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Stability of Bi-Additive Mappings and Bi-Jensen Mappings

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Abstract: Symmetry is repetitive self-similarity. We proved the stability problem by replicating the well-known Cauchy equation and the well-known Jensen equation into two variables. In this paper, we proved the Hyers-Ulam stability of the bi-additive functional equation $f(x+y,z+w) = f(x,z) + f(y,w)$ and the bi-Jensen functional equation $4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x, z) + f(x, w) + f(y, z) + f(y, w)$.

Keywords: stability; bi-additive mapping; bi-Jensen mapping

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1. Introduction

A functional equation is stable if there is a function that exactly satisfies the given equation in the vicinity of a function that approximately satisfies it. Any approximate solution can actually be an exact solution. In Cauchy's equation $f(x+y) = f(x) + f(y)$ we can deal with a class of approximate solutions defined by the functional inequality introduced by Rassias.

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p).$$

It turns out that for $p \neq 1$ each solution of the above inequality can be approximated by an additive function A in such a way that the inequality

$$\|f(x) - A(x)\| \le k\varepsilon\|x\|^p.$$

holds, with a suitable k, on the whole domain (for $p = 0$ it coincides with the classical Hyers–Ulam result).

Let us say $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces. The mapping $h : \mathcal{X} \to \mathcal{Y}$ is called *an additional mapping* (respectively, *an affine mapping*) if h satisfies the Cauchy functional equation $h(x+y) = h(x) + h(y)$ (respectively, the Jensen functional equation $2h\left(\frac{x+y}{2}\right) = h(x) + h(y)$). T. Aoki [1] and Th. M. Rassias [2,3] extended Hyers-Ulam stability taking into account the variables for the Cauchy equation. S.-M. Jung [4] got the result of the Jensen equation. It was also generalized as a functional case by P. Găvruta [5] and S.-M. Jung [6] and Y.-H. Lee and K.-W. Jun [7].

The following functional Equations (1) and (3) are functional equations those combine the existing well-known the Cauchy equation and the Jensen equation.

$$f(x+y,z+w) = f(x,z) + f(y,w). \tag{1}$$

The authors [8] introduce the system of equations

$$\begin{aligned} 2f\left(\tfrac{x+y}{2}, z\right) &= f(x, z) + f(y, z), \\ 2f\left(x, \tfrac{y+z}{2}\right) &= f(x, y) + f(x, z). \end{aligned} \tag{2}$$

and the bi-Jensen functional equation

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x, z) + f(x, w) + f(y, z) + f(y, w). \tag{3}$$

We made the above functional equations with a symmetrical structure. Symmetry is repetitive self-similarity. The solution of (2) is coincide with the solution of (3). The solution of (1) is of the form $A_1(x) + A_2(y)$, where A_1 and A_2 are additive mappings. The solution of (2) is of the form $A_1(x) + A_2(y) + f(0,0)$, where A_1 and A_2 are additive mappings. The solution of (2) contains the solution of (1). The difference of the solutions (1) and (2) is merely a constant, that is, the solutions (1) and (2) are similar.

Jun, Jung, and Lee [9] obtained the stability on a bi-Jensen functional equation in Banach spaces. Additionally, the authors [10] proved the stability on a Cauchy-Jensen functional equation Banach spaces.

In this paper, we investigate the generalized Hyers-Ulam stability of (1) in Banach spaces and 2-Banach spaces. We proved the Hyers-Ulam stability of (2) and (3) in quasi-Banach spaces.

2. Solution and Stability of a Bi-Additive Functional Equation

In the following theorem, we find out the general solution of the bi-additive functional Equation (1).

Theorem 1. *A mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfies (1) if and only if there exist two additive mappings $A_1, A_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that*

$$f(x,y) = A_1(x) + A_2(y)$$

for all $x, y \in \mathcal{X}$.

Proof. We first assume that f is a solution of (1). Define $A_1, A_2 : \mathcal{X} \to \mathcal{Y}$ by $A_1(x) := f(x,0)$ and $A_2(x) := f(0,x)$ for all $x \in \mathcal{X}$. One can easily verify that A_1, A_2 are additive. Letting $y = z = 0$ in (1), we get

$$f(x,w) = f(x,0) + f(0,w) = A_1(x) + A_2(w)$$

for all $x, w \in \mathcal{X}$.

Conversely, we assume that there is two additive mappings $A_1, A_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$, such that $f(x,y) = A_1(x) + A_2(y)$ for all $x, y \in \mathcal{X}$. Since A_1, A_2 are additive, we gain

$$\begin{aligned} f(x+y, z+w) &= A_1(x+y) + A_2(z+w) \\ &= A_1(x) + A_1(y) + A_2(z) + A_2(w) \\ &= A_1(x) + A_2(z) + A_1(y) + A_2(w) \\ &= f(x,z) + f(y,w) \end{aligned}$$

for all $x, y, z, w \in \mathcal{X}$. □

From now on, let $\mathcal{X}$ and $\mathcal{Y}$ be a normed linear space and a Banach space, respectively.

