Explicit Solutions of Initial Value Problems for Fractional Generalized Proportional Differential Equations with and without Impulses

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Abstract: The object of investigation in this paper is a scalar linear fractional differential equation with generalized proportional derivative of Riemann–Liouville type (LFDEGD). The main goal is the obtaining an explicit solution of the initial value problem of the studied equation. Note that the locally solvability, being the same as the existence of solutions to the initial value problem, is connected with the symmetry of a transformation of a system of differential equations. At the same time, several criteria for existence of the initial value problem for nonlinear fractional differential equations with generalized proportional derivative are connected with the linear ones. It leads to the necessity of obtaining an explicit solution of LFDEGD. In this paper two cases are studied: the case of no impulses in the differential equation are presented and the case when instantaneous impulses at initially given points are involved. All obtained formulas are based on the application of Mittag–Leffler function with two parameters. In the case of impulses, initially the appropriate impulsive conditions are set up and later the explicit solutions are obtained.

Keywords: generalized proportional fractional derivatives; Mittag–Leffler function

1. Introduction

Recently, fractional differential equations have appeared strongly in the diffusion process, the process of dynamics, signal and image processing, etc. It has been mainly due to the fact that the mathematical modeling of numerous processes and phenomena in the frame of the fractional operators is capable of tracing the previous effects of the concerned phenomena. For instance, see [1–6] and references therein.

In 2014, Khalil et al. [7] introduced an interesting derivative, called the conformable derivative. Later, many researchers argued that this derivative could not be considered as a fractional derivative because it has no memory property. This new definition seems to be a natural extension of the classical derivative. Unfortunately, this new definition has a point of weakness as it does not tend towards the original function when the order approaches zero. Anderson and Ulness [8,9] proposed a modified conformable derivative by utilizing proportional derivatives. Later, Jarad et al. [10] introduced a new generalized proportional derivative which is well-behaved and has several advantages over the classical derivatives such as meaning that it generalizes formerly known derivatives in the literature. For recent contributions relevant to fractional differential equations via generalized proportional derivatives, see [11–16].

One of the main problems in differential equations is the solvability. At the same time, the local solvability being the same as the existence of solutions to the initial value problem, it is connected with the symmetry of a transformation of a system of differential equations. This paper is the first work to give an explicit formula for the solutions of the initial value problem for scalar linear fractional differential equation with generalized

proportional fractional derivative in terms of the Mittag–Leffler function, which reflects the novelty of the work compared to the aforementioned contributions, which mainly depend on discussing the mild solution of the integral equations corresponding to the differential equation in question.

The rest of the paper is structured as follows: In Section 2, we recall some useful preliminaries and auxiliary results. In Section 3, a scalar linear generalized proportional fractional differential equation with an initial condition expressed by a generalized proportional fractional integral is defined. An explicit formula of the solutions of the studied initial value problem is obtained. In Section 4, a linear generalized proportional fractional differential equation with instantaneous impulses is discussed. Finally, in order to confirm the validity of the theoretical findings, two examples are given in Section 5.

2. Preliminaries and Auxiliary Results

We provide some basic definitions and properties of the fractional proportional derivative and integral (see for example, [10]).

Definition 1 ([8] *Amended conformable derivative*). *Let $\rho \in [0,1]$, $\jmath \subset \mathbb{R}$ and the functions $\kappa_0, \kappa_1 : [0,1] \times [a,b] \to [0,\infty)$ be continuous such that for all $t \in [a,b]$ we have $\lim_{\rho \to 0^+} \kappa_1(\rho,t) = 1$, $\lim_{\rho \to 0^+} \kappa_0(\rho,t) = 0$, $\lim_{\rho \to 1^-} \kappa_1(\rho,t) = 0$, $\lim_{\rho \to 1^-} \kappa_0(\rho,t) = 1$ and $\kappa_1(\rho,t) \neq 0$, $\rho \in [0,1)$, $\kappa_0(\rho,t) \neq 0, \rho \in (0,1]$. Then the amended conformable derivative of order ρ of a function $v(\cdot) : [a,b] \to \mathbb{R}$ is defined by*

$$(\mathscr{D}^\rho v)(t) = \kappa_1(\rho,t)v(t) + \kappa_0(\rho,t)v'(t), \quad t \in [a,b]. \tag{1}$$

The aforesaid amended conformable derivative (1) is said to be a proportional derivative. For more details, see [8].

For the located situation when $\kappa_1(\rho,t) = 1 - \rho$ and $\kappa_0(\rho,t) = \rho$ the equality (1) takes the form

$$(\mathscr{D}^\rho v)(t) = (1-\rho)v(t) + \rho v'(t), \quad t \in [a,b]. \tag{2}$$

Definition 2 ([10] *The generalized proportional fractional integral*). *Let $\rho \in (0,1]$ and $\alpha > 0$. The left generalized proportional fractional integrals of the function $v \in L^1([a,b], \mathbb{R})$ is defined by*

$$({}_a\mathscr{I}^{\alpha,\rho}v)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}v(s)\,ds, \quad t \in [a,b]. \tag{3}$$

Definition 3 ([10] *The generalized proportional fractional derivative*). *Let $\rho, \alpha \in (0,1]$. The left generalized proportional fractional derivative of the function $v \in L^1([a,b], \mathbb{R})$ is defined by*

$$\begin{aligned}({}_a^R\mathscr{D}^{\alpha,\rho}v)(t) &= \mathscr{D}^{1,\rho}{}_a\mathscr{I}^{1-\alpha,\rho}v(t) \\ &= \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\mathscr{D}^{1,\rho}\Big(\int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{-\alpha}v(s)\,ds\Big),\end{aligned} \tag{4}$$

where $(\mathscr{D}^{1,\rho}v)(t) = (\mathscr{D}^\rho v)(t) = (1-\rho)v(t) + \rho v'(t)$.

