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Cayley–Klein Lie Bialgebras: Noncommutative Spaces, Drinfel’d Doubles and Kinematical Applications

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Abstract: The Cayley–Klein (CK) formalism is applied to the real algebra $\mathfrak{so}(5)$ by making use of four graded contraction parameters describing, in a unified setting, 81 Lie algebras, which cover the (anti-)de Sitter, Poincaré, Newtonian and Carrollian algebras. Starting with the Drinfel’d–Jimbo real Lie bialgebra for $\mathfrak{so}(5)$ together with its Drinfel’d double structure, we obtain the corresponding CK bialgebra and the CK r -matrix coming from a Drinfel’d double. As a novelty, we construct the (first-order) noncommutative CK spaces of points, lines, 2-planes and 3-hyperplanes, studying their structural properties. By requiring dealing with real structures, we found that there exist 63 specific real Lie bialgebras together with their sets of four noncommutative spaces. Furthermore, we found 14 classical r -matrices coming from Drinfel’d doubles, obtaining new results for the de Sitter $\mathfrak{so}(4, 1)$ and anti-de Sitter $\mathfrak{so}(3, 2)$ as well as for some of their contractions. These geometric results were exhaustively applied onto the $(3 + 1)$ D kinematical algebras, considering not only the usual $(3 + 1)$ D spacetime but also the 6D space of lines. We established different assignments between the geometrical CK generators and the kinematical ones, which convey physical identifications for the CK contraction parameters in terms of the cosmological constant/curvature Λ and the speed of light c . We, finally, obtained four classes of kinematical r -matrices together with their noncommutative spacetimes and spaces of lines, comprising all κ -deformations as particular cases.

Keywords: quantum groups; classical r -matrices; contractions; symmetric homogeneous spaces; Anti-de Sitter; Carroll; Newton–Hooke; kappa-deformation; noncommutative spacetimes; noncommutative spaces of lines

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1. Introduction

The notion of Cayley–Klein (CK) Lie algebras along with their corresponding Lie groups and symmetric homogeneous spaces date back to early studies of projective metrics. In particular, CK Lie groups appear in a natural way within the context of the consideration by Klein that most geometries are, in fact, subgeometries of projective geometry and also in relation to Cayley’s theory of projective metrics [1–3]. However, the complete classification of CK geometries, understood as geometries endowed with a projective metric, was not given by Klein himself. The two-dimensional (2D) geometries were studied under the name of “quadratic geometries” by Poincaré, who followed a modern group theoretical approach.

The classification of CK geometries for an arbitrary dimension N was finally achieved by Sommerville in 1909 [4], where he showed that there exist exactly 3^N different CK geometries in dimension N , each of them corresponding to a different choice of the kind of measure of distance between points, lines, 2-planes, \dots , $(N - 1)$ -hyperplanes, which can

be either of elliptic, parabolic or hyperbolic type [3]. Then, CK groups are just the motion groups of the CK geometries acting as groups of isometries of the symmetric homogeneous CK spaces. In dimension N , such CK groups are semisimple pseudo-orthogonal groups $SO(p, q)$ ($p + q = N + 1$) and some of their contractions, such as the inhomogeneous $ISO(p', q')$ ($p' + q' = N$).

In order to set up the main ideas and the formalism that we shall follow along the whole paper, let us consider the well-known nine 2D CK geometries [3,5–12]. These emerge as the different possibilities for considering the measures of distance between two points and the measure of an angle between two lines, with each of them being either of elliptic, parabolic or hyperbolic type. The CK groups are the simple real Lie groups $SO(3)$ and $SO(2, 1)$, the non-simple inhomogeneous Euclidean $ISO(2)$ and Poincaré $ISO(1, 1)$ and the twice inhomogeneous Galilean $IISO(1)$ (in this notation $ISO(1) \equiv \mathbb{R}$).

The 2D CK geometries are constructed through the coset spaces of the above 3D Lie groups with a precise 1D isotropy subgroup. Early, the usual procedure for describing these geometries made use of hypercomplex numbers with two hypercomplex units ι_1 and ι_2 [2,3,5,6]. We recall that a hypercomplex number is defined by $z := x + \iota y$, where (x, y) are two real coordinates and ι is a hypercomplex unit such that $\iota^2 \in \{-1, +1, 0\}$.

Hence, there are three possible kind of hypercomplex numbers according to the specific unit ι : (1) If $\iota^2 = -1$, then ι is an *elliptical* unit providing the usual complex numbers; (2) If $\iota^2 = +1$, ι is a *hyperbolic* unit yielding the so-called split complex, double or Clifford numbers; and (3) if $\iota^2 = 0$, ι is a *parabolic* unit leading to the dual or Study numbers. Alternatively, 2D CK geometries can also be studied in terms of two real graded contraction parameters ω_1 and ω_2 , which can take positive, negative or zero values [7–12]. By taking into account the above two approaches, we display the specific 2D CK geometries in Table 1, where they are named in their original geometric form [3], as well as in their physical (or kinematical) terminology (second and third rows).

Table 1. The nine two-dimensional Cayley–Klein geometries as homogeneous spaces according to two real graded contraction parameters (ω_1, ω_2) and to two hypercomplex units (ι_1, ι_2) .

Measure of Angle	Measure of Distance		
	Elliptic $\omega_1 > 0 \quad \iota_1^2 = -1$	Parabolic $\omega_1 = 0 \quad \iota_1^2 = 0$	Hyperbolic $\omega_1 < 0 \quad \iota_1^2 = +1$
Elliptic $\omega_2 > 0 \quad \iota_2^2 = -1$	• Spherical $SO(3)/SO(2)$	• Euclidean $ISO(2)/SO(2)$	• Hyperbolic $SO(2, 1)/SO(2)$
Parabolic $\omega_2 = 0 \quad \iota_2^2 = 0$	• Co-Euclidean or Oscillating NH $ISO(2)/\mathbb{R}$	• Galilean $IISO(1)/\mathbb{R}$	• Co-Minkowskian or Expanding NH $ISO(1, 1)/\mathbb{R}$
Hyperbolic $\omega_2 < 0 \quad \iota_2^2 = +1$	• Co-Hyperbolic or Anti-de Sitter $SO(2, 1)/SO(1, 1)$	• Minkowskian $ISO(1, 1)/SO(1, 1)$	• Doubly Hyperbolic or De Sitter $SO(2, 1)/SO(1, 1)$

Consequently, the family of 2D CK geometries contains nine homogeneous spaces of constant curvature: the three classical Riemannian spaces in the first row of Table 1; the three (Newtonian) spaces with a degenerate metric in the second row; and the three pseudo-Riemannian or Lorentzian spaces in the third row.

In the procedure that makes use of the real parameters (ω_1, ω_2) [7–12], the generic CK Lie algebra is denoted by $\mathfrak{so}_{\omega_1, \omega_2}(3) = \text{span}\{J_{01}, J_{02}, J_{12}\}$, which corresponds to a two-parametric family of Lie algebras with commutation relations given by

$$[J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -\omega_2 J_{01}, \quad [J_{01}, J_{02}] = \omega_1 J_{12}. \quad (1)$$

The vanishing of a given parameter ω_m (i.e., $\omega_m \rightarrow 0$) is equivalent to apply an İnönü–Wigner contraction [13]. The CK algebra has a single quadratic Casimir given by

$$\mathcal{C} = \omega_2 J_{01}^2 + J_{02}^2 + \omega_1 J_{12}^2. \quad (2)$$

The 2D CK geometry is then defined as the following coset space between the CK Lie group $\text{SO}_{\omega_1, \omega_2}(3)$ with Lie algebra $\mathfrak{so}_{\omega_1, \omega_2}(3)$ (1) and the isotropy subgroup of a point H , spanned by J_{12} [9]:

$$\mathbb{S}_{[\omega_1], \omega_2}^2 := \text{SO}_{\omega_1, \omega_2}(3)/H, \quad H = \text{SO}_{\omega_2}(2) = \langle J_{12} \rangle. \quad (3)$$

Hence, J_{12} leaves a point O invariant (the origin) acting as the generator of rotations around O , while J_{01} and J_{02} are the generators of translations that move O along two basic directions on the space. The CK geometry (3) has a metric, provided by the Casimir (2), which is of constant (Gaussian) curvature equal to ω_1 and with metric signature given by $\text{diag}(+1, \omega_2)$ (see Table 1).

Spacetimes of constant curvature in $(1+1)$ dimensions arise as particular cases of the CK geometry (3) through different assignments between the geometrical generators J_{ab} and the kinematical ones, which require appropriate relations between the CK parameters (ω_1, ω_2) and physical quantities. Explicitly, let $\{P_0, P_1, K\}$ be the generators of time translations, space translations and boost transformations. Under the particular identification

$$J_{01} \equiv P_0, \quad J_{02} \equiv P_1, \quad J_{12} \equiv K, \quad \omega_1 \equiv -\Lambda, \quad \omega_2 \equiv -c^{-2}, \quad (4)$$

where Λ is the cosmological constant and c is the speed of light, we find that the CK algebra (1) adopts the form

$$[K, P_0] = P_1, \quad [K, P_1] = \frac{1}{c^2} P_0, \quad [P_0, P_1] = -\Lambda K. \quad (5)$$

Thus, under the relations (4), the CK group $\text{SO}_{\omega_1, \omega_2}(3)$ with Lie algebra (5) becomes a kinematical group acting as the group of isometries of six relevant $(1+1)\text{D}$ spacetime models [14] of constant curvature equal to $-\Lambda$, which are all contained in (3). These are the three Lorentzian spacetimes with the metric signature $\text{diag}(+1, -c^{-2})$, mentioned in the third row of Table 1, along with their non-relativistic limit $c \rightarrow \infty$ leading to the three Newtonian spacetimes with degenerate metric $\text{diag}(+1, 0)$ in the second row of Table 1 (NH means Newton–Hooke). All of the kinematical algebras and spacetimes can also be described within the CK framework except for the static algebra [14]; at this dimension, the latter is just the abelian algebra.

In principle, the very same results can be obtained by making use of the formalism in terms of hypercomplex numbers [2,3,5,6] since, roughly speaking, one finds the relations $\iota^2 \sim -\omega$. Nevertheless, in addition to simply dealing with real numbers instead of hypercomplex ones, the main differences between both approaches clearly appear in the pure contracted case corresponding to consider the parabolic unit $\iota^2 = 0$ and to set $\omega = 0$. In particular, the contraction of exponentials of a Lie generator J could give rise to different results; for instance:

$$\exp(\iota^2 x J) \rightarrow 1, \quad \exp(\iota x J) \rightarrow 1 + \iota x J, \quad \exp(\omega x J) \rightarrow 1, \quad \exp(\sqrt{\omega} x J) \rightarrow 1,$$

where x is a real number. We stress that this kind of exponential often appears in quantum groups, and, moreover, terms depending on some $\sqrt{\omega_m}$ will be omnipresent throughout the paper (within Lie bialgebra structures), so that both procedures could be no longer equivalent in a quantum deformation framework. Thus, we shall make use of the graded contraction approach with real parameters ω_m in such manner that a smooth and well-defined $\omega_m \rightarrow 0$ limit of all the expressions that we shall present here will be always feasible.

In addition, we stress that the same CK group $\text{SO}(2, 1)$ appears three times in Table 1, and two of their CK spaces are “similar” (see [15,16] for a very detailed description of these three geometries in terms of hypercomplex numbers). The structure of the three CK

geometries involved, hyperbolic and the (anti-)de Sitter ones, can be better understood by considering not only the usual CK space (3) shown in Table 1 but also their 2D homogeneous spaces of lines, as it was performed in [8,12] and likewise for the Euclidean and Poincaré groups, which appear twice in Table 1.

In arbitrary dimension N , the CK algebra depends on N real graded contraction parameters ω_m ($m = 1, \dots, N$) and is denoted by $\mathfrak{so}_{\omega_1, \dots, \omega_N}(N+1)$. This family comprises 3^N semisimple and non-semisimple real Lie algebras (being some of them isomorphic), which share common geometric and algebraic properties. The signs of the parameters $\omega_m \neq 0$ determine a specific real form $\mathfrak{so}(p, q)$ and when at least one ω_m vanishes the CK algebra becomes a non-semisimple one.

In algebraic terms, a CK algebra can be defined as a graded contracted Lie algebra from $\mathfrak{so}(N+1)$ [17], which keeps the same number of algebraically independent Casimir invariants as in the semisimple case, regardless of the values of ω_m [18]. This definition implies that all of the 3^N particular CK algebras have the same rank (even for the most contracted case with all $\omega_m = 0$), so that they are also known as quasisimple orthogonal algebras [18,19]. From this viewpoint they can be seen as the “closest” contracted algebras to the semisimple ones.

In this respect, we remark that the CK contraction sequence ensures to always obtain a non-trivial quadratic Casimir (like (2)), which, in turn, means that there always exists a non-trivial metric on the ND CK geometry, although degenerate in many cases. This fact explains the absence of the static algebra in the CK family. Obviously, if one goes beyond the CK Lie algebra contraction sequence, then one can obtain the static algebra, and finally arrive at the abelian algebra [20].