Theorem 2. *Let $0 < p < 1$, $\varepsilon > 0$, $\delta \geq 0$ and $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping such that*

$$\|f(x+y, z+w) - f(x,z) - f(y,w)\| \leq \varepsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \tag{4}$$

for all $x, y, z, w \in \mathcal{X}$. Then there is unique bi-additive mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$, such that

$$\|f(x,y) - F(x,y)\| \leq \varepsilon + \frac{2\delta}{2-2^p}(\|x\|^p + \|y\|^p) \tag{5}$$

for all $x, y \in \mathcal{X}$. *The mapping* F *is given by* $F(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 2^j y)$ *for all* $x, y \in \mathcal{X}$.

Proof. Putting $y = x$ and $w = z$ in (4), we have

$$\left\| f(x,z) - \frac{1}{2} f(2x, 2z) \right\| \le \frac{\varepsilon}{2} + \delta(\|x\|^p + \|z\|^p)$$

for all $x, z \in \mathcal{X}$. Thus, we obtain

$$\left\| \frac{1}{2^j} f(2^j x, 2^j z) - \frac{1}{2^{j+1}} f(2^{j+1} x, 2^{j+1} z) \right\| \le \frac{\varepsilon}{2^{j+1}} + 2^{j(p-1)} \delta(\|x\|^p + \|z\|^p)$$

for all $x, z \in \mathcal{X}$ and all j. Replacing z by y in the above inequality, we see that

$$\left\| \frac{1}{2^j} f(2^j x, 2^j y) - \frac{1}{2^{j+1}} f(2^{j+1} x, 2^{j+1} y) \right\| \le \frac{\varepsilon}{2^{j+1}} + 2^{j(p-1)} \delta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{X}$ and all j. For given integers $l, m (0 \le l < m)$, we get

$$\left\| \frac{1}{2^l} f(2^l x, 2^l y) - \frac{1}{2^m} f(2^m x, 2^m y) \right\| \le \sum_{j=l}^{m-1} \left[\frac{\varepsilon}{2^{j+1}} + 2^{j(p-1)} \delta(\|x\|^p + \|y\|^p) \right] \tag{6}$$

for all $x, y \in \mathcal{X}$. By (6), the sequence $\{\frac{1}{2^j} f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{2^j} f(2^j x, 2^j y)\}$ converges for all $x, y \in \mathcal{X}$. Define $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$F(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 2^j y)$$

for all $x, y \in \mathcal{X}$. By (4), we have

$$\begin{aligned} &\frac{1}{2^j} \left\| f\big(2^j(x+y), 2^j(z+w)\big) - f(2^j x, 2^j z) - f(2^j y, 2^j w) \right\| \\ &\le \frac{\varepsilon}{2^j} + 2^{j(p-1)} \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \end{aligned}$$

for all $x, y, z, w \in \mathcal{X}$ and all $j \in \mathbb{N}$. Letting $j \to \infty$ in the above inequality, we see that F satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (6), one can obtain the inequality (5). If $G : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is another 2-variable additive mapping satisfying (5), we obtain

$$\begin{aligned} &\|F(x,y) - G(x,y)\| \\ &= \frac{1}{2^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\le \frac{1}{2^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y)\| + \frac{1}{2^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y)\| \\ &\le \frac{1}{2^{n-1}} \left[\varepsilon + \frac{2^{np+1}}{2 - 2^p} \delta(\|x\|^p + \|y\|^p) \right] \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

for all $x, y \in \mathcal{X}$. Hence the mapping F is the unique bi-additive mapping, as desired. □

Corollary 1. *Let* $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *be a mapping such that*

$$\|f(x+y, z+w) - f(x,z) - f(y,w)\| \le \varepsilon$$

for all $x, y, z, w \in \mathcal{X}$. *Then, there exists a unique mapping* $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *satisfying* (1), *such that*

$$\|f(x,y) - F(x,y)\| \le \frac{\varepsilon}{2}$$

for all $x, y \in \mathcal{X}$.

Proof. If we insert $\delta = 0$ in Theorem 2, we obtain ε as an estimate of the difference between the exact and the approximate solution of the considered equation. □

In the case $p > 2$ in Theorem 2, one can also obtain the similar result.

We explain some definitions [11,12] on 2-Banach spaces.

Definition 1. *Let $\mathcal{X}$ be a vector space over $\mathbb{R}$ with dimension greater than 1 and $\|\cdot,\cdot\| : \mathcal{X}^2 \to \mathbb{R}$ be a function. Then we say $(\mathcal{X}, \|\cdot,\cdot\|)$ is a linear 2-normed space if*

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x, y\| = \|y, x\|$;
(c) $\|\alpha x, y\| = |\alpha| \|x, y\|$;
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$
for all $\alpha \in \mathbb{R}$ and $x, y, z \in \mathcal{X}$. In this case, the function $\|\cdot,\cdot\|$ is called a 2-norm on $\mathcal{X}$.