Remark 1. *Note in the case $\rho \in (0,1]$ and $\alpha = 0$ it is defined that $({}_a\mathscr{I}^{0,\rho}v)(t) = v(t)$ and $({}_a\mathscr{D}^{0,\rho}v)(t) = v(t)$ (see [10]).*

We will provide some results which will be used in our further considerations.

Lemma 1 ([10])**.** *If $\rho \in (0,1]$, $\beta > 0$, and $\alpha \in (0,1]$ and $v \in L^1([a,b],\mathbb{R})$, we have the following statements:*

$$\left({}_a\mathscr{I}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}\tau} (\tau - a)^{\beta-1} \right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha \Gamma(\beta + \alpha)} e^{\frac{\rho-1}{\rho}t} (t-a)^{\alpha+\beta-1} \tag{5}$$

$$_a\mathscr{I}^{\alpha,\rho}\left({}_a\mathscr{I}^{\beta,\rho} v \right)(t) = {}_a\mathscr{I}^{\beta,\rho}({}_a\mathscr{I}^{\alpha,\rho} v)(t) = \left({}_a\mathscr{I}^{\alpha+\beta,\rho} v \right)(t); \tag{6}$$

$$_a\mathscr{I}^{\alpha,\rho}({}_a^R\mathscr{D}^{\alpha,\rho} v)(t) = v(t) - \frac{({}_a\mathscr{I}^{1-\alpha,\rho} v)(a)}{\rho^{\alpha-1}\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-a)} (t-a)^{\alpha-1}. \tag{7}$$

We will prove the following preliminary result which is similar to Lemma 3.2 [1] for the Riemann–Liouville fractional derivative.

Lemma 2. *Let $\rho, \alpha \in (0,1]$ and $y(t) \in L^1([a,b],\mathbb{R})$. Then*

(i)   If there exists a limit

$$\lim_{t \to a+} \left(e^{\frac{1-\rho}{\rho}t} (t-a)^{1-\alpha} y(t) \right) = c \in \mathbb{R} \tag{8}$$

then also exists a limit

$$({}_a\mathscr{I}^{1-\alpha,\rho} y)(a) := \lim_{t \to a+} ({}_a\mathscr{I}^{1-\alpha,\rho} y)(t) = c\frac{\Gamma(\alpha)}{\rho^{1-\alpha}} e^{\frac{\rho-1}{\rho}a}. \tag{9}$$

(ii) if there exists a limit

$$({}_a\mathscr{I}^{1-\alpha,\rho} y)(a) = k \in \mathbb{R}, \tag{10}$$

then if there exists the limit $\lim_{t \to a+} \left(e^{\frac{1-\rho}{\rho}t} (t-a)^{1-\alpha} y(t) \right)$, then

$$\lim_{t \to a+} \left(e^{\frac{1-\rho}{\rho}t} (t-a)^{1-\alpha} y(t) \right) = \frac{k\rho^{1-\alpha} e^{\frac{1-\rho}{\rho}a}}{\Gamma(\alpha)}. \tag{11}$$

Proof. Let the limit (8) be satisfied and $\epsilon > 0$ be an arbitrary number. From (8) there exists a number $\eta > 0$ such that

$$|\lim_{t \to a+} \left(e^{\frac{1-\rho}{\rho}t} (t-a)^{1-\alpha} y(t) \right) - c| < \epsilon, \quad t \in (a, a+\eta). \tag{12}$$

Moreover, since the exponential function is continuous

$$|e^{\frac{\rho-1}{\rho}t} - e^{\frac{\rho-1}{\rho}a}| < \epsilon, \quad t \in (a, a+\eta). \tag{13}$$

Then according to (7)

$$\left({}_a\mathscr{I}^{1-\alpha,\rho} e^{\frac{\rho-1}{\rho}t} (t-a)^{\alpha-1} \right)$$
$$= \frac{e^{\frac{\rho-1}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(-s)} (t-s)^{-\alpha} \left(e^{\frac{\rho-1}{\rho}s} (s-a)^{\alpha-1} \right) ds \tag{14}$$
$$= \frac{\Gamma(\alpha)}{\rho^{1-\alpha}} e^{\frac{\rho-1}{\rho}t}$$

