = 0.$$

Proof. It suffices to check that $M^{\gamma,\alpha,\beta}_{n,p}(e_i; x)$ converges uniformly to e_i, for $i \in \{0, 1, 2\}$, as n tends to ∞ and applies the well-known weighted Korovkin type theorem, where $e_i(x) = x^i$. The uniform convergence arises from the fact that

$$\lim_n \left\| M^{\gamma,\alpha,\beta}_{n,p} e_i - e_i \right\|_\rho = 0 \qquad (i = 0, 1, 2).$$

Using Theorem 1, the result for $i = 0$ is trivial.

We now prove that the results are true for $i = 1$ and $i = 2$, respectively. Indeed, for $f \in C^*_\rho[0, \infty)$, we obtain

$$\left\| M^{\gamma,\alpha,\beta}_{n,p} e_1 - e_1 \right\|_\rho = \sup_{x \geqq 0} \left\{ \frac{|M^{\gamma,\alpha,\beta}_{n,p} e_1 - e_1|}{\rho(x)} \right\} \leqq \sup_{x \geqq 0} \frac{|\gamma_1(n, p, \alpha, \beta, x)|}{\rho(x)}.$$

By a similar consideration, we have

$$\left\| M^{\gamma,\alpha,\beta}_{n,p} e_2 - e_2 \right\|_\rho = \sup_{x \geqq 0} \left\{ \frac{|M^{\gamma,\alpha,\beta}_{n,p} e_2 - e_2|}{\rho(x)} \right\} \leqq \sup_{x \geqq 0} \left\{ \frac{|\gamma_2(n, p, \alpha, \beta, x)|}{\rho(x)} \right\},$$

where

$$\gamma_1(n,p,\alpha,\beta,x) = \frac{\alpha+1}{n+p+\beta} + \frac{n+p+2\gamma-2}{n+p+\beta}x - x$$

and

$$\gamma_2(n,p,\alpha,\beta,x) = \frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2} + \frac{2((n+p+2\gamma)\alpha+2n+2p-2\alpha+5\gamma-5)}{(n+p+\beta)^2}\,x + \frac{(n+p)(n+4\gamma+p-3)}{(n+p+\beta)^2}\,x^2 - x^2.$$

We thus conclude that

$$\lim_{n\to\infty} \left\| M_{n,p}^{\gamma,\alpha,\beta} e_i - e_i \right\|_\rho = 0 \qquad (i = 0,1,2),$$

which completes the proof of Theorem 13. □

Theorem 14. *Let $f \in C_\rho^*[0,\infty)$. Then, the following inequality holds true*:

$$\sup_{x\in[0,\infty)} \frac{\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right|}{(1+x^2)(1+\mathcal{C}x+\mathcal{D}x^2+\mathcal{E}x^3+\mathcal{F}x^4)} \leqq \mathcal{K}\Omega\Big(f; n^{-\frac{1}{4}}\Big)$$

for a sufficiently large n, where $\mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are positive constants dependent only on n, p, α, β and γ, and $\mathcal{K}$ is a positive constant.

Proof. For $x \in [0,\infty)$, we have

$$M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x) = (n+p)\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)\int_0^\infty \left| f\left(\frac{(n+p)t+\alpha}{n+p+\beta}\right) - f(x)\right| s_{n,p}^k(t)\;\mathrm{d}t.$$

Using the properties of the weighted modulus, we obtain

$$\begin{aligned}&\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)\right| \\ &\quad\leqq (n+p)\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x) 2(1+\delta_n^2)\Omega(f;\delta_n)(1+x^2) \\ &\quad\cdot \int_0^\infty \left(\frac{\left|\frac{(n+p)t+\alpha}{n+p+\beta} - x\right|}{\delta_n} + 1\right)\left[1 + \left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^2\right] s_{n,p}^k(t)\;\mathrm{d}t.\end{aligned}$$

Let us define

$$S(t,x) = \left(\frac{\left|\frac{(n+p)t+\alpha}{n+p+\beta} - x\right|}{\delta_n} + 1\right)\left[1 + \left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^2\right] s_{n,p}^k(t).$$