Definition 2. *Let $\mathcal{X}$ be linear 2-normed space and $\{x_n\}$ a sequence in $\mathcal{X}$. The sequence $\{x_n\}$ is said to convergent in $\mathcal{X}$ if there is an $x \in \mathcal{X}$, such that*

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in \mathcal{X}$. In this case, we say that a sequence $\{x_n\}$ converges to x, simply denoted by $\lim_{n\to\infty} x_n = x$.

Definition 3. *Let $\mathcal{X}$ be linear 2-normed space and $\{x_n\}$ a sequence in $\mathcal{X}$ is called a Cauchy sequence if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ for all $y \in \mathcal{X}$. For convenience, we will write $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$. A 2-Banach space is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.*

In the following lemma, we get some primitive properties in linear 2-normed spaces that will be used to prove our stability results.

Lemma 1 ([13])**.** *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space and $x \in \mathcal{X}$.*

(a) If $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.
(b) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ for all $x, y, z \in \mathcal{X}$.
(c) If a sequence $\{x_n\}$ is convergent in $\mathcal{X}$, then $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for all $y \in \mathcal{X}$.

In the rest of this section, let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a 2-Banach space.

Theorem 3. *Let $p \in (0, 1)$, $\varepsilon > 0$, $\delta, \eta \geq 0$ and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a surjective mapping such that*

$$\begin{aligned} &\|f(x+y, z+w) - f(x,z) - f(y,w), f(u,v)\| \\ &\leq \varepsilon + \delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|) \end{aligned} \tag{7}$$

for all $x, y, z, w, u, v \in \mathcal{X}$. Then there exists a unique mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying (1), such that

$$\|f(x,y) - F(x,y), f(u,v)\| \leq \frac{\varepsilon}{2} + \frac{2\delta}{2 - 2^p}(\|x\|^p + \|y\|^p) + \frac{\eta}{2}(\|u\| + \|v\|) \tag{8}$$

for all $x, y, u, v \in \mathcal{X}$.

Proof. Letting $y = x$ and $w = z$ in (7), we have

$$\left\| f(x,z) - \frac{1}{2} f(2x, 2z), f(u,v) \right\| \le \frac{\varepsilon}{2} + \delta(\|x\|^p + \|z\|^p) + \frac{\eta}{2}(\|u\| + \|v\|)$$

for all $x, z, u, v \in \mathcal{X}$. Thus, we obtain

$$\begin{aligned} &\left\| \frac{1}{2^j} f(2^j x, 2^j z) - \frac{1}{2^{j+1}} f(2^{j+1}x, 2^{j+1}z), f(u,v) \right\| \\ &\le \frac{\varepsilon}{2^{j+1}} + 2^{j(p-1)}\delta(\|x\|^p + \|z\|^p) + \frac{\eta}{2^{j+1}}(\|u\| + \|v\|) \end{aligned}$$

for all $x, z, u, v \in \mathcal{X}$ and all j. Replacing z by y in the above inequality, we see that

$$\begin{aligned} &\left\| \frac{1}{2^j} f(2^j x, 2^j y) - \frac{1}{2^{j+1}} f(2^{j+1}x, 2^{j+1}y), f(u,v) \right\| \\ &\le \frac{\varepsilon}{2^{j+1}} + 2^{j(p-1)}\delta(\|x\|^p + \|y\|^p) + \frac{\eta}{2^{j+1}}(\|u\| + \|v\|) \end{aligned}$$

for all $x, y, u, v \in \mathcal{X}$ and all j. For given integers $l, m (0 \le l < m)$, we get

$$\begin{aligned} &\left\| \frac{1}{2^l} f(2^l x, 2^l y) - \frac{1}{2^m} f(2^m x, 2^m y), f(u,v) \right\| \\ &\le \sum_{j=l}^{m-1} \left[\frac{\varepsilon}{2^{j+1}} + 2^{j(p-1)}\delta(\|x\|^p + \|z\|^p) + \frac{\eta}{2^{j+1}}(\|u\| + \|v\|) \right] \end{aligned} \tag{9}$$

for all $x, y, u, v \in \mathcal{X}$. By (9), the sequence $\{\frac{1}{2^j} f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{2^j} f(2^j x, 2^j y)\}$ converges for all $x, y \in \mathcal{X}$. Define $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by $F(x,y) := \lim_{j \to \infty} \frac{1}{2^j} f(2^j x, 2^j y)$ for all $x, y \in \mathcal{X}$. By (7), we have

$$\begin{aligned} &\left\| \frac{1}{2^j} f(2^j(x+y), 2^j(z+w)) - \frac{1}{2^j} f(2^j x, 2^j z) - \frac{1}{2^j} f(2^j y, 2^j w), f(u,v) \right\| \\ &\le \frac{1}{2^j} \left[\varepsilon + 2^{jp}\delta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) + \eta(\|u\| + \|v\|) \right] \end{aligned}$$