From the CK algebra, the corresponding CK Lie group $SO_{\omega_1, \dots, \omega_N}(N+1)$ can be constructed, and the ND CK geometry is defined as the coset space (see (3))

$$\mathbb{S}_{[\omega_1], \omega_2, \dots, \omega_N}^N := SO_{\omega_1, \dots, \omega_N}(N+1) / SO_{\omega_2, \dots, \omega_N}(N), \quad (6)$$

which has a metric of constant (sectional) curvature equal to ω_1 with signature given by $\text{diag}(+1, \omega_2, \dots, \omega_N)$.

In this paper, we focus on the physically relevant dimension $N = 4$, thus covering the $(3+1)$ D spacetimes of constant curvature. Hence, the CK algebra and group will depend on a set of four real graded contraction parameters $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, thus, comprising $3^4 = 81$ specific Lie algebras. For each CK algebra/group, we shall consider four types of symmetric homogeneous spaces: the usual 4D CK space of points (spacetimes) (6) along with the 6D space of lines, 6D space of 2-planes and 4D space of 3-hyperplanes, with all of them of constant curvature and equal to $\omega_1, \dots, \omega_4$. Therefore, in this paper, by a CK geometry, it will be understood the set of these four homogeneous spaces associated with a given Lie algebra in the CK family and not only the usual space of points (6), which is the one commonly considered in the literature.

In this geometrical setting, we initially study CK Lie bialgebras and their associated noncommutative spaces in order to further develop their physical applications. Thus, we, first review the basics on quantum groups that will be used along the paper in Section 2. Secondly, we present a two-fold work in Sections 3–6 with two main but interrelated parts, whose structure, objectives and results are as follows.

1. Starting with the Drinfel'd–Jimbo Lie bialgebra for $\mathfrak{so}(5)$ and also considering its Drinfel'd double structure in Section 3, we obtain the corresponding CK bialgebra along with the classical CK r -matrix coming from a Drinfel'd double in Section 4. As a novelty, we construct, by means of quantum duality, the first-order (in the quantum coordinates) noncommutative CK spaces of points, lines, 2-planes and 3-hyperplanes. We analyse their properties and always require real structures. We found that, finally, there were 63 specific real Lie bialgebras together with their sets of four (first-order) noncommutative spaces, which are summarized in Tables 2 and 3, respectively. Additionally, we found 14 classical r -matrices coming from Drinfel'd

double real structures: there are four cases (I)–(IV) for the simple algebras and 10 more cases for their contractions. In this way, we obtain new results for de Sitter $\mathfrak{so}(4, 1)$ (case (II)) and anti-de Sitter $\mathfrak{so}(3, 2)$ (case (IV)) Drinfel'd doubles and for some of their contractions, which are displayed in Table 4.

2. The above geometric results are exhaustively applied onto the $(3 + 1)$ D kinematical algebras [14] not only considering the usual $(3 + 1)$ D spacetime but also the 6D space of lines; the classical picture for each kinematical algebra is presented in Section 5 and outlined in Table 5. In Section 6, we establish different assignments between the geometrical CK generators and the kinematical ones, which convey appropriate physical identifications for the CK contraction parameters ω in terms of the cosmological constant/curvature Λ and speed of light c . In this process, we obtain four classes of kinematical r -matrices and, for some algebras, also r -matrices coming from Drinfel'd doubles. These classes are called A, B, C and D, matching, in this order, with the above cases (I)–(IV). The resulting kinematical bialgebras are given in Table 6, while their corresponding first-order noncommutative spacetimes and spaces of lines are shown in Table 7. We stress that class C covers the kappa-deformations.

Although, in this work, we do not construct the complete quantum kinematical algebras and their associated full noncommutative spacetimes and spaces of lines, we comment on related known results and open problems in Sections 6.5 and 6.6, respectively. To finish with, several conclusions and a more exhaustive list of open problems, also concerning the geometric CK setting, are drawn in Section 7.

2. Fundamentals on Quantum Groups

In this Section, we review the basic background on quantum groups necessary for the paper along with updates and physical motivation related to the main results here presented. We shall focus on quantum deformations of Lie algebras (with a Hopf algebra structure) along with their connection to Lie bialgebras, Poisson–Lie groups, Poisson homogeneous spaces, noncommutative spaces and Drinfel'd doubles. More details on these topics can be found in [21–26].

2.1. Lie Bialgebras and Quantum Algebras

Let us consider an n D Lie algebra $\mathfrak{g} = \text{span}\{X_1, \dots, X_n\}$ with commutation relations given by

$$[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k. \quad (7)$$

The Lie algebra \mathfrak{g} is endowed with a Lie bialgebra structure (\mathfrak{g}, δ) [27] if there exists a map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ called the *cocommutator* verifying two conditions:

- (i) δ is a 1-cocycle,

$$\delta([X_i, X_j]) = [\delta(X_i), X_j \otimes 1 + 1 \otimes X_j] + [X_i \otimes 1 + 1 \otimes X_i, \delta(X_j)], \quad \forall X_i, X_j \in \mathfrak{g}. \quad (8)$$

- (ii) The dual map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie bracket on the dual Lie algebra \mathfrak{g}^* of \mathfrak{g} .

Therefore, any cocommutator δ can be written in a skew-symmetric form as

$$\delta(X_i) = \sum_{j,k=1}^n f_i^{jk} X_j \otimes X_k, \quad f_i^{jk} = -f_i^{kj}, \quad (9)$$

in such a manner that the antisymmetric factors f_i^{jk} turn out to be the structure constants of the dual Lie algebra $\mathfrak{g}^* = \text{span}\{\hat{x}_1, \dots, \hat{x}_n\}$:

$$[\hat{x}^j, \hat{x}^k] = \sum_{i=1}^n f_i^{jk} \hat{x}^i. \quad (10)$$

The duality between the generators of \mathfrak{g} and \mathfrak{g}^* is determined by a canonical pairing given by the bilinear form

$$\langle \hat{x}^i, X_j \rangle = \delta_j^i, \quad \forall i, j. \quad (11)$$

The cocycle condition (8) leads to the following compatibility equations among the structure constants c_{ij}^k (7) and f_i^{jk} (10):

$$\sum_{k=1}^n f_k^{lm} c_{ij}^k = \sum_{k=1}^n \left(f_i^{lk} c_{kj}^m + f_i^{km} c_{kj}^l + f_j^{lk} c_{ik}^m + f_j^{km} c_{ik}^l \right), \quad \forall i, j, l, m.$$

For some Lie bialgebras, the 1-cocycle δ is coboundary [27], that is, it can be obtained from an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ in the form

$$\delta(X_i) = [X_i \otimes 1 + 1 \otimes X_i, r], \quad \forall X_i \in \mathfrak{g}. \quad (12)$$

The element r is the so-called *classical r -matrix*, which can always be written in a skew-symmetric form

$$r = \sum_{i,j=1}^n r^{ij} X_i \otimes X_j, \quad r^{ij} = -r^{ji}, \quad (13)$$

and must be a solution of the modified classical Yang–Baxter equation

$$[X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i, [[r, r]]] = 0, \quad \forall X_i \in \mathfrak{g}, \quad (14)$$

where $[[r, r]]$ is the Schouten bracket defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \quad (15)$$

such that

$$r_{12} = \sum_{i,j=1}^n r^{ij} X_i \otimes X_j \otimes 1, \quad r_{13} = \sum_{i,j=1}^n r^{ij} X_i \otimes 1 \otimes X_j, \quad r_{23} = \sum_{i,j=1}^n r^{ij} 1 \otimes X_i \otimes X_j.$$

If the Schouten bracket (15) does not vanish for an r -matrix written in the skew-symmetric form (13), then the Lie algebra \mathfrak{g} is endowed with a *quasitriangular* (or standard) Lie bialgebra structure $(\mathfrak{g}, \delta(r))$. The vanishing of the Schouten bracket corresponds to the classical Yang–Baxter equation

$$[[r, r]] = 0, \quad (16)$$

and $(\mathfrak{g}, \delta(r))$ is called *triangular* (or nonstandard) Lie bialgebra.

We remark that the deformation parameter, that we shall denote as z throughout the paper and such that $q = e^z$, is already contained within δ (9) and r (13) in the factors f_i^{jk} and r^{ij} as a global multiplicative constant. This, in turn, means that the non-deformed or “classical” limit $z \rightarrow 0$ (i.e., $q \rightarrow 1$) leads to a trivial coproduct $\delta = 0$ with all $f_i^{jk} \equiv 0$ and classical r -matrix $r = 0$ with all $r^{ij} \equiv 0$, being \mathfrak{g}^* an abelian Lie algebra, thus, with commutative generators \hat{x}^i (10).

A *quantum algebra* $\mathcal{U}_z(\mathfrak{g})$ is a Hopf algebra deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} constructed as formal power series $\mathbb{C}[[z]]$ in a deformation indeterminate parameter z and coefficients in $\mathcal{U}(\mathfrak{g})$ —that is, $\mathcal{U}_z(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) \hat{\otimes} \mathbb{C}[[z]]$. The Hopf algebra structure of $\mathcal{U}_z(\mathfrak{g})$ is determined by the coproduct Δ_z , counit ϵ and antipode γ mappings [24–26]. In particular, the coproduct $\Delta_z : \mathcal{U}_z(\mathfrak{g}) \rightarrow \mathcal{U}_z(\mathfrak{g}) \otimes \mathcal{U}_z(\mathfrak{g})$ must be an algebra homomorphism and fulfil the coassociativity condition

$$(\text{Id} \otimes \Delta_z) \Delta_z = (\Delta_z \otimes \text{Id}) \Delta_z,$$

where Id is the identity map, giving rise to a coalgebra structure $(\mathcal{U}_z(\mathfrak{g}), \Delta_z)$. Once Δ_z is obtained, the remaining maps, ϵ and γ , can directly be deduced from the Hopf algebra axioms providing the complete Hopf algebra structure. Hence, hereafter, we shall only focus on the coalgebra structure of $\mathcal{U}_z(\mathfrak{g})$ assuming the existence of the corresponding counit and antipode.

The remarkable point is that any quantum algebra $\mathcal{U}_z(\mathfrak{g})$ is determined at the first-order in z by a Lie bialgebra (\mathfrak{g}, δ) . Explicitly, if we write the coproduct Δ_z as a formal power series in z , the cocommutator δ (9) is just the skew-symmetric part of the first-order term Δ_1 in z , namely

$$\begin{aligned}\Delta_z(X_i) &= \Delta_0(X_i) + \Delta_1(X_i) + o[z^2], \\ \Delta_0(X_i) &= X_i \otimes 1 + 1 \otimes X_i, \\ \delta(X_i) &= \Delta_1(X_i) - \sigma \circ \Delta_1(X_i),\end{aligned}\tag{17}$$

where σ is the flip operator $\sigma(X_i \otimes X_j) = X_j \otimes X_i$ and Δ_0 is called the *primitive* (non-deformed) coproduct. Therefore, each Lie bialgebra (\mathfrak{g}, δ) determines a quantum deformation $(\mathcal{U}_z(\mathfrak{g}), \Delta_z)$, and the equivalence classes (under automorphisms) of Lie bialgebra structures on \mathfrak{g} will provide all its possible quantum algebras.

We recall that, for semisimple Lie algebras, all their Lie bialgebra structures are coboundaries, so that all their possible quantum deformations are determined by classical r -matrices. The paradigmatic type of them is provided by the so-called Drinfel'd–Jimbo deformations [28–30]; the corresponding Drinfel'd–Jimbo r -matrix for the compact real form $\mathfrak{so}(5)$ [31] will be our starting point for the detailed study of the CK Lie bialgebras, which will be performed in Sections 3 and 4.

However, even for semisimple Lie algebras, the determination of all the Lie bialgebra structures through classical r -matrices is a cumbersome task and, in fact, there are only classifications for the Lorentz algebra $\mathfrak{so}(3, 1)$ [32] and for the related real forms $\mathfrak{so}(4)$ and $\mathfrak{so}(2, 2)$ [33,34]; from a kinematical viewpoint, these classifications for $\mathfrak{so}(3, 1)$ and $\mathfrak{so}(2, 2)$ correspond to $(2 + 1)\text{D}$ (anti-)de Sitter r -matrices [35]. Therefore, in the $(3 + 1)\text{D}$ case, which is the one that we shall consider throughout this paper, there are no such classifications for the simple algebras $\mathfrak{so}(p, q)$ with $p + q = 5$, although we remark that there are some partial results.

In particular, it was shown in [36] that there exist only two two-parametric classical r -matrices for the (anti-)de Sitter algebras $\mathfrak{so}(4, 1)$ and $\mathfrak{so}(3, 2)$ keeping the primitive (undeformed) time translation generator and a single rotation generator. The classification of their r -matrices, which preserve a Lorentz $\mathfrak{so}(3, 1)$ sub-bialgebra was very recently obtained in [37] starting with the former full $(2 + 1)\text{D}$ classification [32,33].

All the Lie bialgebra structures for inhomogeneous pseudo-orthogonal algebras $\mathfrak{iso}(p, q)$ with $p + q \geq 3$ are also coboundaries [38,39]. Their classification for the $(3 + 1)\text{D}$ Poincaré algebra $\mathfrak{iso}(3, 1)$ was obtained in [38,40,41], while for the $(2 + 1)\text{D}$ Poincaré algebra $\mathfrak{iso}(2, 1)$ and 3D Euclidean $\mathfrak{iso}(3)$, it was performed in [42]; the latter classifications have also been recovered in [43] by contracting the $(2 + 1)\text{D}$ (anti-)de Sitter and $\mathfrak{so}(4)$ r -matrices given in [33,34].