Since $s_{n,p}^k(t) > 0$ for every $t \in (0,\infty)$, we have

$$S(t,x) \leqq \begin{cases} 2(1+\delta_n^2) s_{n,p}^k(t) & \left(\left|\dfrac{(n+p)t+\alpha}{n+p+\beta} - x\right| \leqq \delta_n\right) \\[2ex] 2(1+\delta_n^2)\dfrac{\left(\dfrac{(n+p)t+\alpha}{n+p+\beta} - x\right)^4}{\delta_n^4} s_{n,p}^k(t) & \left(\left|\dfrac{(n+p)t+\alpha}{n+p+\beta} - x\right| \geqq \delta_n\right), \end{cases}$$

which implies that

$$S(t,x) \leqq 2(1+\delta_n^2)\left(1+\frac{\left(\frac{(n+p)t+\alpha}{n+p+\beta}-x\right)^4}{\delta_n^4}\right)s_{n,p}^k(t).$$

Thus, clearly, we get

$$\left|M_{n,p}^{\gamma,\alpha,\beta}(f;x)-f(x)\right| \leqq 4(n+p)\sum_{k=0}^{\infty} b_{n,p}^{k,\gamma}(x)(1+\delta_n^2)\Omega(f;\delta_n)(1+x^2)$$
$$\cdot\int_0^\infty \left(1+\delta_n^2\right)\left(1+\frac{\left(\frac{(n+p)t+\alpha}{n+p+\beta}-x\right)^4}{\delta_n^4}\right)s_{n,p}^k(t)\;\mathrm{d}t.$$

By Lemma 1, we have

$$\begin{aligned}\int_0^\infty &\left(1+\frac{\left(\frac{(n+p)t+\alpha}{n+p+\beta}-x\right)^4}{\delta_n^4}\right)s_{n,p}^k(t)\;\mathrm{d}t\\ &=\int_0^\infty s_{n,p}^k(t)\;\mathrm{d}t+\frac{1}{\delta_n^4(n+p+\beta)^4}\sum_{j=0}^{4}\binom{4}{j}(n+p)^j\\ &\qquad\cdot(\alpha-(n+p+\beta)x)^{4-j}\int_0^\infty t^j s_{n,p}^k(t)\;\mathrm{d}t\\ &=\frac{1}{n+p}+\frac{1}{\delta_n^4(n+p+\beta)^4(n+p)}\sum_{j=0}^{4}(\alpha-(n+p+\beta)x)^{4-j}\;j!\binom{4}{j}\binom{k+j}{j}.\end{aligned}$$

Thus, by using the above Proposition, we have

$$\begin{aligned}\sum_{k=0}^{\infty} b_{n,p}^{k,\alpha}(x)&\int_0^\infty\left(1+\frac{\left(\frac{(n+p)t+\alpha}{n+p+\beta}-x\right)^4}{\delta_n^4}\right)s_{n,p}^k(t)\;\mathrm{d}t\\ &=\frac{1}{n+p}+\sum_{j=0}^{4}\sum_{i=0}^{j}\frac{24(\alpha-(n+p+\beta)x)^{4-j}}{(4-j)!\;\delta_n^4(n+p+\beta)^4(n+p)}\\ &\qquad\cdot\left\{\binom{j}{i}+(1-\gamma)\left[\binom{j}{i}-\binom{j+2}{i+2}\right]x\right\}\binom{n+p-1+i}{i}x^i\\ &=\frac{1}{n+p}+\frac{3x^2(x+2)^2(n+p)-2xA+\frac{B}{n+p}}{(n+p+\beta)^4\delta_n^4},\end{aligned}$$

where

$$\begin{aligned}A=&-6\alpha^2-24\alpha-36+(-3\alpha^2+12\alpha\beta-24\alpha\gamma\\ &+6\alpha+24\beta-84\gamma+48)x+(6\alpha\beta-12\alpha\gamma-6\beta^2+24\beta\gamma+8\alpha-6\beta-54\gamma+38)x^2\\ &+(-3\beta^2+12\beta\gamma-8\beta-8\gamma+5)x^3\end{aligned}$$

and

$$
\begin{aligned}
B = {} & \alpha^4 + 4\alpha^3 + 12\alpha^2 + 24\alpha + 24 + (-4\alpha^3\beta + 8\alpha^3\gamma - 8\alpha^3 - 12\alpha^2\beta + 60\alpha^2\gamma - 60\alpha^2 \\
& - 24\alpha\beta + 216\alpha\gamma - 216\alpha - 24\beta + 336\gamma - 336)x \\
& + [-6\beta(-\alpha^2\beta + 4\alpha^2\gamma - 4\alpha^2 - 2\alpha\beta + 20\alpha\gamma - 20\alpha - 2\beta + 36\gamma - 36)]x^2 \\
& + 4\beta^2(-\alpha\beta + 6\alpha\gamma - 6\alpha - \beta + 15\gamma - 15)x^3 + [-\beta^3(-\beta + 8\gamma - 8)]x^4.
\end{aligned}
$$