for all $x, y, z, w, u, v \in \mathcal{X}$ and all j. Letting $j \to \infty$, we see that F satisfies (1). Setting $l = 0$ and taking $m \to \infty$ in (9), one can obtain the inequality (8). If $G : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is another mapping satisfying (1) and (8), we obtain

$$\begin{aligned} &\|F(x,y) - G(x,y), f(u,v)\| = \frac{1}{2^n} \|F(2^n x, 2^n y) - G(2^n x, 2^n y), f(u,v)\| \\ &\le \frac{1}{2^n} \|F(2^n x, 2^n y) - f(2^n x, 2^n y), f(u,v)\| + \frac{1}{2^n} \|f(2^n x, 2^n y) - G(2^n x, 2^n y), f(u,v)\| \\ &\le \frac{2}{2^n} \left[\frac{\varepsilon}{2} + \frac{2^{np+1}\delta}{2 - 2^p}(\|x\|^p + \|y\|^p) + \frac{\eta(\|u\| + \|v\|)}{2} \right] \\ &\to 0 \ \text{ as } \ n \to \infty \end{aligned}$$

for all $x, y, u, v \in \mathcal{X}$. Hence the mapping F is the unique mapping satisfying (1), as desired. □

Corollary 2. *Let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping, such that*

$$\|f(x+y, z+w) - f(x,z) - f(y,w), f(u,v)\| \le \varepsilon$$

for all $x, y, z, w, u, v \in \mathcal{X}$. Then there exists a unique mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying (1) *such that*

$$\|f(x,y) - F(x,y), f(u,v)\| \le \frac{\varepsilon}{2}$$

for all $x, y, u, v \in \mathcal{X}$.

Proof. Taking $\delta = \eta = 0$ in Theorem 3, we have the desired result. □

In the case $p > 2$ in Theorem 3, one can also obtain the similar result.

3. Solution and Stability of a Bi-Jensen Functional Equation

In [14,15], one can find the concept of quasi-Banach spaces.

Definition 4. *Let* $\mathcal{X}$ *be a real linear space. A quasi-norm is real-valued function on* $\mathcal{X}$ *satisfying the following:*
(i) $\|x\| \geq 0$ *for all* $x \in \mathcal{X}$ *and* $\|x\| = 0$ *if, and only if,* $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $x \in \mathcal{X}$.
(iii) There is a constant $K \geq 1$, *such that* $\|x + y\| \leq K(\|x\| + \|y\|)$ *for all* $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $\mathcal{X}$. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-Banach space is called a *p-Banach space*.

A quasi-norm gives rise to a linear topology on X, namely the least linear topology for which the unit ball $B = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is a neighborhood of zero. This topology is locally bounded, that is, it has a bounded neighborhood of zero. Actually, every locally bounded topology arises in this way.

From now on, assume that $\mathcal{X}$ is a quasi-normed space with quasi-norm $\|\cdot\|_{\mathcal{X}}$ and that $\mathcal{Y}$ is a p-Banach space with p-norm $\|\cdot\|_{\mathcal{Y}}$. Let K be the modulus of concavity of $\|\cdot\|_{\mathcal{Y}}$.

Let $\varphi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and $\psi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be two functions such that

$$\lim_{n\to\infty} \frac{1}{3^n}\varphi(3^n x, 3^n y, z) = 0, \quad \lim_{n\to\infty} \frac{1}{3^n}\psi(3^n x, y, z) = 0 \tag{10}$$

and

$$\lim_{n\to\infty} \frac{1}{3^n}\varphi(x, y, 3^n z) = 0, \quad \lim_{n\to\infty} \frac{1}{3^n}\psi(x, 3^n y, 3^n z) = 0 \tag{11}$$

for all $x, y, z \in \mathcal{X}$, and

$$M(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^{pj}}\varphi(3^j x, 3^j y, z)^p < \infty \tag{12}$$

and

$$N(z, x, y) := \sum_{j=0}^{\infty} \frac{1}{3^{pj}}\psi(z, 3^j x, 3^j y)^p < \infty \tag{13}$$

for all $x, z \in \mathcal{X}$ and all $y \in \{-x, -3x\}$.

We will use the following lemma in order to prove Theorem 4.

Lemma 2 ([16]). *Let* $0 \leq p \leq 1$ *and let* $x_1, x_2, \cdots, x_n$ *be non-negative real numbers. Then*

$$\left(\sum_{j=1}^{n} x_j\right)^p \leq \sum_{j=1}^{n} x_j{}^p.$$

Theorem 4. *Let $0 < p \leq 1$ and suppose that a mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfies the inequalities*

$$\left\|2f\left(\frac{x+y}{2}, z\right) - f(x, z) - f(y, z)\right\|_Y \leq \varphi(x, y, z), \tag{14}$$

$$\left\|2f\left(x, \frac{y+z}{2}\right) - f(x, y) - f(x, z)\right\|_Y \leq \psi(x, y, z) \tag{15}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive-Jensen mapping $J_1 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x, y) - f(0, y) - J_1(x, y)\|_{\mathcal{Y}} \leq \frac{K}{3}[M(x, -x, y) + M(-x, 3x, y)]^{\frac{1}{p}} \tag{16}$$

for all $x, y \in \mathcal{X}$. There exists a unique Jensen-additive mapping $J_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x, y) - f(x, 0) - J_2(x, y)\|_{\mathcal{Y}} \leq \frac{K}{3}[N(x, y, -y) + N(x, -y, 3y)]^{\frac{1}{p}} \tag{17}$$

for all $x, y \in \mathcal{X}$.