Concerning other kinematical algebras, we also recall that the obtention of Lie bialgebras mainly covers low-dimensional cases such as the $(1 + 1)\text{D}$ Galilei algebra (isomorphic to the Heisenberg–Weyl algebra \mathfrak{h}_3) [44–47], the 2D Euclidean algebra $\mathfrak{iso}(2)$ [48], the $(1 + 1)\text{D}$ centrally extended Galilei algebra [49–51] and the $(1 + 1)\text{D}$ centrally extended Poincaré algebra in the light-cone basis (isomorphic to the oscillator algebra \mathfrak{h}_4) [52,53]. With the exception of the latter, all of them have both coboundary and non-coboundary Lie bialgebra structures.

In general, for solvable and nilpotent Lie algebras, many of their Lie bialgebra structures are non-coboundaries; in this respect, see [54,55] and the references therein for the classification of 3D Lie bialgebras. Finally, very recently, the classification of 4D inde-

composable coboundary Lie bialgebras was carried out in [56], which showed how the difficulties of this task grow when the dimensions of the Lie bialgebras increase.

2.2. Quantum Groups and Noncommutative Spaces

Let us consider a quantum algebra $(\mathcal{U}_z(\mathfrak{g}), \Delta_z)$ with underlying Lie bialgebra (\mathfrak{g}, δ) and let G be the Lie group with Lie algebra \mathfrak{g} . A *quantum group* (G_z, Δ_{G_z}) is a noncommutative algebra of functions on G defined as the dual Hopf algebra to the quantum algebra $(\mathcal{U}_z(\mathfrak{g}), \Delta_z)$. Explicitly, let m_{G_z} and m_z be the noncommutative products in G_z and $\mathcal{U}_z(\mathfrak{g})$, respectively. The duality between the Hopf algebras $(G_z, m_{G_z}, \Delta_{G_z})$ and $(\mathcal{U}_z(\mathfrak{g}), m_z, \Delta_z)$ is established by means of a canonical pairing $\langle \cdot, \cdot \rangle : G_z \times \mathcal{U}_z(\mathfrak{g}) \rightarrow \mathbb{R}$ such that

$$\langle m_{G_z}(f \otimes g), X \rangle = \langle f \otimes g, \Delta_z(X) \rangle, \quad (18)$$

$$\langle \Delta_{G_z}(f), X \otimes Y \rangle = \langle f, m_z(X \otimes Y) \rangle, \quad (19)$$

where $X, Y \in \mathcal{U}_z(\mathfrak{g})$, $f, g \in G_z$ and $\langle f \otimes g, X \otimes Y \rangle = \langle f, X \rangle \langle g, Y \rangle$.

The duality relation (18) implies that the noncommutative product m_{G_z} in the quantum group G_z is defined by the coproduct Δ_z in the quantum algebra $\mathcal{U}_z(\mathfrak{g})$ and, conversely, the expression (19) implies that the coproduct Δ_{G_z} in G_z is given by the noncommutative product m_z in $\mathcal{U}_z(\mathfrak{g})$. By taking into account that the first-order term in z of the coproduct Δ_z is defined by the cocommutator δ (17), a straightforward consequence of the above Hopf algebra duality is that the commutation relations for the quantum group G_z at the first-order in the quantum (noncommutative) coordinates \hat{x}^i are given by the dual map δ^* of δ (9), that is, with fundamental Lie brackets (10).

Furthermore, each quantum group (G_z, Δ_{G_z}) can be associated with a *Poisson–Lie group* (G, Π) , with Poisson structure Π , and the latter with a unique Lie bialgebra structure (\mathfrak{g}, δ) . In particular, it is well known [27] that Poisson–Lie structures on a connected and simply connected Lie group G are in one-to-one correspondence with Lie bialgebra structures. Hence, quantum groups are quantizations of Poisson–Lie groups, that is, quantizations of the Poisson–Hopf algebras of multiplicative Poisson structures on Lie groups [24,25,30]. In the case of coboundary Lie bialgebras $(\mathfrak{g}, \delta(r))$, coming from a skew-symmetric classical r -matrix (13), the Poisson structure Π of the Poisson–Lie group is given by the so-called Sklyanin bracket [24,30]

$$\{f, g\} = \sum_{i,j=1}^n r^{ij} (\nabla_i^L f \nabla_j^L g - \nabla_i^R f \nabla_j^R g), \quad f, g \in C^\infty(G), \quad (20)$$

where ∇_i^L and ∇_i^R are left- and right-invariant vector fields on G .

Next, a *Poisson homogeneous space* of a Poisson–Lie group (G, Π) is a Poisson manifold (M, π) endowed with a transitive group action $\triangleright : G \times M \rightarrow M$, which is a Poisson map with respect to the Poisson structure on the manifold M and the product $\Pi \oplus \pi$ of the Poisson structures on G and M . In this paper, we shall consider that the manifold M is an ℓ D homogeneous space

$$M = G/H \quad (21)$$

of a Lie group G (the motion group of M) with isotropy subgroup H whose Lie algebras are \mathfrak{g} and \mathfrak{h} , respectively. Moreover, throughout this paper, we will be interested in pointed Poisson homogeneous spaces, i.e., Poisson homogeneous spaces in which the origin is fixed, and we will not study how the Poisson structure is modified when this origin is changed. The Lie algebra \mathfrak{g} , understood as a vector space, can be written as the sum of two vector subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}.$$

The generators of \mathfrak{h} leave a point of M invariant, which is taken as the origin O of the space, and thus they play the role of rotations around O , while the ℓ generators belonging to \mathfrak{t} move O along ℓ basic directions, behaving as translations on M . The group parameters

(u^1, \dots, u^ℓ) of the generators of \mathfrak{t} lead to ℓ coordinates of M and they span the annihilator \mathfrak{h}_\perp of the vector subspace \mathfrak{h} in the dual Lie algebra \mathfrak{g}^* [57].

In principle, the Lie algebra \mathfrak{g} of G may admit several coboundary Lie bialgebra structures $(\mathfrak{g}, \delta(r))$ —that is, different classical r -matrices. Then, a particular Poisson homogeneous space (M, π) can be constructed by endowing the motion group G with the Poisson–Lie structure Π (20) for a given classical r -matrix and the homogeneous space M (21) with a Poisson bracket π that has to be compatible with the group action $\triangleright : G \times M \rightarrow M$. Therefore, according to the possible classical r -matrices of \mathfrak{g} , it follows that the Lie group G may be endowed with several Poisson–Lie structures Π (20), each of them leading to a different Poisson homogeneous space [58].

A distinguished type of Poisson homogeneous space is that in which the Poisson bracket π is obtained as the canonical projection on M with the coordinates (u^1, \dots, u^ℓ) of the Poisson–Lie bracket Π [57,59] (see also [60,61]). In terms of the underlying Lie bialgebra, this requirement corresponds to imposing the so-called *coisotropy condition* for the cocommutator δ with respect to the isotropy subalgebra \mathfrak{h} of H given by [57,59]

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \quad (22)$$

A particular and very restrictive case of the above condition is for the subalgebra \mathfrak{h} to be a sub-Lie bialgebra,

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}, \quad (23)$$

which implies that the Poisson homogeneous space is constructed through an isotropy subgroup H , which is a Poisson subgroup (H, Π) of (G, Π) . Furthermore, since the quantum group (G_z, Δ_{G_z}) is the quantization of the Poisson–Lie group (G, Π) , the quantization of a coisotropic Poisson homogeneous space (M, π) fulfilling (22) provides a quantum homogeneous space M_z with the quantum coordinates $(\hat{u}^1, \dots, \hat{u}^\ell)$, onto which the quantum group G_z co-acts covariantly [62]. The coisotropy condition (22) ensures that the commutation relations that define M_z at the first-order in all the quantum coordinates close a Lie subalgebra, which is simply the annihilator \mathfrak{h}_\perp of \mathfrak{h} on the dual Lie algebra \mathfrak{g}^* , and such relations determine a Lie subalgebra of \mathfrak{g}^* (10). Generically, M_z is called a *noncommutative space*.

Probably, the best known and most studied example of noncommutative spaces is the so-called κ -Minkowski spacetime coming from the κ -Poincaré algebra [63–69] where κ is the quantum deformation parameter, which is proportional to the Planck mass and here related to z as $\kappa = z^{-1}$. The quantum algebra $\mathcal{U}_\kappa(\mathfrak{g})$ and quantum group G_z correspond to the κ -Poincaré algebra and κ -Poincaré group. In this case, the underlying homogeneous space (21) is the flat $(3+1)$ D Minkowski spacetime constructed as the coset space of the Poincaré group $G = \text{ISO}(3, 1)$ with the Lorentz isotropy subgroup $H = \text{SO}(3, 1)$:

$$\mathbf{M}^{3+1} = \text{ISO}(3, 1)/\text{SO}(3, 1). \quad (24)$$

Thus, the dimension is $\ell = 4$, and the coordinates (u^1, \dots, u^4) are identified with the time and spatial ones (x^0, x^i) ($i = 1, 2, 3$). The κ -Poincaré classical r -matrix [67] provides a quasitriangular quantum deformation of the Poincaré algebra [64–66] such that the Lorentz subalgebra $\mathfrak{h} = \mathfrak{so}(3, 1)$ fulfils the coisotropy condition (22), thus, giving rise to the subalgebra \mathfrak{h}_\perp whose generators are the quantum coordinates (\hat{x}^0, \hat{x}^i) dual to the time and space translation generators. The complete quantization of \mathfrak{h}_\perp is the κ -Minkowski spacetime \mathbf{M}_κ^{3+1} , which is defined by the commutation relations given by [67]:

$$[\hat{x}^0, \hat{x}^i] = -\frac{1}{\kappa} \hat{x}^i, \quad [\hat{x}^i, \hat{x}^j] = 0, \quad i, j = 1, 2, 3, \quad (25)$$

which are covariant under the κ -Poincaré quantum group [69]. We remark that \mathbf{M}_κ^{3+1} is a linear algebra that coincides exactly with the one obtained through the Sklyanin bracket (20) of the underlying classical r -matrix [67], which provides the (linear) Poisson homogeneous

spacetime. Therefore, no higher-order terms in the classical and quantum coordinates arise. By contrast, when the κ -deformation is applied to a curved manifold instead of (24), higher-order terms in the coordinates appear in the Poisson homogeneous spacetime, so that the corresponding quantization is not straightforward at all, as the recent constructions of the κ -noncommutative (anti-)de Sitter [70], Newtonian and Carrollian [71] spacetimes explicitly show.

It is worth stressing that the κ -Minkowski spacetime \mathbf{M}_κ^{3+1} (25) has become the paradigmatic noncommutative space in the same way that Drinfel'd–Jimbo deformations are for quantum algebras; in fact, we recall that the κ -Poincaré algebra was formerly obtained as a real quantum algebra coming from a contraction of the Drinfel'd–Jimbo deformation of $\mathfrak{sp}(4, \mathbb{C})$ [63,64].

Among other issues and within the vast literature, let us mention that κ -Minkowski space along with the κ -Poincaré algebra have been studied in relation to noncommutative differential calculi [72,73], wave propagation on noncommutative spacetimes [74], deformed or doubly special relativity at the Planck scale [75–80], noncommutative field theory [81–83], representation theory on Hilbert spaces [84,85], generalized κ -Minkowski spacetimes through twisted κ -Poincaré deformations [86,87], deformed dispersion relations [88–90], curved momentum spaces [91–95], relative locality phenomena [96], star products [97], deformed phase spaces [98], noncommutative spaces of worldlines [99,100] and light cones [101] (in all cases see the references therein).

In Section 6.3, we shall recover the κ -Minkowski space \mathbf{M}_κ^{3+1} (25) and the κ -Poincaré Lie bialgebra as a particular case of “time-like” deformations within the CK family of Lie bialgebras, a fact that is already well known [31,102]. However, as we shall show in Section 4.3, what is a striking point is that a formally similar structure to \mathbf{M}_κ^{3+1} arises as the first-order noncommutative CK space of points, which is shared by 63 CK bialgebras. Moreover, the complete (in all orders in the quantum coordinates) noncommutative CK space of points is kept linear and shared by 27 CK bialgebras.

Consequently, a linear noncommutative space similar to (25) is somewhat “ubiquitous”, which, in turn, suggests that additional “structures” should be taken into account. In fact, this is one of the main aims of this paper, and, as a novel result, we shall explicitly show in Section 4.3 that the consideration of other noncommutative spaces beyond the space of points (kinematically, spacetimes) associated with a given quantum algebra (namely, noncommutative spaces of lines, 2-planes and 3-hyperplanes) does allow one to distinguish mathematical and physical properties between two quantum algebras with the same underlying noncommutative space of points.