From the above relation, we obtain

$$
\begin{aligned}
& |M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)| \\
& \quad \leqq 4(n+p)(1+\delta_n^2)^2\Omega(f;\delta_n)(1+x^2) \\
& \quad\quad \cdot \sum_{k=0}^{\infty} b_{n,p}^{k,\alpha}(x) \int_0^{\infty} \left(1 + \frac{\left(\frac{(n+p)t+\alpha}{n+p+\beta} - x\right)^4}{\delta_n^4}\right) s_{n,p}^k(t)\,\mathrm{d}t \\
& \quad = 4(n+p)(1+\delta_n^2)^2\Omega(f;\delta_n)(1+x^2) \\
& \quad\quad \cdot \left(\frac{1}{n+p} + \frac{3x^2(x+2)^2(n+p) - 2xA + \frac{B}{n+p}}{(n+p+\beta)^4\delta_n^4}\right).
\end{aligned}
$$

In addition, for $\delta_n = n^{-\frac{1}{4}}$, we have

$$
\begin{aligned}
& |M_{n,p}^{\gamma,\alpha,\beta}(f;x) - f(x)| \\
& \quad \leqq \mathcal{K}\Omega(f;\delta_n)(1+x^2)(1 + \mathcal{C}x + \mathcal{D}x^2 + \mathcal{E}x^3 + \mathcal{F}x^4),
\end{aligned}
$$

where $\mathcal{C}, \mathcal{D}, \mathcal{E}$, and $\mathcal{F}$ are positive constants depending only on n, p, α, β, and γ, and $\mathcal{K}$ is a positive constant. This proves Theorem 14. □

6. Shape-Preserving Properties

In this section, we will present some shape-preserving properties by proving that the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators preserves the convexity under certain conditions.

Theorem 15. *Let $f \in C[0,\infty)$. If $f(x)$ is convex on $[0,\infty)$ and $n + p + 4\gamma > 3$ for $\alpha, \beta, \gamma \in \mathbb{R}$, then, the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators are also convex.*

Proof. Let us suppose that $f(x)$ is convex and that x_0 and x_1 are distinct points in the interval $[x, y]$, where $x < x_0 < x_1 < y$ and $x, y \in [a, b] \subset [0, \infty)$. Then, the Lagrangian interpolation polynomial through the points $\big(x_0, f(x_0)\big)$ and $\big(x_1, f(x_1)\big)$ is given by

$$
P(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).
$$

Then, based upon Theorem 1, we have

$$
\begin{aligned}
\left[M_{n,p}^{\gamma,\alpha,\beta}(P;x)\right]'' &= \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\left(\frac{\alpha+1}{n+p+\beta} + \frac{n+p+2\gamma-2}{n+p+\beta}\,x\right) + \frac{x_1 f(x_0) - x_0 f(x_1)}{x_1 - x_0}\right]'' \\
&= 0.
\end{aligned}
$$

On the other hand, we have

$$M_{n,p}^{\gamma,\alpha,\beta}(f;x) = M_{n,p}^{\gamma,\alpha,\beta}(P;x) + \frac{f''(\xi_t)}{2!}\left[M_{n,p}^{\gamma,\alpha,\beta}(t^2;x) - (x_0+x_1)M_{n,p}^{\gamma,\alpha,\beta}(t,x) + x_0x_1\right]$$
$$= M_{n,p}^{\gamma,\alpha,\beta}(P;x) + \frac{f''(\xi_t)}{2!}\left[\frac{\alpha^2+2\alpha+2}{(n+p+\beta)^2}\right.$$
$$+ \frac{2[(n+p+2\gamma)\alpha+2n+2p-2\alpha+5\gamma-5]}{(n+p+\beta)^2}x + \frac{(n+p)(n+4\gamma+p-3)}{(n+p+\beta)^2}x^2$$
$$\left. - (x_0+x_1)\left(\frac{\alpha+1}{n+p+\beta} + \frac{n+p+2\gamma-2}{n+p+\beta}x\right) + x_0x_1\right].$$