Proof. Let $g(x,y) := f(x,y) - f(0,y)$ for all $x, y \in \mathcal{X}$. Then $g(0,y) = 0$ for all $y \in \mathcal{X}$. Letting y by $-x$ in (14), we get

$$\|g(x, z) + g(-x, z)\|_{\mathcal{Y}} \leq \varphi(x, -x, z)$$

for all $x, z \in \mathcal{X}$. Replacing x by $-x$ and y by $3x$ in (14), we have

$$\|2g(x, z) - g(-x, z) - g(3x, z)\|_{\mathcal{Y}} \leq \varphi(-x, 3x, z)$$

for all $x, z \in \mathcal{X}$. By two above inequalities and replacing z by y, we get

$$\|3g(x,y) - g(3x,y)\|_{\mathcal{Y}} \leq K[\varphi(x,-x,y) + \varphi(-x,3x,y)]$$

for all $x, y \in \mathcal{X}$. Thus we have

$$\left\|\frac{1}{3^j}g(3^jx,y) - \frac{1}{3^{j+1}}g(3^{j+1}x,y)\right\|_{\mathcal{Y}} \leq \frac{K}{3^{j+1}}[\varphi(3^jx,-3^jx,y) + \varphi(-3^jx,3^{j+1}x,y)]$$

for all $x, y \in \mathcal{X}$ and all j. For given integer l, m $(0 \leq l < m)$, by Lemma 2, we get

$$\left\|\frac{1}{3^l}g(3^lx,y) - \frac{1}{3^m}g(3^mx,y)\right\|_{\mathcal{Y}}^p \leq \left(\frac{K}{3}\right)^p \sum_{j=l}^{m-1} \frac{1}{3^{pj}}\left[\varphi(3^jx,-3^jx,y)^p + \varphi(-3^jx,3^{j+1}x,y)^p\right] \tag{18}$$

for all $x, y \in \mathcal{X}$. By (12), the sequence $\{\frac{1}{3^j}g(3^jx,y)\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{3^j}g(3^jx,y)\}$ converges for all $x, y \in \mathcal{X}$. Define $J_1 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$J_1(x,y) := \lim_{j\to\infty} \frac{1}{3^j}g(3^jx,y)$$

for all $x, y \in \mathcal{X}$. Putting $l = 0$ and taking $m \to \infty$ in (18), one can obtain the inequality (16). From the definition of J_1, we get

$$3^jJ_1(x,y) = J_1(3^jx,y) \quad \text{and} \quad J_1(0,y) = 0 \tag{19}$$

for all $x, y \in \mathcal{X}$ and all j. By (14), (16) and (19), we gain

$$\begin{aligned}
&\|2J_1(2x,y) - 4J_1(x,y)\| \; Y \\
&= \|2J_1(2x,y) - J_1(3x,y) - J_1(x,y)\| \; Y \\
&\le 3^{-j}\left\|2J_1(3^j \cdot 2x,y) - J_1(3^j \cdot 3x,y) - J_1(3^j x,y)\right\| \; Y \\
&\le 3^{-j}\left[\left\|2J_1(3^j \cdot 2x,y) - 2f(3^j \cdot 2x,y)\right\|_{\mathcal{Y}} + \left\|J_1(3^j \cdot 3x,y) - f(3^j \cdot 3x,y)\right\|_{\mathcal{Y}}\right] \\
&\quad + 3^{-j}\left\|J_1(3^j x,y) - f(3^j x,y)\right\|_{\mathcal{Y}} + 3^{-j}\left\|2f\left(\frac{3^j(3x+x)}{2},y\right) - f(3^j \cdot 3x,y) - f(3^j x,y)\right\| \; Y \\
&\le 2 \cdot 3^{-j-1}K[M(3^j \cdot 2x,\, 3^j(-2x),\, y) + M(3^j \cdot (-2x),\, 3^{j+1} \cdot 2x,\, y)]^{\frac{1}{p}} \\
&\quad + 3^{-j-1}K[M(3^j \cdot 3x,\, 3^j(-3x),\, y) + M(3^j \cdot (-3x),\, 3^{j+1} \cdot 3x,\, y)]^{\frac{1}{p}} \\
&\quad + 3^{-j-1}K[M(3^j x,\, 3^j(-x),\, y) + M(3^j \cdot (-x),\, 3^{j+1} x,\, y)]^{\frac{1}{p}} \\
&\quad + 3^{-j}\varphi(3^j x, 3^{j+1} x, y)
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all j. From this and (19), we obtain