Then applying (12)–(14) we obtain

$$
\left|({}_a\mathscr{I}^{1-\alpha,\rho}y)(t) - c\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}e^{\frac{\rho-1}{\rho}a}\right|
$$

$$
\leq \left|({}_a\mathscr{I}^{1-\alpha,\rho}y)(t) - c\,{}_a\mathscr{I}^{1-\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\alpha-1}\right| + |c|\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\left|e^{\frac{\rho-1}{\rho}t} - e^{\frac{\rho-1}{\rho}a}\right|
$$

$$
= \frac{e^{\frac{\rho-1}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(-s)}(t-s)^{-\alpha}(s-a)^{\alpha-1}e^{\frac{\rho-1}{\rho}s}
$$

$$
\times \left|e^{\frac{1-\rho}{\rho}s}(s-a)^{1-\alpha}y(s) - c\right|ds + |c|\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\left|e^{\frac{\rho-1}{\rho}t} - e^{\frac{\rho-1}{\rho}a}\right|
$$

$$
\leq \epsilon\frac{e^{\frac{\rho-1}{\rho}t}}{\rho^{1-\alpha}\Gamma(1-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}(-s)}(t-s)^{-\alpha}(s-a)^{\alpha-1}e^{\frac{\rho-1}{\rho}s}\,ds + |c|\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\epsilon
$$

$$
= \epsilon\,{}_a\mathscr{I}^{1-\alpha,\rho}e^{\frac{\rho-1}{\rho}t}(t-a)^{\alpha-1} + |c|\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\epsilon
$$

$$
= \epsilon\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}e^{\frac{\rho-1}{\rho}t} + |c|\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\epsilon \leq \epsilon\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\left(e^{\frac{\rho-1}{\rho}a} + |c|\right)
$$

which proves the claim (i) of Lemma 2.

Assume the limit $\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}t}(t-a)^{1-\alpha}y(t)\right)$ exists and is equal to c. Then , according to Lemma 2(i) the equality

$$
({}_a\mathscr{I}^{1-\alpha,\rho}y)(a) = c\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}e^{\frac{\rho-1}{\rho}a}
$$

holds and, hence, in accordance with (10) the validity of (11) follows.　□

3. Linear Generalized Proportional Fractional Differential Equation

Consider the linear scalar fractional equation with generalized proportional fractional derivative and initial value conditions (PIVP)

$$
\begin{aligned}
({}_a^R\mathscr{D}^{\alpha,\rho}u)(t) &= \lambda u(t) + f(t), \quad t \in (a,b], \\
({}_a\mathscr{I}^{1-\alpha,\rho}u)(a) &= \eta
\end{aligned}
\tag{15}
$$

where $u(\cdot): [a,b] \to \mathbb{R}$, $\rho \in (0,1]$, $\alpha \in (0,1)$, λ is a real constant, $f \in C([a,b])$.

Remark 2. *According to Lemma 2 the initial value condition in Equation (15) could be replaced by*

$$
\lim_{t\to a+}\left(e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}u(t)\right) = \frac{\eta\rho^{1-\alpha}}{\Gamma(\alpha)}.
\tag{16}
$$

Define the set

$$
C_{1-\alpha,\rho}([a,b]) = \{x(t): (a,b] \to \mathbb{R}: x \in C((a,b],\mathbb{R}),
$$

$$
\lim_{t\to a+}e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}x(t) < \infty\}
$$

with the norm

$$
\|x\|_{C_{1-\alpha,\rho}} = \max_{t\in[a,b]}|e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}x(t)|.
$$

Note that $C_{1-\alpha,\rho}([a,b])$ is a Banach space. If $u_n \in C_{1-\alpha,\rho}([a,b])$, $n = 1, 2, \ldots$ and $\|u_n - u\|_{C_{1-\alpha,\rho}[a,b]} \to 0$ then $u \in C_{1-\alpha,\rho}([a,b])$.

Theorem 1. *The PIVP (15) has a unique solution* $u \in C_{1-\alpha,\rho}[a,b]$ *given by*

$$
u(t) = \eta e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^{\alpha}\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1}
$$

$$
+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^{\alpha}\right) f(s)\, ds, \tag{17}
$$

for $t \in (a,b]$.

Proof. Apply the generalized fractional proportional integral ${}_a\mathscr{I}^{\alpha,\rho}(\cdot)$ to the first equation of (15) and use equality (7) in Lemma 1 and obtain the following integral equation

$$
u(t) = ({}_a\mathscr{I}^{\alpha,\rho}f)(t) + \lambda({}_a\mathscr{I}^{\alpha,\rho}u)(t)
$$

$$
+ \frac{({}_a\mathscr{I}^{1-\alpha,\rho}u)(0)}{\rho^{\alpha-1}\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}
$$

$$
= \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t \frac{e^{\frac{\rho-1}{\rho}(t-s)}f(s)}{(t-s)^{1-\alpha}}\, ds + \frac{\lambda}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t \frac{e^{\frac{\rho-1}{\rho}(t-s)}u(s)}{(t-s)^{1-\alpha}}\, ds \tag{18}
$$

$$
+ \frac{\eta}{\rho^{\alpha-1}\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1}, \quad t \in (a,b].
$$

We will apply the method of successive approximations to obtain the solution of the integral Equation (18).