From this last relation, we find that

$$\left[M_{n,p}^{\gamma,\alpha,\beta}(f;x)\right]'' = f''(\xi_t)\cdot\frac{(n+p)(n+4\gamma+p-3)}{(n+p+\beta)^2} > 0$$

under the given conditions. This completes the proof of Theorem 15. □

Corollary. *The classical Baskakov-Schurer-Szász-Stancu operators preserve the property of convexity.*

Proof. We know that, for $\gamma = 1$, we are led to the Baskakov-Schurer-Szász-Stancu operators from the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators. Since $n, p \in \mathbb{N}$, in the special case when $\gamma = 1$, we have $n + p + 4\gamma > 3$. The proof now follows from Theorem 15. □

7. Concluding Remarks and Observations

In our present investigation, we have introduced, and systematically studied the properties and relations associated with, a new class of the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators. Our findings have considerably and significantly extended the well-known family of the classical Baskakov-Schurer-Szász-Stancu approximation operators. For our new class of the Baskakov-Schurer-Szász-Stancu approximation operators, we have established a Korovkin type theorem and a Grüss-Voronovskaya type theorem. We have also studied the rate of its convergence. Moreover, we have proved several results which are related to the parametric generalization of the Baskakov-Schurer-Szász-Stancu operators in the weighted spaces. Finally, we have derived a number of shape-preserving properties for the parametric generalization of the Baskakov-Schurer-Szász-Stancu approximation operators. We have also appropriately specialized our results in order to deduce the corresponding shape-preserving properties for the classical Baskakov-Schurer-Szász-Stancu approximation operators.

The various results and their consequences, which we have presented in this article, will potentially motivate and encourage further researches on the subject dealing with the parametric generalization of the Baskakov-Schurer-Szász-Stancu approximation operators.