2.3. Drinfel'd Double Structures

Let us assume that the dimension of the Lie algebra \mathfrak{g} is the even $n = 2d$. In this case, \mathfrak{g} is a Lie algebra of a Drinfel'd double group [30] if there exists a basis $\{Y_1, \dots, Y_d, y^1, \dots, y^d\}$ of \mathfrak{g} such that the commutation relations (7) can be written as

$$\begin{aligned} [Y_\alpha, Y_\beta] &= \sum_{\gamma=1}^d C_{\alpha\beta}^\gamma Y_\gamma, & [y^\alpha, y^\beta] &= \sum_{\gamma=1}^d F_\gamma^{\alpha\beta} y^\gamma, \\ [y^\alpha, Y_\beta] &= \sum_{\gamma=1}^d (C_{\beta\gamma}^\alpha y^\gamma - F_\beta^{\alpha\gamma} Y_\gamma). \end{aligned} \quad (26)$$

Hence, \mathfrak{g} can be split into two Lie subalgebras

$$\mathfrak{g}_1 = \text{span}\{Y_1, \dots, Y_d\}, \quad \mathfrak{g}_2 = \text{span}\{y^1, \dots, y^d\}$$

with the structure constants $C_{\alpha\beta}^\gamma$ and $F_\gamma^{\alpha\beta}$, respectively. Both subalgebras are dual to each other, $\mathfrak{g}_2^* = \mathfrak{g}_1$, by means of the duality defined with respect to the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ given by

$$\langle Y_\alpha, Y_\beta \rangle = 0, \quad \langle y^\alpha, y^\beta \rangle = 0, \quad \langle y^\alpha, Y_\beta \rangle = \delta_\beta^\alpha, \quad \forall \alpha, \beta, \quad (27)$$

which is “associative” or invariant in the sense that

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle, \quad \forall X, Y, Z \in \mathfrak{g}.$$

The triple $(\mathfrak{g}_1, \mathfrak{g}_2 = \mathfrak{g}_1^*, \mathfrak{g})$ is called a Manin triple, and the Drinfel’d double Lie group is the unique connected and simply connected Lie group G with Lie algebra \mathfrak{g} . Therefore, the Lie algebra \mathfrak{g} , verifying (26), is the double Lie algebra of \mathfrak{g}_1 and of its dual algebra $\mathfrak{g}_1^* = \mathfrak{g}_2$.

By construction, each Drinfel’d double structure for \mathfrak{g} has a canonical classical r -matrix

$$r_{\text{can}} = \sum_{\alpha=1}^d y^\alpha \otimes Y_\alpha, \quad (28)$$

which is a solution of the classical Yang–Baxter Equation (16). Moreover, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of \mathfrak{g} always has a quadratic Casimir element given by

$$C = \frac{1}{2} \sum_{\alpha=1}^d (y^\alpha Y_\alpha + Y_\alpha y^\alpha),$$

which is directly related to the bilinear form (27). The tensorized form of C reads

$$\Omega = \frac{1}{2} \sum_{\alpha=1}^d (y^\alpha \otimes Y_\alpha + Y_\alpha \otimes y^\alpha), \quad (29)$$

which is ad-invariant under the action of \mathfrak{g} , that is

$$[X \otimes 1 + 1 \otimes X, \Omega] = 0, \quad \forall X \in \mathfrak{g}.$$

The element Ω leads to a skew-symmetric classical r -matrix for the Drinfel’d double Lie algebra \mathfrak{g} from the canonical one (28) in the form:

$$r_D = r_{\text{can}} - \Omega = \frac{1}{2} \sum_{\alpha=1}^d y^\alpha \wedge Y_\alpha, \quad (30)$$

which is a solution of the modified classical Yang–Baxter Equation (14) (its Schouten bracket does not vanish now), and thus r_D defines a quasitriangular or standard quantum deformation of \mathfrak{g} with a coboundary Lie bialgebra $(\mathfrak{g}, \delta_D(r_D))$ determined by a cocommutator δ_D through the relation (12).

Concerning the Drinfel’d–Jimbo quantum deformations of semisimple algebras [28–30], it is known that they are closely related to quantum deformations of Drinfel’d doubles, that is, quantum doubles, in such a manner that they are “almost” but not strictly speaking quantum doubles [24]. Nevertheless, it is remarkable that proper Drinfel’d double structures for the four Cartan series of semisimple Lie algebras on \mathbb{C} have been obtained in [103,104] by enlarging the Lie algebras with an appropriate number of central extensions.

From a physical viewpoint, it is worth stressing that Drinfel’d double structures are naturally related to $(2+1)$ D gravity, which is a quite different theory from the full $(3+1)$ D one [105,106]. In particular, $(2+1)$ D gravity is a topological theory that admits a description as a Chern–Simons theory with the gauge group given by the group of isometries of the corresponding spacetime of constant curvature [107,108]. The phase space structure of

(2 + 1)D gravity is related to the moduli space of flat connections on a Riemann surface, the symmetries of which are given by certain Poisson–Lie groups [109,110] such that the Poisson structure on this space admits a description in terms of coboundary Lie bialgebras associated with the gauge group.

Hence, quantum group symmetries arise as the quantum counterparts of the (semi-classical) Poisson–Lie symmetries of the classical theory. The essential fact in the (2 + 1)D gravity framework is that the relevant quantum group symmetries are those coming from some classical r -matrices corresponding to Drinfel’d double structures [111–117], which ensures that the Fock–Rosly condition [110] is fulfilled. The symmetric component of such admissible classical r -matrices, which is just the element Ω (29) when these are written in the symmetric form (28), must be dual to the Ad-invariant symmetric bilinear form in the Chern–Simons action.

As a consequence, the κ -Poincaré and κ -(anti-)de Sitter symmetries are not compatible [111,112] with the Chern–Simons formulation of (2 + 1)D gravity. Furthermore, the Chern–Simons approach to non-relativistic (2 + 1)D quantum gravity has also been developed in [118,119] by making use of a two-fold central extension of the Galilei [120] and Newton–Hooke algebras, and their full quantum deformation was obtained in [121]. Additionally, Drinfel’d doubles also play a prominent role in the state sum or spin foam models for (2 + 1)D gravity as shown in [122,123] in the context of the Turaev–Viro model and invariant.

We recall that the classifications of non-isomorphic 4D and 6D real Drinfel’d double structures were carried out in [124] and [125], respectively, while their Hopf algebra quantizations were constructed in [126]. From these results, and also from [54], there were obtained the classifications of Drinfel’d double structures for the (2 + 1)D (anti-)de Sitter algebras in [115], (2 + 1)D Poincaré algebra and centrally extended (1 + 1)D Poincaré algebra in [127] and 3D Euclidean algebra in [128].

In contrast, results concerning Drinfel’d double structures in the (3 + 1)D case are very scarce, only covering the real $\mathfrak{so}(5)$ and anti-de Sitter $\mathfrak{so}(3, 2)$ algebras given in [129]. In this respect, we advance that, in Section 4.4, we shall obtain two new classical r -matrices coming from Drinfel’d doubles: one for the de Sitter $\mathfrak{so}(4, 1)$ and another for the anti-de Sitter $\mathfrak{so}(3, 2)$.

3. The Drinfel’d–Jimbo Lie Bialgebra for $\mathfrak{so}(5)$

Let us consider the real orthogonal Lie algebra $\mathfrak{so}(5)$ with generators $\{J_{ab}\}$ ($a < b$; $a, b = 0, 1, \dots, 4$) fulfilling the Lie brackets

$$[J_{ab}, J_{ac}] = J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = J_{ab}, \quad a < b < c, \quad (31)$$

and such that those commutators involving four different indices are equal to zero. The universal enveloping algebra of the Lie algebra $\mathfrak{so}(5)$ is endowed with two (second- and fourth-order) Casimir operators [18,130]. The quadratic one, coming from the Killing–Cartan form, is given by

$$\mathcal{C} = \sum_{0 \leq a < b \leq 4} J_{ab}^2. \quad (32)$$

A fine grading group $\mathbb{Z}_2^{\otimes 4}$ of $\mathfrak{so}(5)$ is spanned by the four commuting involutive automorphisms $\Theta^{(m)}$ ($m = 1, \dots, 4$) of (31) defined by [17,20]:

$$\Theta^{(m)}(J_{ab}) := \begin{cases} J_{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ -J_{ab}, & \text{if } a < m \leq b. \end{cases} \quad (33)$$

Each involution $\Theta^{(m)}$ provides a Cartan decomposition of $\mathfrak{so}(5)$ in invariant and anti-invariant subspaces denoted $\mathfrak{h}^{(m)}$ and $\mathfrak{t}^{(m)}$ respectively:

$$\mathfrak{so}(5) = \mathfrak{h}^{(m)} \oplus \mathfrak{t}^{(m)}, \quad m = 1, \dots, 4. \quad (34)$$

These subspaces verify that

$$[\mathfrak{h}^{(m)}, \mathfrak{h}^{(m)}] \subset \mathfrak{h}^{(m)}, \quad [\mathfrak{h}^{(m)}, \mathfrak{t}^{(m)}] \subset \mathfrak{t}^{(m)}, \quad [\mathfrak{t}^{(m)}, \mathfrak{t}^{(m)}] \subset \mathfrak{h}^{(m)}, \quad (35)$$

where $\mathfrak{h}^{(m)}$ is a Lie subalgebra such that

$$\mathfrak{h}^{(1)} = \mathfrak{so}(4), \quad \mathfrak{h}^{(2)} = \mathfrak{so}(2) \oplus \mathfrak{so}(3), \quad \mathfrak{h}^{(3)} = \mathfrak{so}(3) \oplus \mathfrak{so}(2), \quad \mathfrak{h}^{(4)} = \mathfrak{so}(4), \quad (36)$$

while the vector subspace $\mathfrak{t}^{(m)}$ is not a subalgebra.

A faithful matrix representation of $\mathfrak{so}(5)$, $\rho : \mathfrak{so}(5) \rightarrow \text{End}(\mathbb{R}^5)$, is given by

$$\rho(J_{ab}) = -e_{ab} + e_{ba}, \quad (37)$$

where e_{ab} is the 5×5 matrix with a single non-zero entry 1 at row a and column b ($a, b = 0, 1, \dots, 4$), fulfilling the orthogonal matrix condition

$$\rho(J_{ab})^T \mathbf{I} + \mathbf{I} \rho(J_{ab}) = 0, \quad \mathbf{I} = \text{diag}(1, 1, 1, 1, 1), \quad (38)$$

where $\rho(J_{ab})^T$ is the transpose matrix of $\rho(J_{ab})$.

3.1. Symmetric Homogeneous Spaces

According to each automorphism $\Theta^{(m)}$ (33) and its associated Cartan decomposition (34), we construct four symmetric homogeneous spaces, of the type (21), as the coset spaces [131–133]

$$\mathbb{S}^{(m)} = \text{SO}(5)/H^{(m)}, \quad m = 1, \dots, 4, \quad (39)$$

where $H^{(m)}$ is the isotropy subgroup with Lie algebra $\mathfrak{h}^{(m)}$ in (34) (see (36)). We briefly describe their structure.

1. *4D space of points.* We write the ten generators of $\mathfrak{so}(5)$ in array form, and the decomposition (34) for $\Theta^{(1)}$ gives

$$\begin{array}{c|ccc} J_{01} & J_{02} & J_{03} & J_{04} \\ \hline & J_{12} & J_{13} & J_{14} \\ & & J_{23} & J_{24} \\ & & & J_{34} \end{array} \quad (40)$$

where the four generators in the rectangle span the subspace $\mathfrak{t}^{(1)}$. Hence, we obtain the coset space

$$\mathbb{S}^{(1)} = \text{SO}(5)/\text{SO}(4), \quad H^{(1)} \equiv \text{SO}(4) = \langle J_{12}, J_{13}, J_{14}, J_{23}, J_{24}, J_{34} \rangle,$$

which is identified with the symmetric homogeneous space of points. The subgroup $H^{(1)}$ is the isotropy (or stabilizer) group of a point, which is taken as the origin in $\mathbb{S}^{(1)}$ so that its generators play the role of rotations on $\mathbb{S}^{(1)}$ (leaving the origin invariant), while the generators of $\mathfrak{t}^{(1)}$ play the role of translations on $\mathbb{S}^{(1)}$ (so moving the origin along four basic directions).

2. *6D space of lines.* The decomposition (34) for $\Theta^{(2)}$ can be represented as

$$\begin{array}{c|ccc} J_{01} & J_{02} & J_{03} & J_{04} \\ \hline & J_{12} & J_{13} & J_{14} \\ & & J_{23} & J_{24} \\ & & & J_{34} \end{array} \quad (41)$$

where now the six generators in the rectangle span the subspace $\mathfrak{t}^{(2)}$. We find the coset space

$$\mathbb{S}^{(2)} = \text{SO}(5)/(\text{SO}(2) \otimes \text{SO}(3)), \quad H^{(2)} \equiv \text{SO}(2) \otimes \text{SO}(3) = \langle J_{01} \rangle \otimes \langle J_{23}, J_{24}, J_{34} \rangle,$$

which is interpreted as the symmetric homogeneous space of lines. The subgroup $H^{(2)}$ is the isotropy group of a line (the origin in $\mathbb{S}^{(2)}$), while the six generators in $\mathfrak{t}^{(2)}$ play the role of translations on $\mathbb{S}^{(2)}$ (moving the origin-line).