Consider the sequence of functions $\{u_m(t)\}_{m=0}^{\infty}$ defined by the equalities

$$
u_0(t) = \frac{\eta}{\Gamma(\alpha)} e^{(\rho-1)\frac{t-a}{\rho}}\left(\frac{t-a}{\rho}\right)^{\alpha-1}, \quad t \in (a,b],
$$

$$
\lim_{t \to a+}\left(e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}u_0(t)\right) = \frac{\eta\rho^{1-\alpha}}{\Gamma(\alpha)}, \tag{19}
$$

and

$$
u_m(t) = u_0(t) + \frac{1}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t \frac{e^{\frac{\rho-1}{\rho}(t-s)}f(s)}{(t-s)^{1-\alpha}}\, ds
$$

$$
+ \frac{\lambda}{\rho^{\alpha}\Gamma(\alpha)} \int_a^t \frac{e^{\frac{\rho-1}{\rho}(t-s)}u_{m-1}(s)}{(t-s)^{1-\alpha}}\, ds \tag{20}
$$

$$
= u_0(t) + \lambda({}_a\mathscr{I}^{\alpha,\rho}u_{m-1})(t) + ({}_a\mathscr{I}^{\alpha,\rho}f)(t), \quad t \in (a,b]
$$

$$
\lim_{t \to a+}\left(e^{\frac{1-\rho}{\rho}(t-a)}(t-a)^{1-\alpha}u_m(t)\right) = \frac{\eta\rho^{1-\alpha}}{\Gamma(\alpha)}, \quad m = 1,2,\ldots.
$$

For any $m = 0,1,2,\ldots$ the function $u_m \in C_{1-\alpha,\rho}[a,b]$.

For $m = 1$ from equalities (19), (20) and (5) with $\beta = \alpha$ we obtain

$$
u_1(t) = \frac{\eta}{\rho^{\alpha-1}\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + ({}_a\mathscr{I}^{\alpha,\rho}f)(t)
$$

$$
+ \frac{\lambda\eta}{\rho^{\alpha-1}\Gamma(\alpha)}\left({}_a\mathscr{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^{\alpha-1}\right)(t) \tag{21}
$$

$$
= \eta e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{1} \frac{\lambda^k(t-a)^{(k+1)\alpha-1}}{\rho^{(k+1)\alpha-1}\Gamma((k+1)\alpha)} + ({}_a\mathscr{I}^{\alpha,\rho}f)(t),
$$

for $t \in (a,b]$.

Similarly, for $m = 2$ from equalities (20) and (7), (6) with $\beta = (k+1)\alpha$ we obtain

$$u_2(t) = u_0(t) + (_a\mathscr{I}^{\alpha,\rho}f)(t) + \lambda(_a\mathscr{I}^{\alpha,\rho}u_1)(t)$$

$$= \frac{\eta}{\rho^{\alpha-1}\Gamma(\alpha)}e^{\frac{\rho-1}{\rho}(t-a)}(t-a)^{\alpha-1} + (_a\mathscr{I}^{\alpha,\rho}f)(t)$$

$$+ \eta\sum_{k=0}^{1}\frac{\lambda^k}{\rho^{(k+1)\alpha-1}\Gamma((k+1)\alpha)}\left(_a\mathscr{I}^{\alpha,\rho}e^{\frac{\rho-1}{\rho}t}t^{(k+1)\alpha-1}\right)(t)$$

$$+ \lambda\left(_a\mathscr{I}^{\alpha,\rho}(_a\mathscr{I}^{\alpha,\rho}f)\right)(t)$$

$$= \eta e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{2}\frac{\lambda^k(t-a)^{(k+1)\alpha-1}}{\rho^{(k+1)\alpha-1}\Gamma((k+1)\alpha)}$$

$$+ (_a\mathscr{I}^{\alpha,\rho}f)(t) + \lambda\left(_a\mathscr{I}^{2\alpha,\rho}f\right)(t)$$

$$= \eta e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{2}\frac{\lambda^k(t-a)^{(k+1)\alpha-1}}{\rho^{(k+1)\alpha-1}\Gamma((k+1)\alpha)}$$

$$+ \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t\frac{e^{\frac{\rho-1}{\rho}(t-s)}f(s)}{(t-s)^{1-\alpha}}\,ds + \lambda\frac{1}{\rho^{2\alpha}\Gamma(2\alpha)}\int_a^t\frac{e^{\frac{\rho-1}{\rho}(t-s)}f(s)}{(t-s)^{1-2\alpha}}\,ds$$

$$= \eta e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{2}\frac{\lambda^k(t-a)^{(k+1)\alpha-1}}{\rho^{(k+1)\alpha-1}\Gamma((k+1)\alpha)}$$

$$+ \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t\sum_{k=1}^{2}\frac{\lambda^{k-1}}{\rho^{k\alpha}\Gamma(k\alpha)(t-s)^{1-k\alpha}}e^{\frac{\rho-1}{\rho}(t-s)}f(s)\,ds.$$

Continuing this process we obtain

$$u_m(t)$$

$$= \eta e^{\frac{\rho-1}{\rho}(t-a)}\sum_{k=0}^{m}\frac{\lambda^k(t-a)^{(k+1)\alpha-1}}{\rho^{(k+1)\alpha-1}\Gamma((k+1)\alpha)}$$

$$+ \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t\sum_{k=1}^{m}\frac{\lambda^{k-1}}{\rho^{k\alpha}\Gamma(k\alpha)(t-s)^{1-k\alpha}}e^{\frac{\rho-1}{\rho}(t-s)}f(s)\,ds$$

$$= \eta e^{\frac{\rho-1}{\rho}(t-a)}\left(\frac{t-a}{\rho}\right)^{\alpha-1}\sum_{k=0}^{m}\frac{\left(\lambda\left(\frac{t-a}{\rho}\right)^{k\alpha}\right)^k}{\Gamma(k\alpha+\alpha)}$$