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References

1. Szász, O. Generalization of S. Bernstein's polynomials to the infinite interval. *J. Res. Nat. Bur. Stand.* **1950**, *45*, 239–245. [CrossRef]
2. Baskakov, V.A. An instance of a sequence of linear positive operators in the space of continuous functions. *Dokl. Akad. Nauk SSSR* **1957**, *113*, 249–251.
3. Schurer, F. *Linear Positive Operators in Approximation Theory*; Report of the Mathematcal Institute of the Technical University of Delft: Delft, The Netherlands, 1962.
4. Stancu, D.D. Approximation of function by means of a new generalized Bernstein operator. *Calcolo* **1983**, *20*, 211–229. [CrossRef]
5. Sofyalioglu, M.; Kanat, K. Approximation properties of generalized Baskakov-Schurer-Szász–Stancu operators preserving e^{-2ax}, $a > 0$. *J. Inequal. Appl.* **2019**, *2019*, 112. [CrossRef]
6. Atlihan, O.G.; Unver, M.; Duman, O. Korovkin theorems on weighted spaces: Revisited. *Period. Math. Hungar* **2017**, *75*, 201–209. [CrossRef]
7. Braha, N.L. Some weighted equi-statistical convergence and Korovkin type-theorem. *Results Math.* **2016**, *70*, 433–446. [CrossRef]
8. Braha, N.L. Some properties of new modified Szász-Mirakyan operators in polynomial weight spaces via power summability method. *Bull. Math. Anal. Appl.* **2018**, *10*, 53–65.
9. Braha, N.L. Some properties of Baskakov-Schurer-Szász operators via power summability methods. *Quaest. Math.* **2019**, *42*, 1411–1426. [CrossRef]
10. Braha, N.L. Korovkin type theorem for Bernstein-Kantorovich operators via power summability method. *Anal. Math. Phys.* **2020**, *10*, 62. [CrossRef]
11. Braha, N.L. Some properties of modified Szász-Mirakyan operators in polynomial spaces via the power summability method. *J. Appl. Anal.* **2020**, *26*, 79–90. [CrossRef]
12. Braha, N.L.; Loku, V. Korovkin type theorems and its applications via $\alpha\beta$-statistically convergence. *J. Math. Inequal.* **2020**, *14*, 951–966. [CrossRef]
13. Braha, N.L.; Loku, V.; Srivastava, H.M. Λ^2-Weighted statistical convergence and Korovkin and Voronovskaya type theorems. *Appl. Math. Comput.* **2015**, *266*, 675–686. [CrossRef]
14. Braha, N.L.; Mansour, T.; Mursaleen, M. Some properties of Kantorovich-Stancu-type generalization of Szász operators including Brenke-type polynomials via power series summability method. *J. Funct. Spaces* **2020**, *2020*, 3480607. [CrossRef]
15. Braha, N.L.; Mansour, T.; Mursaleen, M.; Acar, T. Some properties of λ-Bernstein operators via power summability method. *J. Appl. Math. Comput.* **2020**, *65*, 125–146. [CrossRef]
16. Braha, N.L.; Srivastava, H.M.; Mohiuddine, S.A. A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean. *Appl. Math. Comput.* **2014**, *228*, 162–169. [CrossRef]
17. Gavrea, I.; Raşa, I. Remarks on some quantitative Korovkin-type results. *Rev. Anal. Numér. Théor. Approx.* **1993**, *22*, 173–176.
18. Kadak, U.; Braha, N.L.; Srivastava, H.M. Statistical weighted $\mathcal{B}$-summability and its applications to approximation theorems. *Appl. Math. Comput.* **2017**, *302*, 80–96.
19. Loku, V.; Braha, N.L. Some weighted statistical convergence and Korovkin type-theorem. *J. Inequal. Spec. Funct.* **2017**, *8*, 139–150.
20. Mursaleen, M.; Al-Abied, A.A.H.; Ansari, K.J. On approximation properties of Baskakov-Schurer-Szász–Stancu operators based on q-integers. *Filomat* **2018**, *32*, 1359–1378. [CrossRef]
21. Braha, N.L.; Srivastava, H.M.; Et, M. Some weighted statistical convergence and associated Korovkin and Voronovskaya type theorems. *J. Appl. Math. Comput.* **2021**, *65*, 429–450. [CrossRef]
22. Gupta, V.; Acu, A.M.; Srivastava, H.M. Difference of some positive linear approximation operators for higher-order derivatives, *Symmetry* **2020**, *12*, 915. [CrossRef]
23. Srivastava, H.M.; Ícoz, G.; Çekim, B. Approximation properties of an extended family of the Szász-Mirakjan Beta-type operators, *Axioms* **2019**, *8*, 111. [CrossRef]
24. Srivastava, H.M.; Mursaleen, M.; Alotaibi, A.M.; Al-Abied, A.A.H. Some approximation results involving the q-Szász-Mirakjan-Kantorovich type operators via Dunkl's generalization. *Math. Methods Appl. Sci.* **2017**, *40*, 5437–5452. [CrossRef]
25. Altomare, F.; Campiti, M. *Korovkin-Type Approximation Theory and Its Application*; Walter de Gruyter Studies in Mathematics; De Gruyter & Company: Berlin, Germany, 1994; Volume 17.
26. Ditzian, Z.; Totik, V. *Moduli of Smoothness*; Springer-Verlag: Berlin/Heidelberg, Germany; New York, NY, USA, 1987.
27. Zhuk, V.V. Functions of the Lip 1 class and S. N. Bernstein's polynomials (Russian, with English summary). *Vestn. Leningr. Univ. Mat. Mekh. Astronom.* **1989**, *1989*, 25–30. 122–123; English translation: *Vestn. Leningr. Univ. Math.* **1989**, *22*, 38–44.
28. Landau, E. Einige Ungleichungen für zweímal differentzierban funktionen. *Proc. London Math. Soc.* **1913**, *13*, 43–49.
29. Peetre, J. Theory of interpolation of normed spaces. *Notas Mat. Rio Jan.* **1963**, *39*, 1–86.
30. Gal, S.G.; Gonska, H. Grüss and Grüss-Voronovskaya-type estimates for some Bernstein-type polynomials of real and complex variables. *Jaen J. Approx.* **2015**, *7*, 97–122.