3. *6D space of 2-planes.* The decomposition (34) for $\Theta^{(3)}$ is displayed as

$$\begin{array}{cc|cc} J_{01} & J_{02} & J_{03} & J_{04} \\ & J_{12} & J_{13} & J_{14} \\ & & J_{23} & J_{24} \\ \hline & & & J_{34} \end{array} \quad (42)$$

where the six generators in the rectangle span the subspace $\mathfrak{t}^{(3)}$. The coset space reads

$$\mathbb{S}^{(3)} = \text{SO}(5)/(\text{SO}(3) \otimes \text{SO}(2)), \quad H^{(3)} \equiv \text{SO}(3) \otimes \text{SO}(2) = \langle J_{01}, J_{02}, J_{12} \rangle \otimes \langle J_{34} \rangle,$$

which corresponds to the symmetric homogeneous space of 2-planes. The six generators in $\mathfrak{t}^{(3)}$ play the role of translations on $\mathbb{S}^{(3)}$, while $H^{(3)}$ is the isotropy group of a 2-plane.

4. *4D space of 3-hyperplanes.* Finally, the decomposition (34) for $\Theta^{(4)}$ yields

$$\begin{array}{ccc|c} J_{01} & J_{02} & J_{03} & J_{04} \\ & J_{12} & J_{13} & J_{14} \\ & & J_{23} & J_{24} \\ & & & J_{34} \end{array} \quad (43)$$

where the four generators in the rectangle span the subspace $\mathfrak{t}^{(4)}$. The coset space is given by

$$\mathbb{S}^{(4)} = \text{SO}(5)/\text{SO}(4), \quad H^{(4)} \equiv \text{SO}(4) = \langle J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23} \rangle,$$

which is interpreted as the symmetric homogeneous space of 3-hyperplanes. The four generators in $\mathfrak{t}^{(4)}$ play the role of translations on $\mathbb{S}^{(4)}$, while $H^{(4)}$ is the isotropy group of a 3-hyperplane.

Some remarks are in order. First, the four spaces (39) are of positive constant curvature in the sense that their sectional curvature K is equal to $+1$ and they are endowed with a Riemmanian metric (thus with a positive definite signature). Secondly, the 4D spaces of points and 3-hyperplanes are of rank 1, that is, there is a single invariant under the action of $\text{SO}(5)$ for a pair of points (the ordinary distance) or 3-hyperplanes. The 6D spaces of lines and 2-planes are of rank 2 [132,133] so that there are two independent invariants under the action of $\text{SO}(5)$ for a pair of lines (an angle and the distance between two lines) or 2-planes (see [134] for the Euclidean case). Thirdly, there is a relevant automorphism of $\mathfrak{so}(5)$ defined by

$$\mathcal{D}(J_{ab}) := -J_{4-b\,4-a}, \quad (44)$$

that, in the array display of the generators, is visualized as

$$\begin{array}{cccc} J_{01} & J_{02} & J_{03} & \boxed{J_{04}} \\ & J_{12} & \boxed{J_{13}} & J_{14} \\ & & J_{23} & J_{24} \\ & & & J_{34} \end{array} \quad \xleftrightarrow{\mathcal{D}} \quad \begin{array}{cccc} -J_{34} & -J_{24} & -J_{14} & \boxed{-J_{04}} \\ & -J_{23} & \boxed{-J_{13}} & -J_{03} \\ & & -J_{12} & -J_{02} \\ & & & -J_{01} \end{array} \quad (45)$$

leaving J_{04} and J_{13} invariant (up to the minus sign). Thus, the map \mathcal{D} interchanges the spaces of points and 3-hyperplanes and the spaces of lines and 2-planes:

$$\mathbb{S}^{(1)} \xleftrightarrow{\mathcal{D}} \mathbb{S}^{(4)}, \quad \mathbb{S}^{(2)} \xleftrightarrow{\mathcal{D}} \mathbb{S}^{(3)}. \quad (46)$$

Note that $\mathcal{D}^2 = \text{Id}$. The map \mathcal{D} will be called *polarity*, since for $\mathfrak{so}(3)$ reduces to the well known duality in projective geometry interchanging the 2D space of points with the 2D space of lines (see [8] and the references therein), which, only at this dimension, are both of rank 1. Note that this map is sometimes called ordinary duality [8], although in this paper we will always call it polarity in order to avoid confusion with the completely unrelated notion of quantum duality.

3.2. Lie Bialgebra

Let us consider the Drinfel'd–Jimbo quantum deformation of the real compact form $\mathfrak{g} = \mathfrak{so}(5)$ [28–30], which, in the basis (31), is generated by the following classical r -matrix [31]

$$r_{04,13} = z(J_{14} \wedge J_{01} + J_{24} \wedge J_{02} + J_{34} \wedge J_{03} + J_{23} \wedge J_{12}). \quad (47)$$

Recall that z is the quantum deformation parameter (such that $q = e^z$) and, hereafter, it will be assumed that z is an *indeterminate real parameter*. We remark that $r_{04,13}$ is a solution of the modified classical Yang–Baxter Equation (14) so that this underlies a quasitriangular Hopf algebra structure. Therefore, the corresponding cocommutator is coboundary [27], $\delta = \delta(r)$, so that this is obtained from (47) through the relation (12), yielding

$$\begin{aligned} \delta(J_{04}) &= 0, & \delta(J_{13}) &= 0, \\ \delta(J_{12}) &= zJ_{12} \wedge J_{13}, & \delta(J_{23}) &= zJ_{23} \wedge J_{13}, \\ \delta(J_{01}) &= z(J_{01} \wedge J_{04} + J_{24} \wedge J_{12} + J_{34} \wedge J_{13} + J_{02} \wedge J_{23}), \\ \delta(J_{02}) &= z(J_{02} \wedge J_{04} + J_{12} \wedge J_{14} + J_{34} \wedge J_{23} + J_{23} \wedge J_{01} + J_{12} \wedge J_{03}), \\ \delta(J_{03}) &= z(J_{03} \wedge J_{04} + J_{13} \wedge J_{14} + J_{23} \wedge J_{24} + J_{02} \wedge J_{12}), \\ \delta(J_{14}) &= z(J_{14} \wedge J_{04} + J_{13} \wedge J_{03} + J_{12} \wedge J_{02} + J_{24} \wedge J_{23}), \\ \delta(J_{24}) &= z(J_{24} \wedge J_{04} + J_{23} \wedge J_{03} + J_{01} \wedge J_{12} + J_{12} \wedge J_{34} + J_{23} \wedge J_{14}), \\ \delta(J_{34}) &= z(J_{34} \wedge J_{04} + J_{02} \wedge J_{23} + J_{01} \wedge J_{13} + J_{24} \wedge J_{12}). \end{aligned} \quad (48)$$

The resulting real Lie bialgebra $(\mathfrak{so}(5), \delta(r_{04,13}))$ is determined by the commutation rules (31) and cocommutator (48). The indices in $r_{04,13}$ (47) indicate the primitive generators J_{04} and J_{13} . The first primitive generator J_{04} is the “main” one in the sense that, once the CK scheme of contractions is introduced and further applied to kinematical algebras, it will provide dimensions of the deformation parameter z since the product zJ_{04} must be dimensionless [7,135]; this fact will be studied in detail in Section 6.

The last term of $r_{04,13}$ is a classical r -matrix $r_{13} = zJ_{23} \wedge J_{12}$ giving rise to the Drinfel'd–Jimbo Lie bialgebra $(\mathfrak{so}(3), \delta(r_{13}))$, with $\mathfrak{so}(3) = \text{span}\{J_{12}, J_{13}, J_{23}\}$ and primitive generator J_{13} , which is a sub-Lie bialgebra of $(\mathfrak{so}(5), \delta(r_{04,13}))$; thus, J_{13} plays the role of a “secondary” primitive generator in $r_{04,13}$ [31].

Now, we analyse how to implement the $\mathbb{Z}_2^{\otimes 4}$ -grading of $\mathfrak{so}(5)$ into $(\mathfrak{so}(5), \delta(r_{04,13}))$. This requires generalizing the action of the automorphisms $\Theta^{(m)} : \mathfrak{so}(5) \rightarrow \mathfrak{so}(5)$ (33) onto the cocommutator $\delta : \mathfrak{so}(5) \rightarrow \mathfrak{so}(5) \otimes \mathfrak{so}(5)$ and also to consider a possible action on the quantum deformation parameter [7,135]. Recall that δ (48) is the skew-symmetric part of the first-order term in z (17) of the full coproduct Δ of the real quantum algebra $\mathcal{U}_z(\mathfrak{so}(5)) = \mathcal{U}(\mathfrak{so}(5)) \hat{\otimes} \mathbb{R}[[z]]$ such that, as mentioned above, z is an indeterminate parameter.

Since z is linked to the “main” primitive generator J_{04} through the product zJ_{04} , both elements must be transformed in the same way. By taking into account that $J_{04} \rightarrow -J_{04}$

under the four maps $\Theta^{(m)}$, then $z \rightarrow -z$ as well. Hence, we define four z -maps ($m = 1, \dots, 4$)

$$\begin{aligned}\Theta_z^{(m)}(\delta(J_{ab})) &:= \delta(\Theta^{(m)}(J_{ab})), \\ \Theta_z^{(m)}(zJ_{ab} \otimes J_{cd}) &:= (-z\Theta^{(m)}(J_{ab}) \otimes \Theta^{(m)}(J_{cd})),\end{aligned}\quad (49)$$

where $\Theta^{(m)}$ is given in (33). Notice that the second relation in (49) can directly be applied to the r -matrix (47), consistent with the relation (12), and extended to higher-order tensor product spaces. When the z -maps (49) are applied either to δ (48) or to the r -matrix (47) one finds that only $\Theta_z^{(2)}$ and $\Theta_z^{(3)}$ remain as involutive automorphisms of $(\mathfrak{so}(5), \delta(r_{04,13}))$, meanwhile $\Theta_z^{(1)}$ and $\Theta_z^{(4)}$ are no longer involutions.

This is a consequence of the presence of the term $r_{13} = zJ_{23} \wedge J_{12}$ in $r_{04,13}$, which does not appear in either the Drinfel'd–Jimbo quantum deformation of $\mathfrak{so}(3)$ [7] or in $\mathfrak{so}(4)$ [135]; for these latter deformations, the whole initial $\mathbb{Z}_2^{\otimes 2}$ - and $\mathbb{Z}_2^{\otimes 3}$ -grading is kept, respectively; however, for $(\mathfrak{so}(5), \delta(r_{04,13}))$ there only remains a $\mathbb{Z}_2^{\otimes 2}$ -grading spanned by $\Theta_z^{(2)}$ and $\Theta_z^{(3)}$.

Likewise, the polarity (44) can be implemented into $(\mathfrak{so}(5), \delta(r_{04,13}))$ by also considering an action on the deformation parameter defined by [7,135]

$$\begin{aligned}\mathcal{D}_z(\delta(J_{ab})) &:= \delta(\mathcal{D}(J_{ab})), \\ \mathcal{D}_z(zJ_{ab} \otimes J_{cd}) &:= (-z\mathcal{D}(J_{ab}) \otimes \mathcal{D}(J_{cd})),\end{aligned}\quad (50)$$

with \mathcal{D} given in (44), so that $\mathcal{D}_z^2 = \text{Id}$. It can be checked that the r -matrix (47) remains invariant under this “ z -polarity” (and, clearly, the cocommutator (48) as well). Note that both primitive generators J_{04} and J_{13} are kept unchanged under \mathcal{D} (up to the minus sign) as shown in (45).

From (48), it is straightforward to prove that this deformation fulfils the coisotropy condition (22) [57,59] for the four isotropy subalgebras $\mathfrak{h}^{(m)}$ (36):

$$\delta(\mathfrak{h}^{(m)}) \subset \mathfrak{h}^{(m)} \wedge \mathfrak{so}(5), \quad m = 1, \dots, 4. \quad (51)$$

Thus, each of them would provide a Poisson homogeneous space. Notice, however, that none of them lead to a Poisson subgroup since the condition (23) is not satisfied:

$$\delta(\mathfrak{h}^{(m)}) \not\subset \mathfrak{h}^{(m)} \wedge \mathfrak{h}^{(m)}, \quad m = 1, \dots, 4.$$