$$+ \frac{(t-s)^{\alpha-1}}{\rho^\alpha\Gamma(\alpha)}\int_a^t\sum_{k=0}^{m}\frac{\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right)^k}{\Gamma(k\alpha+\alpha)}e^{\frac{\rho-1}{\rho}(t-s)}f(s)\,ds.$$

$$(22)$$

Taking the limit as $m \to \infty$ in (22), denote $\lim_{m\to\infty}u_m(t) = u(t)$, $t \in (a,b]$ and applying the Mittag–Leffler function with two parameters $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}$ we obtain

$$u(t) = \eta e^{(\rho-1)\frac{t-a}{\rho}}E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^\alpha\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1}$$

$$+ \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}e^{(\rho-1)\left(\frac{t-s}{\rho}\right)}E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right)f(s)\,ds.$$

$$(23)$$

Therefore, the function $u(t)$ satisfies the equality (17).

Furthermore, taking a limit as $m \to \infty$ in (20) for $t \in (a,b]$ it follows that the function $u(t)$ satisfies the integral equality (18) which is equivalent to (15). In addition, from the limit conditions in (20) and Remark 2 it follows that the function $u(t)$ satisfies the initial condition in (15). $\quad\square$

Remark 3. *Note in the case $\rho = 1$ the generalized proportional fractional integral and the generalized proportional fractional derivative are reduced to Riemann–Liouville fractional integral and derivatives, respectively, and the formula (17) is reduced to formula (4.1.14) [1] for the linear Riemann–Liouville fractional differential equation.*

4. Linear Generalized Proportional Fractional Differential Equation with Instantaneous Impulses

Assume the impulsive points $\{t_i\}_{i=1}^m$ are given, such that $t_i < t_{i+1}$, $i = 1, 2, \ldots, m - 1$, and $T : a < b \leq \infty$. We denote $a = t_0$, $b = t_{m+1}$ (in the case $b = \infty$ we have $m = \infty$).

The impulse at a point τ means that there is a jump of the solution at this point and after the jump for $t > \tau$ the solution is determined by the same differential equation but with a new initial value. Therefore, we need an initial condition at the impulsive point τ. Following the idea of Section 3 we will define two equivalent types of the impulsive conditions at the point τ (see Remark 2):

- *integral form of the impulsive condition*

$$(_{\tilde{t}}\mathscr{I}^{1-\alpha,\rho}u)(\tau) = P(u(\tau - 0))$$

- *weighted form of the impulsive condition*

$$\lim_{t \to \tau+} \left(e^{\frac{1-\rho}{\rho}t}(t - \tau)^{1-\alpha}u(t) \right) = G(u(\tau - 0))$$

where $P, G : \mathbb{R} \to \mathbb{R}$ are given functions.

Note that according to Lemma 2 the equality $G(u(\tau - 0)) = \frac{P(u(\tau-0))\rho^{1-\alpha}e^{\frac{1-\rho}{\rho}a}}{\Gamma(\alpha)}$ holds.

Since the generalized proportional fractional derivative significant depends on its lower limit, we will consider the case of the generalized proportional fractional derivative with changed lower limit at each impulsive time. It is reasonable because each impulsive time is considered as an initial time of the fractional differential equation.

Remark 4. *Note for $\alpha \to 1$ and $\rho \in (0, 1]$ the limit $\lim_{t \to \tilde{t}+} \left(e^{\frac{1-\rho}{\rho}t}(t - \tilde{t})^{1-\alpha}u(t) \right)$ is reduced to $\lim_{t \to \tilde{t}+} \left(e^{\frac{1-\rho}{\rho}t}u(t) \right)$ and the impulsive condition $\lim_{t \to \tilde{t}+} \left(e^{\frac{1-\rho}{\rho}t}(t - \tilde{t})^{1-\alpha}u(t) \right) = G(u(\tilde{t} - 0))$ is reduced to the well known impulsive condition $u(\tilde{t} + 0) = \mathcal{G}_{\tilde{t}}(u(\tilde{t} - 0))$ at the impulsive time $\tilde{t}$ for differential equations with ordinary derivatives where $\mathcal{G}_{\tilde{t}}(u) = G(u)e^{\frac{\rho-1}{\rho}\tilde{t}}$.*

Define the set

$$PC_{1-\alpha,\rho}([a,b]) = \left\{ u : [a,b] \to \mathbb{R} : u \in C(\cup_{i=0}^m (t_k, t_{k+1}], \mathbb{R}), \right.$$

$$\left. \lim_{t \to t_k+} e^{\frac{1-\rho}{\rho}(t-t_k)}(t - t_k)^{1-\alpha}u(t) < \infty, \text{ for } k = 0, 1, 2, \ldots, m \right\}$$

with the norm

$$||u||PC_{1-\alpha,\rho} = \max_{k=0,1,2,\ldots,m} \max_{t \in [t_k, t_{k+1}]} |e^{\frac{1-\rho}{\rho}(t-t_k)}(t - t_k)^{1-\alpha}u(t)|.$$

Consider the linear scalar impulsive fractional equation with generalized proportional fractional derivative and initial value conditions (IPIVP)

$$({}_{t_k}^R\mathscr{D}^{\alpha,\rho}u)(t) = \lambda u(t) + f(t), \quad t \in (t_k, t_{k+1}], \ k = 0, 1, \ldots, m - 1$$

$$({}_{t_k}\mathscr{I}^{1-\alpha,\rho}u)(t_k) = P_k(u(t_k - 0)), \quad k = 1, 2, \ldots, m - 1, \tag{24}$$

$$({}_a\mathscr{I}^{1-\alpha,\rho}u)(a) = \eta$$

where $u(\cdot) : [a, b] \to \mathbb{R}$, $\rho \in (0, 1]$, $\alpha \in (0, 1)$, λ is a real constant, $f \in C([a, b])$, $P_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m - 1$.