$$2J_1(x,y) = J_1(2x,y) \tag{20}$$

for all $x, y \in \mathcal{X}$. From (12) and (14),

$$\begin{aligned}
&\left\|2J_1\left(\frac{x+y}{2},z\right) - J_1(x,z) - J_1(y,z)\right\| \; Y \\
&= \lim_{j\to\infty} 3^{-j}\left\|2J_1\left(\frac{3^j x + 3^j y}{2},z\right) - J_1(3^j x,z) - J_1(3^j y,z)\right\| \; Y \\
&\le \lim_{j\to\infty} 3^{-j}\varphi(3^j x, 3^j y, z) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. From (20) and the above inequality,

$$J_1(x+y,z) = 2J_1\left(\frac{x+y}{2},z\right) = J_1(x,z) + J_1(y,z)$$

for all $x, y, z \in \mathcal{X}$. Hence

$$J_1(x+y,z) = J_1(x,z) + J_1(y,z)$$

for all $x, y, z \in \mathcal{X}$. That is, J_1 is an additive mapping with respect to the first variable. By (15), we get

$$\left\|\frac{2}{3^j}g\left(3^j x, \frac{y+z}{2}\right) + \frac{1}{3^j}g(3^j x,y) - \frac{1}{3^j}g(3^j x,z)\right\|_{\mathcal{Y}} \le \frac{1}{3^j}\psi(3^j x,y,z)$$

for all $x, y, z \in \mathcal{X}$ and all j. Letting $j \to \infty$ in the above inequality and using (10), J_1 is a Jensen mapping with respect to the second variable. To prove the uniqueness of J_1, let S_1 be another additive-Jensen mapping satisfying (16). Then we obtain

$$\begin{aligned}
&\|2S_1(2x,y) - 4S_1(x,y)\|_{\mathcal{Y}}^p \\
&= \|2S_1(2x,y) - S_1(3x,y) - S_1(x,y)\|_{\mathcal{Y}}^p \\
&= 3^{-jp}\|2S_1(2 \cdot 3^j x,y) - S_1(3 \cdot 3^j x,y) - S_1(3^j x,y)\|Y^p \\
&\le 3^{-jp}\|2S_1(2 \cdot 3^j x,\, y) - 2g(2 \cdot 3^j x,\, y)\|_{\mathcal{Y}}^p \\
&\quad + 3^{-jp}\|S_1(3 \cdot 3^j x,\, y) - g(3 \cdot 3^j x,\, y)\|_Y^p + 3^{-jp}\|S_1(3^j x,\, y) - g(3^j x,\, y)\|_{\mathcal{Y}}^p \\
&\quad + 3^{-jp}\left\|2g\left(3^j \cdot \frac{3x+x}{2},y\right) - g(3 \cdot 3^j x,y) - g(3^j x,y)\right\|_{\mathcal{Y}}^p
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all j. It follows from (16), we have

$$\begin{aligned}
&\|J_1(x,y) - S_1(x,y)\|_{\mathcal{Y}}^p \\
&= \left\| \frac{1}{3^j} J_1(3^j x, y) - \frac{1}{3^j} S_1(3^j x, y) \right\|_{\mathcal{Y}}^p \\
&\le \left\| \frac{1}{3^j} J_1(3^j x, y) - \frac{1}{3^j} f(3^j x, y) + \frac{1}{3^j} f(0,y) \right\|_{\mathcal{Y}}^p + \left\| \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(0,y) - \frac{1}{3^j} S_1(3^j x, y) \right\|_{\mathcal{Y}}^p \\
&\le \frac{2K^p}{3^{p(j+1)}} [M(3^j x, -3^j x, y) + M(-3^j x, 3^{j+1} x, y)]
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all j. Taking $j \to \infty$ in the above inequality and using (12), we get $J_1 = S_1$.

Define $J_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$J_2(x,y) := \lim_{j\to\infty} \frac{1}{3^j} f(x, 3^j y)$$

for all $x, y \in \mathcal{X}$. By the same method in the above arguments, J_2 is a unique Jensen-additive mapping satisfying (17). □

Corollary 3. *Let $0 < p \le 1$ and $\varepsilon, \delta > 0$ be fixed. Suppose that a mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfies the inequalities*

$$\begin{aligned}
\left\| 2f\left(\frac{x+y}{2}, z\right) - f(x, z) - f(y, z) \right\|_{\mathcal{Y}} &\le \varepsilon, \\
\left\| 2f\left(x, \frac{y+z}{2}\right) - f(x, y) - f(x, z) \right\|_{\mathcal{Y}} &\le \delta
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive-Jensen mapping $J_1 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x, y) - f(0, y) - J_1(x, y)\|_{\mathcal{Y}} \le K\varepsilon \left(\frac{2}{3^p - 1} \right)^{\frac{1}{p}}$$

for all $x, y \in \mathcal{X}$. There exists a unique Jensen-additive mapping $J_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x, y) - f(x, 0) - J_2(x, y)\|_{\mathcal{Y}} \le K\delta \left(\frac{2}{3^p - 1} \right)^{\frac{1}{p}}$$

for all $x, y \in \mathcal{X}$.