3.3. Dual Lie Algebra and Noncommutative Spaces

According to Section 2.1, we denote, by \hat{x}^{ab} ($a < b$; $a, b = 0, 1, \dots, 4$), the generators in $\mathfrak{g}^* = \mathfrak{so}(5)^*$ dual to J_{ab} in $\mathfrak{g} = \mathfrak{so}(5)$ with canonical pairing defined by (11):

$$\langle \hat{x}^{ab}, J_{cd} \rangle = \delta_c^a \delta_d^b. \quad (52)$$

From the cocommutator (48), read as (9), we obtain the commutation relations of the dual Lie algebra $\mathfrak{so}(5)^*$ (10), which are given by

$$\begin{aligned}
[\hat{x}^{01}, \hat{x}^{02}] &= 0, & [\hat{x}^{01}, \hat{x}^{12}] &= z\hat{x}^{24}, & [\hat{x}^{02}, \hat{x}^{12}] &= z(\hat{x}^{03} - \hat{x}^{14}), \\
[\hat{x}^{01}, \hat{x}^{03}] &= 0, & [\hat{x}^{01}, \hat{x}^{13}] &= z\hat{x}^{34}, & [\hat{x}^{03}, \hat{x}^{13}] &= -z\hat{x}^{14}, \\
[\hat{x}^{01}, \hat{x}^{04}] &= z\hat{x}^{01}, & [\hat{x}^{01}, \hat{x}^{14}] &= 0, & [\hat{x}^{04}, \hat{x}^{14}] &= -z\hat{x}^{14}, \\
[\hat{x}^{02}, \hat{x}^{03}] &= 0, & [\hat{x}^{02}, \hat{x}^{23}] &= z(\hat{x}^{01} + \hat{x}^{34}), & [\hat{x}^{03}, \hat{x}^{23}] &= -z\hat{x}^{24}, \\
[\hat{x}^{02}, \hat{x}^{04}] &= z\hat{x}^{02}, & [\hat{x}^{02}, \hat{x}^{24}] &= 0, & [\hat{x}^{04}, \hat{x}^{24}] &= -z\hat{x}^{24}, \\
[\hat{x}^{03}, \hat{x}^{04}] &= z\hat{x}^{03}, & [\hat{x}^{03}, \hat{x}^{34}] &= 0, & [\hat{x}^{04}, \hat{x}^{34}] &= -z\hat{x}^{34}, \\
[\hat{x}^{12}, \hat{x}^{13}] &= z\hat{x}^{12}, & [\hat{x}^{12}, \hat{x}^{23}] &= 0, & [\hat{x}^{13}, \hat{x}^{23}] &= -z\hat{x}^{23}, \\
[\hat{x}^{12}, \hat{x}^{14}] &= z\hat{x}^{02}, & [\hat{x}^{12}, \hat{x}^{24}] &= -z(\hat{x}^{01} + \hat{x}^{34}), & [\hat{x}^{14}, \hat{x}^{24}] &= 0, \\
[\hat{x}^{13}, \hat{x}^{14}] &= z\hat{x}^{03}, & [\hat{x}^{13}, \hat{x}^{34}] &= -z\hat{x}^{01}, & [\hat{x}^{14}, \hat{x}^{34}] &= 0, \\
[\hat{x}^{23}, \hat{x}^{24}] &= z(\hat{x}^{03} - \hat{x}^{14}), & [\hat{x}^{23}, \hat{x}^{34}] &= -z\hat{x}^{02}, & [\hat{x}^{24}, \hat{x}^{34}] &= 0,
\end{aligned} \tag{53}$$

$$\begin{aligned}
[\hat{x}^{01}, \hat{x}^{23}] &= -z\hat{x}^{02}, & [\hat{x}^{01}, \hat{x}^{24}] &= 0, & [\hat{x}^{01}, \hat{x}^{34}] &= 0, \\
[\hat{x}^{02}, \hat{x}^{13}] &= 0, & [\hat{x}^{02}, \hat{x}^{14}] &= 0, & [\hat{x}^{02}, \hat{x}^{34}] &= 0, \\
[\hat{x}^{03}, \hat{x}^{12}] &= -z\hat{x}^{02}, & [\hat{x}^{03}, \hat{x}^{14}] &= 0, & [\hat{x}^{03}, \hat{x}^{24}] &= 0, \\
[\hat{x}^{04}, \hat{x}^{12}] &= 0, & [\hat{x}^{04}, \hat{x}^{13}] &= 0, & [\hat{x}^{04}, \hat{x}^{23}] &= 0, \\
[\hat{x}^{12}, \hat{x}^{34}] &= z\hat{x}^{24}, & [\hat{x}^{13}, \hat{x}^{24}] &= 0, & [\hat{x}^{14}, \hat{x}^{23}] &= -z\hat{x}^{24}.
\end{aligned} \tag{54}$$

In contrast to the commutation rules of $\mathfrak{so}(5)$ (31), the Lie brackets of $\mathfrak{so}(5)^*$ involving four different indices (54) are no longer equal to zero.

Next, we express the dual Lie algebra $\mathfrak{so}(5)^*$ as the sum of two vector spaces

$$\mathfrak{so}(5)^* = \mathfrak{h}_{\perp}^{(m)} \oplus \mathfrak{t}_{\perp}^{(m)}, \quad m = 1, \dots, 4, \tag{55}$$

where $\mathfrak{h}_{\perp}^{(m)}$ and $\mathfrak{t}_{\perp}^{(m)}$ are, in this order, the annihilators in $\mathfrak{so}(5)^*$ of the vector subspaces $\mathfrak{h}^{(m)}$ and $\mathfrak{t}^{(m)}$ of $\mathfrak{so}(5)$ given in (34) and verifying (35). From the results presented in Section 2.2, each $\mathfrak{h}_{\perp}^{(m)}$ leads to a linear noncommutative space that is the first-order in the quantum coordinates of the full noncommutative space associated with the homogeneous space (39). We shall denote such a *first-order noncommutative space* by $\mathbb{S}_z^{(m)} \equiv \mathfrak{h}_{\perp}^{(m)}$.

Following [60,61], we analyse the relations $[\mathfrak{h}_{\perp}^{(m)}, \mathfrak{h}_{\perp}^{(m)}]$, $[\mathfrak{h}_{\perp}^{(m)}, \mathfrak{t}_{\perp}^{(m)}]$ and $[\mathfrak{t}_{\perp}^{(m)}, \mathfrak{t}_{\perp}^{(m)}]$ for each m . Such structures do depend on the chosen quantum deformation (here determined by (47)) and they are directly deduced from (53) and (54):

1. *Noncommutative space of points* $\mathbb{S}_z^{(1)} \equiv \mathfrak{h}_{\perp}^{(1)} = \langle \hat{x}^{01}, \hat{x}^{02}, \hat{x}^{03}, \hat{x}^{04} \rangle$:

$$[\mathfrak{h}_{\perp}^{(1)}, \mathfrak{h}_{\perp}^{(1)}] \subset \mathfrak{h}_{\perp}^{(1)}, \quad [\mathfrak{h}_{\perp}^{(1)}, \mathfrak{t}_{\perp}^{(1)}] \subset \mathfrak{t}_{\perp}^{(1)} + \mathfrak{h}_{\perp}^{(1)}, \quad [\mathfrak{t}_{\perp}^{(1)}, \mathfrak{t}_{\perp}^{(1)}] \subset \mathfrak{h}_{\perp}^{(1)} + \mathfrak{t}_{\perp}^{(1)}. \tag{56}$$

2. *Noncommutative space of lines* $\mathbb{S}_z^{(2)} \equiv \mathfrak{h}_{\perp}^{(2)} = \langle \hat{x}^{02}, \hat{x}^{03}, \hat{x}^{04}, \hat{x}^{12}, \hat{x}^{13}, \hat{x}^{14} \rangle$:

$$[\mathfrak{h}_{\perp}^{(2)}, \mathfrak{h}_{\perp}^{(2)}] \subset \mathfrak{h}_{\perp}^{(2)}, \quad [\mathfrak{h}_{\perp}^{(2)}, \mathfrak{t}_{\perp}^{(2)}] \subset \mathfrak{t}_{\perp}^{(2)}, \quad [\mathfrak{t}_{\perp}^{(2)}, \mathfrak{t}_{\perp}^{(2)}] \subset \mathfrak{h}_{\perp}^{(2)}. \tag{57}$$

3. *Noncommutative space of 2-planes* $\mathbb{S}_z^{(3)} \equiv \mathfrak{h}_{\perp}^{(3)} = \langle \hat{x}^{03}, \hat{x}^{04}, \hat{x}^{13}, \hat{x}^{14}, \hat{x}^{23}, \hat{x}^{24} \rangle$:

$$[\mathfrak{h}_{\perp}^{(3)}, \mathfrak{h}_{\perp}^{(3)}] \subset \mathfrak{h}_{\perp}^{(3)}, \quad [\mathfrak{h}_{\perp}^{(3)}, \mathfrak{t}_{\perp}^{(3)}] \subset \mathfrak{t}_{\perp}^{(3)}, \quad [\mathfrak{t}_{\perp}^{(3)}, \mathfrak{t}_{\perp}^{(3)}] \subset \mathfrak{h}_{\perp}^{(3)}. \tag{58}$$

4. *Noncommutative space of 3-hyperplanes* with $\mathbb{S}_z^{(4)} \equiv \mathfrak{h}_{\perp}^{(4)} = \langle \hat{x}^{04}, \hat{x}^{14}, \hat{x}^{24}, \hat{x}^{34} \rangle$:

$$[\mathfrak{h}_{\perp}^{(4)}, \mathfrak{h}_{\perp}^{(4)}] \subset \mathfrak{h}_{\perp}^{(4)}, \quad [\mathfrak{h}_{\perp}^{(4)}, \mathfrak{t}_{\perp}^{(4)}] \subset \mathfrak{t}_{\perp}^{(4)} + \mathfrak{h}_{\perp}^{(4)}, \quad [\mathfrak{t}_{\perp}^{(4)}, \mathfrak{t}_{\perp}^{(4)}] \subset \mathfrak{h}_{\perp}^{(4)} + \mathfrak{t}_{\perp}^{(4)}. \tag{59}$$

Consequently, the four first-order noncommutative spaces $\mathbb{S}_z^{(m)}$ close on a Lie subalgebra of $\mathfrak{so}(5)^*$

$$[\mathfrak{h}_{\perp}^{(m)}, \mathfrak{h}_{\perp}^{(m)}] \subset \mathfrak{h}_{\perp}^{(m)},$$

as it should be, since this is a direct consequence of the coisotropy condition (51) [57,59–61]. Furthermore, the noncommutative spaces of lines $\mathbb{S}_z^{(2)}$ and 2-planes $\mathbb{S}_z^{(3)}$ are both reductive and symmetric

$$[\mathfrak{h}_{\perp}^{(l)}, \mathfrak{t}_{\perp}^{(l)}] \subset \mathfrak{t}_{\perp}^{(l)}, \quad [\mathfrak{t}_{\perp}^{(l)}, \mathfrak{t}_{\perp}^{(l)}] \subset \mathfrak{h}_{\perp}^{(l)}, \quad l = 2, 3. \quad (60)$$

The pairing (52) allows us to define the following maps in $\mathfrak{so}(5)^*$ from (33) by considering again an action on z :

$$\begin{aligned} \theta^{(m)}(\hat{x}^{ab}) &:= \begin{cases} \hat{x}^{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ -\hat{x}^{ab}, & \text{if } a < m \leq b. \end{cases} \\ \theta_z^{(m)}(\hat{x}^{ab}, z) &:= (\theta^{(m)}(\hat{x}^{ab}), -z). \end{aligned} \quad (61)$$

Similarly to (49), the action of $\theta_z^{(m)}$ can be extended to the tensor product space $\mathfrak{so}(5)^* \otimes \mathfrak{so}(5)^*$, although we shall not make use of it in this paper. It can be checked that only $\theta_z^{(2)}$ and $\theta_z^{(3)}$ are involutive automorphisms of the commutation relations (53) and (54) in agreement with the symmetric property of $\mathbb{S}_z^{(2)}$ and $\mathbb{S}_z^{(3)}$ (60).

Moreover, a dual polarity d_z can also be defined in $\mathfrak{so}(5)^*$ in the form

$$d(\hat{x}^{ab}) := -\hat{x}^{4-b, 4-a}, \quad d_z(\hat{x}^{ab}, z) := (d(\hat{x}^{ab}), -z), \quad d_z^2 = \text{Id}, \quad (62)$$

such that the map d is dual to \mathcal{D} (44) through the pairing (52) and d_z is an automorphism of $\mathfrak{so}(5)^*$, which, as expected, interchanges the noncommutative spaces of points and 3-hyperplanes and the noncommutative spaces of lines and 2-planes:

$$\mathbb{S}_z^{(1)} \xleftrightarrow{d_z} \mathbb{S}_z^{(4)}, \quad \mathbb{S}_z^{(2)} \xleftrightarrow{d_z} \mathbb{S}_z^{(3)}, \quad (63)$$

to be compared with (46).

3.4. Drinfel'd Double Structure

In [129], it was shown that the real Lie algebra $\mathfrak{so}(5)$ has a classical r -matrix coming from a Drinfel'd double structure for the classical complex Lie algebra \mathfrak{c}_2 . We now review the main results according to the notation introduced in Section 2.3.