Remark 5. *According to Lemma 2 the initial value condition in Equation (15) could be replaced by equality (16) and the impulsive conditions could be replaced by*

$$\lim_{t \to t_k+} \left(e^{\frac{1-\rho}{\rho}(t-t_k)}(t - t_k)^{1-\alpha}u(t) \right) = G_k(u(t_k - 0)), \quad k = 1, 2 \ldots, m - 1,$$

where $G_k(u) = \frac{P_k(u)\rho^{1-\alpha}}{\Gamma(\alpha)}$.

Theorem 2. *The IPIVP (24) has a unique solution* $u \in PC_{1-\alpha,\rho}[a, b]$ *given by*

$$u(t) = P_k(u(t_k - 0))e^{(\rho-1)\frac{t-t_k}{\rho}}E_{\alpha,\alpha}\left(\lambda\left(\frac{t - t_k}{\rho}\right)^{\alpha}\right)\left(\frac{t - t_k}{\rho}\right)^{\alpha-1}$$

$$+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}e^{(\rho-1)(\frac{t-s}{\rho})}E_{\alpha,\alpha}(\lambda\left(\frac{t - s}{\rho}\right)^{\alpha})f(s)\,ds, \tag{25}$$

$$\text{for } t \in (t_k, t_{k+1}], \ k = 0, 1, 2, \ldots, m - 1,$$

where the notation $P_0(u(t_0 - 0)) \equiv \eta$ *is used.*

In the partial case $P_k(u) = C_k u$, $C_k = const, k = 1, 2, \ldots, m - 1$, *the solution of IPIVP (24) is given by*

$$u(t) = \eta e^{(\rho-1)\frac{t-t_p}{\rho}}E_{\alpha,\alpha}\left(\lambda\left(\frac{t - t_p}{\rho}\right)^{\alpha}\right)\left(\frac{t - t_p}{\rho}\right)^{\alpha-1}$$

$$\times \prod_{k=1}^{p}\left[C_k e^{(\rho-1)\frac{t_k-t_{k-1}}{\rho}}E_{\alpha,\alpha}\left(\lambda\left(\frac{t_k - t_{k-1}}{\rho}\right)^{\alpha}\right)\left(\frac{t_k - t_{k-1}}{\rho}\right)^{\alpha-1}\right]$$

$$+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\left[\sum_{k=1}^{p}\left(\prod_{j=k}^{p}C_j\right)\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}e^{(\rho-1)(\frac{t_k-s}{\rho})}E_{\alpha,\alpha}(\lambda\left(\frac{t_k - s}{\rho}\right)^{\alpha})f(s)\,ds\right] \tag{26}$$

$$\times e^{(\rho-1)\frac{t-t_p}{\rho}}E_{\alpha,\alpha}\left(\lambda\left(\frac{t - t_p}{\rho}\right)^{\alpha}\right)\left(\frac{t - t_p}{\rho}\right)^{\alpha-1}$$

$$+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{t_p}^{t}(t - s)^{\alpha-1}e^{(\rho-1)(\frac{t-s}{\rho})}E_{\alpha,\alpha}(\lambda\left(\frac{t - s}{\rho}\right)^{\alpha})f(s)\,ds,$$

$$\text{for } t \in (t_p, t_{p+1}], p = 0, 1, 2, \ldots, m - 1.$$

Proof. We use an induction w.r.t. as the intervals to prove the claim.

For any $k = 0, 1, 2, \ldots, m - 1$, the IPIVP (24) is reduced to an initial value problem of the type (15) with $\eta = P_k(u(t_k - 0))$, $a = t_k$, $b = t_{k+1}$ and $P_0(u) \equiv \eta$. According to Lemma 1 and Equation (17) it has a solution $u_k \in C_{1-\alpha,\rho}[t_k, t_{k+1}]$ given by

$$u_k(t) = P_k(u_{k-1}(t_k - 0))e^{(\rho-1)\frac{t-t_k}{\rho}}E_{\alpha,\alpha}\left(\lambda\left(\frac{t - t_k}{\rho}\right)^{\alpha}\right)\left(\frac{t - t_k}{\rho}\right)^{\alpha-1}$$

$$+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}e^{(\rho-1)(\frac{t-s}{\rho})}E_{\alpha,\alpha}(\lambda\left(\frac{t - s}{\rho}\right)^{\alpha})f(s)\,ds, \tag{27}$$

$$\text{for } t \in (t_k, t_{k+1}].$$

Define the function $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m - 1$. Then the function $u \in PC_{1-\alpha,\rho}[t_k, t_{k+1}]$ and satisfies the IPIVP (24).