Proof. Let $\varphi(x, y, z) := \varepsilon$ and $\psi(x, y, z) := \delta$ for all $x, y, z \in \mathcal{X}$. By Theorem 4, we have an additive-Jensen mapping J_1 and a Jensen-additive mapping J_2, as desired. □

From now on, let $\chi : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{1}{4^n} \chi(2^n x, 2^n y, 2^n z, 2^n w) = 0 \tag{21}$$

and

$$L(x,y,z,w) := \sum_{j=0}^{\infty} \frac{1}{4^{pj}} \chi(2^j x, 2^j y, 2^j z, 2^j w)^p < \infty \tag{22}$$

for all $x, y, z, w \in \mathcal{X}$.

Theorem 5. *Let $0 < p \leq 1$ and suppose that a mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfies $f(x, 0) = 0$ and the inequality*

$$\left\| 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w) \right\|_{\mathcal{Y}} \leq \chi(x, y, z, w) \quad (23)$$

for all $x, y, z, w \in \mathcal{X}$. Then the limit $F(x, y) := \lim_{j\to\infty} \frac{1}{4^j} f(2^j x, 2^j y)$ exists for all $x, y \in X$ and the mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is the unique bi-Jensen mapping satisfying

$$\|f(x, y) - f(0, y) - F(x, y)\|_{\mathcal{Y}} \leq \tilde{\chi}(x, y)^{\frac{1}{p}}, \quad (24)$$

where

$$\tilde{\chi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{4^{p(j+1)}} \left[\chi(2^{j+1}x, 0, 2^{j+1}y, 0)^p + \chi(0, 0, 2^{j+1}y, 0)^p \right]$$

for all $x, y \in \mathcal{X}$.

Proof. Replacing x by $2^{j+1}x$ and putting $y = 0$, $z = 2^{j+1}y$, $w = 0$ in (23), we gain

$$\left\| \frac{1}{4^j} f(2^j x, 2^j y) - \frac{1}{4^{j+1}} f(2^{j+1}x, 2^{j+1}y) - \frac{1}{4^{j+1}} f(0, 2^{j+1}y) \right\|_{\mathcal{Y}} \leq \frac{1}{4^{j+1}} \chi(2^{j+1}x, 0, 2^{j+1}y, 0) \quad (25)$$

for all $x, y \in \mathcal{X}$ and all j. Letting $x = 0$ in (25), we get

$$\left\| \frac{1}{4^j} f(0, 2^j y) - \frac{2}{4^{j+1}} f(0, 2^{j+1}y) \right\|_{\mathcal{Y}} \leq \frac{1}{4^{j+1}} \chi(0, 0, 2^{j+1}y, 0) \quad (26)$$

for all $y \in \mathcal{X}$ and all j. By (25) and(26), we have

$$\begin{aligned} &\left\| \frac{1}{4^j} \left[f(2^j x, 2^j y) - f(0, 2^j y) \right] - \frac{1}{4^{j+1}} \left[f(2^{j+1}x, 2^{j+1}y) - f(0, 2^{j+1}y) \right] \right\|_{\mathcal{Y}}^p \\ &\leq \frac{1}{4^{p(j+1)}} \left[\chi(2^{j+1}x, 0, 2^{j+1}y, 0)^p + \chi(0, 0, 2^{j+1}y, 0)^p \right] \end{aligned} \quad (27)$$

for all $x, y \in \mathcal{X}$ and all j. Thus we have

$$\begin{aligned} &\left\| \frac{1}{4^l} \left[f(2^l x, 2^l y) - f(0, 2^l y) \right] - \frac{1}{4^m} [f(2^m x, 2^m y) - f(0, 2^m y)] \right\|_{\mathcal{Y}}^p \\ &\leq \sum_{j=l}^{m-1} \frac{1}{4^{p(j+1)}} \left[\chi(2^{j+1}x, 0, 2^{j+1}y, 0)^p + \chi(0, 0, 2^{j+1}y, 0)^p \right] \end{aligned} \quad (28)$$

for all integers l, m $(0 \leq l < m)$ and all $x, y \in \mathcal{X}$. By (22), the sequence $\{\frac{1}{4^j}[f(2^j x, 2^j y) - f(0, 2^j y)]\}$ is a Cauchy sequence for all $x, y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{4^j}[f(2^j x, 2^j y) - f(0, 2^j y)]\}$ converges for all $x, y \in \mathcal{X}$. So one can define the mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$F(x, y) := \lim_{n\to\infty} \frac{1}{4^n} [f(2^n x, 2^n y) - f(0, 2^n y)] \quad (29)$$

for all $x, y \in \mathcal{X}$. Letting $l = 0$ and taking the limit $m \to \infty$ in (28), we get (24). Now, we show that F is a bi-Jensen mapping.