Let us consider the complex Lie algebra \mathfrak{c}_2 in a Chevalley basis with generators $\{h_l, e_{\pm l}\}$ ($l = 1, 2$) fulfilling the Lie brackets given by

$$\begin{aligned} [h_1, e_{\pm 1}] &= \pm e_{\pm 1}, & [h_1, e_{\pm 2}] &= \mp e_{\pm 2}, & [e_{+1}, e_{-1}] &= h_1, \\ [h_2, e_{\pm 1}] &= \mp e_{\pm 1}, & [h_2, e_{\pm 2}] &= \pm 2e_{\pm 2}, & [e_{+2}, e_{-2}] &= h_2, \\ [h_1, h_2] &= [e_{-1}, e_{+2}] = [e_{+1}, e_{-2}] = 0. \end{aligned}$$

We define four new generators $e_{\pm 3}, e_{\pm 4}$ as

$$[e_{+1}, e_{+2}] := e_{+3}, \quad [e_{-2}, e_{-1}] := e_{-3}, \quad [e_{+1}, e_{+3}] := e_{+4}, \quad [e_{-3}, e_{-1}] := e_{-4},$$

such that the Serre relations read

$$[e_{+1}, e_{+4}] = [e_{+2}, e_{+3}] = 0, \quad [e_{-1}, e_{-4}] = [e_{-2}, e_{-3}] = 0.$$

Then, the 10 generators $\{h_l, e_{\pm m}\}$ with $l = 1, 2$ and $m = 1, \dots, 4$ span the Lie algebra \mathfrak{c}_2 in the Cartan–Weyl basis. As a shorthand notation, we denote $e_m \equiv e_{+m}$ and $f_m \equiv e_{-m}$ so that the full commutation rules of \mathfrak{c}_2 read

$$\begin{aligned}
[h_1, h_2] &= 0, & [h_1, e_1] &= e_1, & [h_1, f_1] &= -f_1, \\
[h_1, e_2] &= -e_2, & [h_1, f_2] &= f_2, & [h_1, e_3] &= 0, \\
[h_1, f_3] &= 0, & [h_1, e_4] &= e_4, & [h_1, f_4] &= -f_4, \\
[h_2, e_1] &= -e_1, & [h_2, f_1] &= f_1, & [h_2, e_2] &= 2e_2, \\
[h_2, f_2] &= -2f_2, & [h_2, e_3] &= e_3, & [h_2, f_3] &= -f_3, \\
[h_2, e_4] &= 0, & [h_2, f_4] &= 0, & [e_1, f_1] &= h_1, \\
[e_1, e_2] &= e_3, & [e_1, f_2] &= 0, & [e_1, e_3] &= e_4, \\
[e_1, f_3] &= -f_2, & [e_1, e_4] &= 0, & [e_1, f_4] &= -f_3, \\
[f_1, e_2] &= 0, & [f_1, f_2] &= -f_3, & [f_1, e_3] &= e_2, \\
[f_1, f_3] &= -f_4, & [f_1, e_4] &= e_3, & [f_1, f_4] &= 0, \\
[e_2, f_2] &= h_2, & [e_2, e_3] &= 0, & [e_2, f_3] &= f_1, \\
[e_2, e_4] &= 0, & [e_2, f_4] &= 0, & [f_2, e_3] &= -e_1, \\
[f_2, f_3] &= 0, & [f_2, e_4] &= 0, & [f_2, f_4] &= 0, \\
[e_3, f_3] &= h_1 + h_2, & [e_3, e_4] &= 0, & [e_3, f_4] &= f_1, \\
[f_3, e_4] &= -e_1, & [f_3, f_4] &= 0, & [e_4, f_4] &= 2h_1 + h_2.
\end{aligned} \tag{64}$$

To unveil the Drinfel'd double structure for \mathfrak{c}_2 , we consider the linear combination of the two generators h_1 and h_2 belonging to the Cartan subalgebra given by [129]:

$$e_0 := \frac{1}{\sqrt{2}}((1+i)h_1 + ih_2), \quad f_0 := \frac{1}{\sqrt{2}}((1-i)h_1 - ih_2). \tag{65}$$

Finally, the identification

$$Y_a \equiv e_a, \quad y^a \equiv f_a, \quad a = 0, \dots, 4,$$

allows us to express the commutation relations (64) with the new Cartan generators (65) in the required form (26), thus, obtaining a Drinfel'd double structure for \mathfrak{c}_2 with two 5D subalgebras $\mathfrak{g}_1 = \text{span}\{Y_a \equiv e_a\}$ and $\mathfrak{g}_2 = \text{span}\{y^a \equiv f_a\}$ ($a = 0, \dots, 4$), which are dual to each other by means of the canonical pairing (27).

- Lie subalgebra $\mathfrak{g}_1 = \text{span}\{e_0, \dots, e_4\}$:

$$\begin{aligned}
[e_0, e_1] &= \frac{1}{\sqrt{2}}e_1, & [e_0, e_2] &= -\frac{1}{\sqrt{2}}(1-i)e_2, \\
[e_0, e_3] &= \frac{i}{\sqrt{2}}e_3, & [e_0, e_4] &= \frac{1}{\sqrt{2}}(1+i)e_4, \\
[e_1, e_2] &= e_3, & [e_1, e_3] &= e_4, & [e_1, e_4] &= 0, \\
[e_2, e_3] &= 0, & [e_2, e_4] &= 0, & [e_3, e_4] &= 0.
\end{aligned}$$

- Lie subalgebra $\mathfrak{g}_2 = \mathfrak{g}_1^* = \text{span}\{f_0, \dots, f_4\}$:

$$\begin{aligned}
[f_0, f_1] &= -\frac{1}{\sqrt{2}}f_1, & [f_0, f_2] &= \frac{1}{\sqrt{2}}(1+i)f_2, \\
[f_0, f_3] &= \frac{i}{\sqrt{2}}f_3, & [f_0, f_4] &= -\frac{1}{\sqrt{2}}(1-i)f_4, \\
[f_1, f_2] &= -f_3, & [f_1, f_3] &= -f_4, & [f_1, f_4] &= 0, \\
[f_2, f_3] &= 0, & [f_2, f_4] &= 0, & [f_3, f_4] &= 0.
\end{aligned}$$

• Crossed relations $[e_a, f_b]$:

$$\begin{aligned}
 [e_0, f_0] &= 0, & [e_1, f_1] &= \frac{1}{\sqrt{2}}(e_0 + f_0), & [e_3, f_3] &= -\frac{i}{\sqrt{2}}(e_0 - f_0), \\
 [e_2, f_2] &= -\frac{1}{\sqrt{2}}((1+i)e_0 + (1-i)f_0), & [e_4, f_4] &= \frac{1}{\sqrt{2}}((1-i)e_0 + (1+i)f_0), \\
 [f_1, e_0] &= \frac{1}{\sqrt{2}}f_1, & [f_2, e_0] &= \frac{1}{\sqrt{2}}(i-1)f_2, & [f_3, e_0] &= \frac{i}{\sqrt{2}}f_3, & [f_4, e_0] &= \frac{1}{\sqrt{2}}(1+i)f_4, \\
 [e_1, f_0] &= -\frac{1}{\sqrt{2}}e_1, & [e_2, f_0] &= \frac{1}{\sqrt{2}}(1+i)e_2, & [e_3, f_0] &= \frac{i}{\sqrt{2}}e_3, & [e_4, f_0] &= \frac{1}{\sqrt{2}}(i-1)e_4, \\
 [e_1, f_2] &= 0, & [e_2, f_1] &= 0, & [e_3, f_1] &= -e_2, & [e_4, f_1] &= -e_3, \\
 [e_1, f_3] &= -f_2, & [e_2, f_3] &= f_1, & [e_3, f_2] &= e_1, & [e_4, f_2] &= 0, \\
 [e_1, f_4] &= -f_3, & [e_2, f_4] &= 0, & [e_3, f_4] &= f_1, & [e_4, f_3] &= e_1.
 \end{aligned}$$

From these results, the real Lie algebra $\mathfrak{so}(5) \sim \mathfrak{c}_2$ is obtained on the basis with generators $\{J_{ab}\}$ obeying the commutation rules (31) through the following change of basis [129]:

$$\begin{aligned}
 e_0 &= -\frac{1}{\sqrt{2}}(J_{04} - iJ_{13}), & f_0 &= \frac{1}{\sqrt{2}}(J_{04} + iJ_{13}), \\
 e_1 &= \frac{1}{\sqrt{2}}(J_{23} + iJ_{12}), & f_1 &= -\frac{1}{\sqrt{2}}(J_{23} - iJ_{12}), \\
 e_2 &= \frac{1}{2}(J_{01} - J_{34} - i(J_{03} + J_{14})), & f_2 &= -\frac{1}{2}(J_{01} - J_{34} + i(J_{03} + J_{14})), \\
 e_3 &= \frac{1}{\sqrt{2}}(J_{24} + iJ_{02}), & f_3 &= -\frac{1}{\sqrt{2}}(J_{24} - iJ_{02}), \\
 e_4 &= \frac{1}{2}(J_{01} + J_{34} + i(J_{03} - J_{14})), & f_4 &= -\frac{1}{2}(J_{01} + J_{34} - i(J_{03} - J_{14})),
 \end{aligned}$$

whose inverse reads

$$\begin{aligned}
 J_{01} &= \frac{1}{2}(e_2 - f_2 + e_4 - f_4), & J_{13} &= -\frac{i}{\sqrt{2}}(e_0 + f_0), \\
 J_{02} &= -\frac{i}{\sqrt{2}}(e_3 + f_3), & J_{14} &= \frac{i}{2}(e_2 + f_2 + e_4 + f_4), \\
 J_{03} &= \frac{i}{2}(e_2 + f_2 - e_4 - f_4), & J_{23} &= \frac{1}{\sqrt{2}}(e_1 - f_1), \\
 J_{04} &= -\frac{1}{\sqrt{2}}(e_0 - f_0), & J_{24} &= \frac{1}{\sqrt{2}}(e_3 - f_3), \\
 J_{12} &= -\frac{i}{\sqrt{2}}(e_1 + f_1), & J_{34} &= -\frac{1}{2}(e_2 - f_2 - e_4 + f_4).
 \end{aligned}$$

The canonical pairing (27) now reads

$$\langle J_{ab}, J_{cd} \rangle = -\delta_{ac}\delta_{bd}.$$

Then, the canonical classical r -matrix (28) turns out to be

$$r_{\text{can}} = \sum_{a=0}^4 f_a \otimes e_a = \frac{1}{2}i(J_{01} \wedge J_{14} + J_{02} \wedge J_{24} + J_{03} \wedge J_{34} + J_{12} \wedge J_{23} + J_{04} \wedge J_{13}) + \Omega,$$

where

$$\Omega = -\frac{1}{2} \sum_{0 \leq a < b \leq 4} J_{ab} \otimes J_{ab},$$

is the ad-invariant element (29) corresponding to the tensorized expression of the Casimir \mathcal{C} (32). Hence, the skew-symmetric form for r_{can} is obtained by subtracting Ω , as in (30). We explicitly introduce the quantum deformation parameter z , multiplying this result by $2iz$ as $\tilde{r}_D = 2iz(r_{\text{can}} - \Omega)$, obtaining the real r -matrix

$$\tilde{r}_D = z(J_{14} \wedge J_{01} + J_{24} \wedge J_{02} + J_{34} \wedge J_{03} + J_{23} \wedge J_{12} + J_{13} \wedge J_{04}), \quad (66)$$

which, in terms of the Drinfel'd–Jimbo classical r -matrix (47) considered for $\mathfrak{so}(5)$, reads

$$\tilde{r}_D = r_{04,13} + zJ_{13} \wedge J_{04}.$$

Hence, a Reshetikhin twist with the commuting primitive generators must be added to (47) in order to obtain a classical r -matrix coming from a Drinfel'd double structure. Recall that it is possible to consider a generalized two-parametric r -matrix [136]

$$r_{z,\vartheta} = z(J_{14} \wedge J_{01} + J_{24} \wedge J_{02} + J_{34} \wedge J_{03} + J_{23} \wedge J_{12}) + \vartheta J_{13} \wedge J_{04}, \quad (67)$$

showing the effects of the twist with the quantum deformation parameter ϑ on the former deformation determined by $r_{04,13}$ and properly recovering the Drinfel'd double r -matrix whenever $\vartheta = z$. We remark that both \tilde{r}_D and $r_{z,\vartheta}$ are quasitriangular classical r -matrices (like $r_{04,13}$), so that they are solutions of the modified classical Yang–Baxter Equation (14), while the twist itself determines a triangular r -matrix with vanishing Schouten bracket (16). In this sense, \tilde{r}_D and $r_{z,\vartheta}$ can be regarded as “hybrid” classical r -matrices [114,137].

4. The Drinfel'd–Jimbo Lie Bialgebra for the Cayley–Klein Algebra $\mathfrak{so}_\omega(5)$

The $\mathbb{Z}_2^{\otimes 4}$ -grading of $\mathfrak{so}(5)$ generated by the four automorphisms $\Theta^{(m)}$ (33) enables one to obtain a particular set of contracted real Lie algebras [17,20] through the graded contraction formalism [138,139]. These are the so-called orthogonal Cayley–Klein (CK) algebras or quasisimple orthogonal algebras [18,19,31,133] (see [140] for their description in terms of hypercomplex units).