Let $P_k(u) = C_k u$, $C_k = const, k = 1, 2, \ldots, m - 1$,. Then from (27), we obtain inductively for $t \in (a, t_1]$

$$
\begin{aligned}
u_0(t) = {}& \eta e^{(\rho-1)\frac{t-a}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-a}{\rho}\right)^\alpha\right)\left(\frac{t-a}{\rho}\right)^{\alpha-1} \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right) f(s) \, ds,
\end{aligned}
\tag{28}
$$

and for $t \in (t_1, t_2]$

$$
\begin{aligned}
u_1(t) = {}& C_1\Bigg(\eta e^{(\rho-1)\frac{t_1-t_0}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_1-t_0}{\rho}\right)^\alpha\right)\left(\frac{t_1-t_0}{\rho}\right)^{\alpha-1} \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t_1-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_1-s}{\rho}\right)^\alpha\right) f(s) \, ds\Bigg) \\
& \times e^{(\rho-1)\frac{t-t_1}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-t_1}{\rho}\right)^\alpha\right)\left(\frac{t-t_1}{\rho}\right)^{\alpha-1} \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right) f(s) \, ds \\
= {}& C_1 \eta e^{(\rho-1)\frac{t_1-t_0}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_1-t_0}{\rho}\right)^\alpha\right)\left(\frac{t_1-t_0}{\rho}\right)^{\alpha-1} e^{(\rho-1)\frac{t-t_1}{\rho}} \\
& \times E_{\alpha,\alpha}\left(\lambda\left(\frac{t-t_1}{\rho}\right)^\alpha\right)\left(\frac{t-t_1}{\rho}\right)^{\alpha-1} \\
& + C_1 e^{(\rho-1)\frac{t-t_1}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-t_1}{\rho}\right)^\alpha\right)\left(\frac{t-t_1}{\rho}\right)^{\alpha-1} \frac{1}{\rho^\alpha \Gamma(\alpha)} \\
& \times \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t_1-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_1-s}{\rho}\right)^\alpha\right) f(s) \, ds \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right) f(s) \, ds,
\end{aligned}
\tag{29}
$$

and for $t \in (t_2, t_3]$

$$
\begin{aligned}
u_2(t) = {}& C_2\Bigg(C_1 \eta e^{(\rho-1)\frac{t_1-t_0}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_1-t_0}{\rho}\right)^\alpha\right)\left(\frac{t_1-t_0}{\rho}\right)^{\alpha-1} \\
& \times e^{(\rho-1)\frac{t_2-t_1}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_2-t_1}{\rho}\right)^\alpha\right)\left(\frac{t_2-t_1}{\rho}\right)^{\alpha-1} \\
& + C_1 e^{(\rho-1)\frac{t_2-t_1}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_2-t_1}{\rho}\right)^\alpha\right)\left(\frac{t_2-t_1}{\rho}\right)^{\alpha-1} \frac{1}{\rho^\alpha \Gamma(\alpha)} \\
& \times \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t_1-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_1-s}{\rho}\right)^\alpha\right) f(s) \, ds \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t_2-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_2-s}{\rho}\right)^\alpha\right) f(s) \, ds\Bigg) \\
& \times e^{(\rho-1)\frac{t-t_2}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-t_2}{\rho}\right)^\alpha\right)\left(\frac{t-t_2}{\rho}\right)^{\alpha-1} \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right) f(s) \, ds \\
= {}& \eta e^{(\rho-1)\frac{t-t_2}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-t_2}{\rho}\right)^\alpha\right)\left(\frac{t-t_2}{\rho}\right)^{\alpha-1} \\
& \times \prod_{k=1}^2 \left[C_k e^{(\rho-1)\frac{t_k-t_{k-1}}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t_k-t_{k-1}}{\rho}\right)^\alpha\right)\left(\frac{t_k-t_{k-1}}{\rho}\right)^{\alpha-1}\right] \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \Bigg[\sum_{k=1}^2 \left(\prod_{j=k}^2 C_j\right) \int_{t_{k-1}}^{t_k} (t_k-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t_k-s}{\rho}\right)} \\
& \times E_{\alpha,\alpha}\left(\lambda\left(\frac{t_k-s}{\rho}\right)^\alpha\right) f(s) \, ds\Bigg] e^{(\rho-1)\frac{t-t_2}{\rho}} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-t_2}{\rho}\right)^\alpha\right)\left(\frac{t-t_2}{\rho}\right)^{\alpha-1} \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} e^{(\rho-1)\left(\frac{t-s}{\rho}\right)} E_{\alpha,\alpha}\left(\lambda\left(\frac{t-s}{\rho}\right)^\alpha\right) f(s) \, ds.
\end{aligned}
\tag{30}
$$

Following this process we inductively prove the explicit form of the solution (26). □

Remark 6. *The IPIVP (24) could not be considered as a partial case of PIVP (15) because of the presence of the generalized proportional fractional derivative and its deep dependence on the lower limit. Therefore, formula (17) could not be considered as a partial case of (26). Note that it is totally different to the case of ordinary derivatives.*

5. Applications

Example 1. *Consider the PIVP (15) in the partial case $\lambda = 1$, $f(t) = e^{(\rho-1)(\frac{t}{\rho})}$, $a = 0$, $b = \infty$. Then applying*