On the other hand it follows from (22), (23) and (29) that

$$
\begin{aligned}
&\left\|4F\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - F(x, z) - F(x, w) - F(y, z) - F(y, w)\right\|_{\mathcal{Y}}^{p} \\
&= \lim_{n\to\infty} \frac{1}{4^{pn}} \left\|4f\left(\frac{2^n x + 2^n y}{2}, \frac{2^n z + 2^n w}{2}\right) - f(2^n x, 2^n z) - f(2^n x, 2^n w) - f(2^n y, 2^n z)\right. \\
&\qquad \left. - f(2^n y, 2^n w) - 4f\left(0, \frac{2^n z + 2^n w}{2}\right) + 2f(0, 2^n z) + 2f(0, 2^n w)\right\|_{\mathcal{Y}}^{p} \\
&= \lim_{n\to\infty} \frac{1}{4^{pn}} [\chi(2^n x, 2^n y, 2^n z, 2^n w)^p + \chi(0, 0, 2^n z, 2^n w)^p] = 0
\end{aligned}
$$

for all $x, y, z, w \in \mathcal{X}$. Hence the mapping F satisfies (3).

To prove the uniqueness of F, let $G : \mathcal{X} \to \mathcal{Y}$ be another bi-Jensen mapping satisfying (24). It follows from (22) that

$$
\lim_{n\to\infty} \frac{1}{4^{pn}} L(2^n x, 2^n y, 2^n z, 2^n w) = \lim_{n\to\infty} \sum_{j=n}^{\infty} \frac{1}{4^{pj}} \chi(2^j x, 2^j y, 2^j z, 2^j w)^p = 0
$$

for all $x, y, z, w \in \mathcal{X}$. Hence $\lim_{n\to\infty} \frac{1}{4^{pn}} \tilde{\chi}(2^n x, 2^n y) = 0$ for all $x, y \in \mathcal{X}$. It follows from (21), (27) and (29) the above equality that

$$
\begin{aligned}
&\|F(2x, 2y) - 4F(x, y)\|\ Y \\
&= \lim_{n\to\infty} \left\| \frac{1}{4^n} f(2^{n+1} x, 2^{n+1} y) - f(0, 2^{n+1} y) - \frac{1}{4^{n-1}} f(2^n x, 2^n y) + f(0, 2^n y) \right\|\ Y \\
&= 4 \lim_{n\to\infty} \left\| \frac{1}{4^n} [f(2^n x, 2^n y) - f(0, 2^n y)] - \frac{1}{4^{n+1}} [f(2^{n+1} x, 2^{n+1} y) - f(0, 2^{n+1} y)] \right\|\ Y \\
&\leq 4 \lim_{n\to\infty} \frac{1}{4^{p(n+1)}} \left[\chi(2^{n+1} x, 0, 2^{n+1} y, 0)^p + \chi(0, 0, 2^{n+1} y, 0)^p\right] = 0
\end{aligned}
$$

for all $x, y \in \mathcal{X}$. So $F(2x, 2y) = 4F(x, y)$ for all $x, y \in \mathcal{X}$. Thus it follows from (24) and (29) that

$$
\begin{aligned}
\|F(x, y) - G(x, y)\|_{\mathcal{Y}}^{p} &= \lim_{n\to\infty} \frac{1}{4^{pn}} \|f(2^n x, 2^n y) - f(0, 2^n y) - G(2^n x, 2^n y)\|_{\mathcal{Y}}^{p} \\
&\leq \lim_{n\to\infty} \frac{1}{4^{pn}} \tilde{\chi}(2^n x, 2^n y) = 0
\end{aligned}
$$

for all $x, y \in \mathcal{X}$. So $F = G$. □

Corollary 4. *Let $0 < p \leq 1$ and $\varepsilon > 0$ be fixed. Suppose that a mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfies the inequalities*

$$
\left\|4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w)\right\|_{\mathcal{Y}} \leq \varepsilon
$$

for all $x, y, z, w \in \mathcal{X}$. Then there exists a unique bi-Jensen mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$
\|f(x, y) - f(0, y) - F(x, y)\|_{\mathcal{Y}} \leq \varepsilon\left(\frac{2}{4^p - 1}\right)^{\frac{1}{p}}
$$

for all $x, y \in \mathcal{X}$.

Proof. Taking $\chi(x, y, z, w) := \varepsilon$ for all $x, y, z, w \in \mathcal{X}$ in Theorem 5, we obtain $\tilde{\chi}(x, y) = \frac{2\varepsilon^p}{4^p - 1}$ for all $x, y \in \mathcal{X}$. Thus we obtain the estimate value $\tilde{\chi}(x, y)^{\frac{1}{p}} = \varepsilon(\frac{2}{4^p - 1})^{\frac{1}{p}}$ for all $x, y \in \mathcal{X}$. □

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