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\frac{t-s}{\rho}\right)^\alpha\right) ds = \sum_{k=1}^\infty \frac{\int_0^t (t-s)^{\alpha k-1} ds}{\rho^{\alpha(k-1)}\Gamma(k\alpha)}$$

$$= \rho^\alpha \sum_{k=1}^\infty \frac{\left(\frac{t}{\rho}\right)^{\alpha k}}{\Gamma(k\alpha+1)} = \rho^\alpha \left(E_\alpha\left(\left(\frac{t}{\rho}\right)^\alpha\right) - 1\right)$$

and formula (17), the solution is given by

$$u(t) = e^{(\rho-1)\frac{t}{\rho}} \left(\eta E_{\alpha,\alpha}\left(\left(\frac{t}{\rho}\right)^\alpha\right)\left(\frac{t}{\rho}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\left(E_\alpha\left(\left(\frac{t}{\rho}\right)^\alpha\right) - 1\right)\right), \quad t > 0.$$

Example 2. *Consider the IPIVP (24) in the partial case $\lambda = 1$, $f(t) = e^{(\rho-1)(\frac{t}{\rho})}$, $a = 0$, $b = \infty$, $t_k = k$ and $P_k(u) = Cu$, $C = const, k = 1, 2, \dots$.*
Then applying (see (31))

$$\int_k^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\frac{t-s}{\rho}\right)^\alpha\right) ds = \rho^\alpha \left(E_\alpha\left(\left(\frac{t-k}{\rho}\right)^\alpha\right) - 1\right)$$

and Formula (26), the solution is given by

$$u(t) = \eta e^{(\rho-1)\frac{t-p}{\rho}} E_{\alpha,\alpha}\left(\left(\frac{t-p}{\rho}\right)^\alpha\right)\left(\frac{t-p}{\rho}\right)^{\alpha-1}$$

$$\times \prod_{k=1}^p \left[Ce^{\frac{\rho-1}{\rho}} E_{\alpha,\alpha}\left(\left(\frac{1}{\rho}\right)^\alpha\right)\left(\frac{1}{\rho}\right)^{\alpha-1}\right]$$

$$+ \frac{1}{\rho^\alpha\Gamma(\alpha)} \left[\sum_{k=1}^p \left(\prod_{j=k}^p C\right) e^{\frac{k(\rho-1)}{\rho}} \int_{k-1}^k (k-s)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\frac{k-s}{\rho}\right)^\alpha\right) ds\right]$$

$$\times e^{(\rho-1)\frac{t-p}{\rho}} E_{\alpha,\alpha}\left(\left(\frac{t-p}{\rho}\right)^\alpha\right)\left(\frac{t-p}{\rho}\right)^{\alpha-1}$$

$$+ \frac{e^{(\rho-1)(\frac{t}{\rho})}}{\rho^\alpha\Gamma(\alpha)} \int_p^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\left(\frac{t-s}{\rho}\right)^\alpha\right) ds$$

$$= \eta e^{\frac{(\rho-1)t}{\rho}} E_{\alpha,\alpha}\left(\left(\frac{t-p}{\rho}\right)^\alpha\right)\left(\frac{t-p}{\rho}\right)^{\alpha-1} C^p \left[E_{\alpha,\alpha}\left(\left(\frac{1}{\rho}\right)^\alpha\right)\left(\frac{1}{\rho}\right)^{\alpha-1}\right]^p$$

$$+ C^{p+1} \frac{e^{\frac{(\rho-1)(t-p)}{\rho}} E_{\alpha,\alpha}\left(\left(\frac{t-p}{\rho}\right)^\alpha\right)\left(\frac{t-p}{\rho}\right)^{\alpha-1}}{\Gamma(\alpha)}\left(E_\alpha\left(\left(\frac{1}{\rho}\right)^\alpha\right) - 1\right)$$

$$\times \left[\sum_{k=1}^p \left(\frac{e^{\frac{\rho-1}{\rho}}}{C}\right)^k\right] + \frac{e^{\frac{(\rho-1)t}{\rho}}}{\Gamma(\alpha)}\left(E_\alpha\left(\left(\frac{t-p}{\rho}\right)^\alpha\right) - 1\right),$$

for $t \in (p, p+1]$, $p = 0, 1, 2, \dots$.

6. Conclusions

In this paper a scalar linear fractional differential equation with a generalized proportional fractional derivative of Riemann–Liouville type (LFDEGD) on a finite interval is studied. Two different cases are investigated. The object of investigation in the first case is the initial value problem of LFDEGD with an initial condition expressed by a generalized proportional fractional integral. An explicit formula of the solution of the studied initial value problem is obtained. In the second case the case when instantaneous impulses occur at fixed initially given points is considered. We study the case of a changeable lower limit of the generalized proportional fractional derivative at each impulsive time. It is reasonable because each impulsive time is considered as an initial time of the fractional differential equation. An appropriate impulsive conditions by generalized proportional fractional integrals are set up. An explicit solution is given.

Note that in the case of ordinary derivatives, the impulsive case is a generalization of the case without impulses. However, it is not the situation of the generalized proportional fractional derivative of Riemann–Liouville type. It is mainly because the solution has a singularity at each impulsive point. It requires the study of both cases, impulsive and non-impulsive, neither of which is a partial case of the other one.

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