We collectively denote them by $\mathfrak{so}_\omega(5)$, as this family of contracted algebras depends explicitly on four real graded contraction parameters $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$. Alternatively, each contraction parameter ω_m ($m = 1, 2, 3, 4$) can be introduced in the initial commutation rules of $\mathfrak{so}(5)$ (31) by means of the following mapping provided by the involution $\Theta^{(m)}$ (33):

$$\phi^{(m)}(J_{ab}) := \begin{cases} J_{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ \sqrt{\omega_m} J_{ab}, & \text{if } a < m \leq b. \end{cases}$$

The composition of the four (commuting) mappings gives [31]

$$\Phi(J_{ab}) := \phi^{(1)} \circ \phi^{(2)} \circ \phi^{(3)} \circ \phi^{(4)}(J_{ab}) = \sqrt{\omega_{ab}} J_{ab}, \quad (68)$$

where the contraction parameter with two indices ω_{ab} is defined by

$$\omega_{ab} := \prod_{s=a+1}^b \omega_s, \quad 0 \leq a < b \leq 4. \quad (69)$$

Next, we apply the map (68) with all the factors $\sqrt{\omega_{ab}} \neq 0$ onto the commutation rules of $\mathfrak{so}(5)$ (31) obtaining the Lie brackets corresponding to the CK family $\mathfrak{so}_\omega(5)$, which are given by

$$[J_{ab}, J_{ac}] = \omega_{ab} J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = \omega_{bc} J_{ab}, \quad a < b < c, \quad (70)$$

without sum over repeated indices and with all the remaining brackets being equal to zero. This is just the same result coming from a particular solution of the $\mathbb{Z}_2^{\otimes 4}$ -graded contraction equations for $\mathfrak{so}(5)$ [17] (see [20] for the general solution). Explicitly, the non-vanishing commutation relations of $\mathfrak{so}_\omega(5)$ read

$$\begin{aligned}
[J_{01}, J_{02}] &= \omega_1 J_{12}, & [J_{01}, J_{12}] &= -J_{02}, & [J_{02}, J_{12}] &= \omega_2 J_{01}, \\
[J_{01}, J_{03}] &= \omega_1 J_{13}, & [J_{01}, J_{13}] &= -J_{03}, & [J_{03}, J_{13}] &= \omega_2 \omega_3 J_{01}, \\
[J_{01}, J_{04}] &= \omega_1 J_{14}, & [J_{01}, J_{14}] &= -J_{04}, & [J_{04}, J_{14}] &= \omega_2 \omega_3 \omega_4 J_{01}, \\
[J_{02}, J_{03}] &= \omega_1 \omega_2 J_{23}, & [J_{02}, J_{23}] &= -J_{03}, & [J_{03}, J_{23}] &= \omega_3 J_{02}, \\
[J_{02}, J_{04}] &= \omega_1 \omega_2 J_{24}, & [J_{02}, J_{24}] &= -J_{04}, & [J_{04}, J_{24}] &= \omega_3 \omega_4 J_{02}, \\
[J_{03}, J_{04}] &= \omega_1 \omega_2 \omega_3 J_{34}, & [J_{03}, J_{34}] &= -J_{04}, & [J_{04}, J_{34}] &= \omega_4 J_{03}, \\
[J_{12}, J_{13}] &= \omega_2 J_{23}, & [J_{12}, J_{23}] &= -J_{13}, & [J_{13}, J_{23}] &= \omega_3 J_{12}, \\
[J_{12}, J_{14}] &= \omega_2 J_{24}, & [J_{12}, J_{24}] &= -J_{14}, & [J_{14}, J_{24}] &= \omega_3 \omega_4 J_{12}, \\
[J_{13}, J_{14}] &= \omega_2 \omega_3 J_{34}, & [J_{13}, J_{34}] &= -J_{14}, & [J_{14}, J_{34}] &= \omega_4 J_{13}, \\
[J_{23}, J_{24}] &= \omega_3 J_{34}, & [J_{23}, J_{34}] &= -J_{24}, & [J_{24}, J_{34}] &= \omega_4 J_{23}.
\end{aligned} \tag{71}$$

We remark that, although the factor $\sqrt{\omega_{ab}} \neq 0$ in the map (68) can be an imaginary number, enabling to change the real form of the algebra, the resulting commutation relations (70) of $\mathfrak{so}_\omega(5)$ only comprise real Lie algebras. Moreover, the zero value for ω_{ab} is consistently allowed in (70), which is equivalent to apply an Inönü–Wigner contraction [13,31], leading to a more abelian (contracted) Lie algebra. Consequently, each graded contraction parameter ω_m can take a positive, negative or zero value in (70), and, when $\omega_m \neq 0$, it can be reduced to ± 1 through scaling of the Lie generators. Hence, $\mathfrak{so}_\omega(5)$ contains $3^4 = 81$ specific real Lie algebras, with some of them being isomorphic.

Moreover, the CK algebra $\mathfrak{so}_\omega(5)$ (70) is always endowed with two non-trivial Casimirs regardless of the values of ω . One of them is the quadratic Casimir coming from the Killing–Cartan form, which is given by [18]

$$\begin{aligned}
\mathcal{C} &= \omega_2 \omega_3 \omega_4 J_{01}^2 + \omega_3 \omega_4 J_{02}^2 + \omega_4 J_{03}^2 + J_{04}^2 + \omega_1 \omega_3 \omega_4 J_{12}^2 + \omega_1 \omega_4 J_{13}^2 \\
&\quad + \omega_1 J_{14}^2 + \omega_1 \omega_2 \omega_4 J_{23}^2 + \omega_1 \omega_2 J_{24}^2 + \omega_1 \omega_2 \omega_3 J_{34}^2,
\end{aligned} \tag{72}$$

to be compared with (32). Observe that, in the most contracted case, with all $\omega_m = 0$, $\mathcal{C} = J_{04}^2$. The second Casimir is a fourth-order one that can be found explicitly in [18], and this is related to the Pauli–Lubanski operator. In the most contracted case, the fourth-order Casimir does not vanish. In this respect, we recall that the CK Lie algebras are the only graded contracted algebras from $\mathfrak{so}(N+1)$ [17,18] that preserve the rank of the semisimple algebra, understood as the number of algebraically independent Casimirs, which, at this dimension, is equal to two.

To unveil the structure of the CK family $\mathfrak{so}_\omega(5)$, let us recall that the vector representation of the CK algebra in terms of 5×5 real matrices, $\rho : \mathfrak{so}_\omega(5) \rightarrow \text{End}(\mathbb{R}^5)$, is given by [18,19]

$$\rho(J_{ab}) = -\omega_{ab} e_{ab} + e_{ba}, \tag{73}$$

which fulfils that

$$\begin{aligned}
\rho(J_{ab})^T \mathbf{I}_\omega + \mathbf{I}_\omega \rho(J_{ab}) &= 0, \\
\mathbf{I}_\omega &= \text{diag}(1, \omega_{01}, \omega_{02}, \omega_{03}, \omega_{04}) \\
&= \text{diag}(1, \omega_1, \omega_1 \omega_2, \omega_1 \omega_2 \omega_3, \omega_1 \omega_2 \omega_3 \omega_4),
\end{aligned} \tag{74}$$

to be compared with (37) and (38). The value of the parameter ω_{ab} (69) determines the Lie subalgebra generated by J_{ab} (73), denoted $\mathfrak{so}_{\omega_{ab}}(2)$, i.e., $\mathfrak{so}(2)$ for $\omega_{ab} > 0$, $\mathfrak{so}(1,1)$ for $\omega_{ab} < 0$ and $\mathfrak{iso}(1) \equiv \mathbb{R}$ for the pure contracted case with $\omega_{ab} = 0$.

According to the values of $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, we mention the most relevant mathematical and physical Lie algebras contained within $\mathfrak{so}_\omega(5)$ [18,19]:

- If all $\omega_m \neq 0$, $\mathfrak{so}_\omega(5)$ is a pseudo-orthogonal algebra $\mathfrak{so}(p, q)$ ($p + q = 5$) where (p, q) are the number of positive and negative terms in the invariant quadratic form with matrix \mathbf{I}_ω (74). Clearly, for all $\omega_m > 0$, we recover $\mathfrak{so}(5)$; otherwise, we find either

$\mathfrak{so}(3, 2)$ (isomorphic to the $(3 + 1)$ D anti-de Sitter algebra) or $\mathfrak{so}(4, 1)$ (isomorphic to the $(3 + 1)$ D de Sitter algebra or to the 4D hyperbolic one).

- When only $\omega_1 = 0$, we find the inhomogeneous pseudo-orthogonal algebras with semidirect sum structure

$$\mathfrak{so}_{0,\omega_2,\omega_3,\omega_4}(5) \equiv \mathbb{R}^4 \oplus_S \mathfrak{so}_{\omega_2,\omega_3,\omega_4}(4) \equiv \mathfrak{iso}(p, q), \quad p + q = 4, \quad (75)$$

where the abelian subalgebra \mathbb{R}^4 is spanned by $\langle J_{01}, J_{02}, J_{03}, J_{04} \rangle$ and $\mathfrak{so}_{\omega_2,\omega_3,\omega_4}(4)$ is a pseudo-orthogonal algebra, preserving the quadratic form with a 4×4 matrix

$$\text{diag}(1, \omega_{12}, \omega_{13}, \omega_{14}) = \text{diag}(1, \omega_2, \omega_2\omega_3, \omega_2\omega_3\omega_4),$$

that acts on \mathbb{R}^4 through the vector representation (73) (see (40)). Hence, the 4D Euclidean $\mathfrak{iso}(4)$, the $(3 + 1)$ D Poincaré $\mathfrak{iso}(3, 1)$ and $\mathfrak{iso}(2, 2)$ algebras belong to this class.

- If only $\omega_4 = 0$, we again obtain inhomogeneous pseudo-orthogonal algebras with semidirect sum structure

$$\mathfrak{so}_{\omega_1,\omega_2,\omega_3,0}(5) \equiv \mathbb{R}'^4 \oplus_S \mathfrak{so}_{\omega_1,\omega_2,\omega_3}(4) \equiv \mathfrak{i}'\mathfrak{so}(p, q), \quad p + q = 4, \quad (76)$$

where, now, the abelian subalgebra $\mathbb{R}'^4 = \langle J_{04}, J_{14}, J_{24}, J_{34} \rangle$ and $\mathfrak{so}_{\omega_1,\omega_2,\omega_3}(4)$, that preserves the quadratic form with a 4×4 matrix

$$\text{diag}(1, \omega_{01}, \omega_{02}, \omega_{03}) = \text{diag}(1, \omega_1, \omega_1\omega_2, \omega_1\omega_2\omega_3),$$

acts on \mathbb{R}'^4 through the contragredient of the vector representation (73) (see (43)). These algebras are isomorphic to the previous ones with structure (75), e.g., $\mathfrak{iso}(4) \simeq \mathfrak{i}'\mathfrak{so}(4)$.

- For $\omega_1 = \omega_2 = 0$, we obtain a “twice-inhomogeneous” pseudo-orthogonal algebra

$$\mathfrak{so}_{0,0,\omega_3,\omega_4}(5) \equiv \mathbb{R}^4 \oplus_S \left(\mathbb{R}^3 \oplus_S \mathfrak{so}_{\omega_3,\omega_4}(3) \right) \equiv \mathfrak{ii}\mathfrak{so}(p, q), \quad p + q = 3, \quad (77)$$

where $\mathbb{R}^4 = \langle J_{01}, J_{02}, J_{03}, J_{04} \rangle$, $\mathbb{R}^3 = \langle J_{12}, J_{13}, J_{14} \rangle$ and $\mathfrak{so}_{\omega_3,\omega_4}(3) = \langle J_{23}, J_{24}, J_{34} \rangle$ is a pseudo-orthogonal algebra that preserves the quadratic form with a 3×3 matrix

$$\text{diag}(1, \omega_{23}, \omega_{24}) = \text{diag}(1, \omega_3, \omega_3\omega_4).$$

Here, we find the $(3 + 1)$ D Galilean algebra $\mathfrak{ii}\mathfrak{so}(3)$ as well as $\mathfrak{ii}\mathfrak{so}(2, 1)$.

- If $\omega_1 = \omega_4 = 0$, we obtain that

$$\mathfrak{so}_{0,\omega_2,\omega_3,0}(5) \equiv \mathbb{R}^4 \oplus_S \left(\mathbb{R}'^3 \oplus_S \mathfrak{so}_{\omega_2,\omega_3}(3) \right) \equiv \mathfrak{ii}'\mathfrak{so}(p, q), \quad p + q = 3, \quad (78)$$

where $\mathbb{R}^4 = \langle J_{01}, J_{02}, J_{03}, J_{04} \rangle$, $\mathbb{R}'^3 = \langle J_{14}, J_{24}, J_{34} \rangle$ are abelian subalgebras and the pseudo-orthogonal algebra $\mathfrak{so}_{\omega_2,\omega_3}(3) = \langle J_{12}, J_{13}, J_{23} \rangle$, preserving

$$\text{diag}(1, \omega_{12}, \omega_{13}) = \text{diag}(1, \omega_2, \omega_2\omega_3),$$

acts on \mathbb{R}'^3 through the contragredient of the vector representation, while $\mathbb{R}'^3 \oplus_S \mathfrak{so}_{\omega_2,\omega_3}(3)$ acts on \mathbb{R}^4 through the vector representation. Alternatively, the structure (78) can also be expressed as

$$\mathfrak{so}_{0,\omega_2,\omega_3,0}(5) \equiv \mathbb{R}'^4 \oplus_S \left(\mathbb{R}^3 \oplus_S \mathfrak{so}_{\omega_2,\omega_3}(3) \right) \equiv \mathfrak{i}'\mathfrak{ii}\mathfrak{so}(p, q), \quad p + q = 3, \quad (79)$$

where $\mathbb{R}'^4 = \langle J_{04}, J_{14}, J_{24}, J_{34} \rangle$ and $\mathbb{R}^3 = \langle J_{01}, J_{02}, J_{03} \rangle$; note that $\mathbb{R}'^4 \simeq \mathbb{R}^4$ and $\mathbb{R}^3 \simeq \mathbb{R}^3$ via \mathcal{D} (44). As particular algebras, we obtain the $(3 + 1)$ D Carroll algebra $\mathfrak{ii}'\mathfrak{so}(3) \simeq \mathfrak{i}'\mathfrak{ii}\mathfrak{so}(3)$ formerly introduced in [14,141] (see also [142–147] and the references therein) and $\mathfrak{ii}'\mathfrak{so}(2, 1) \simeq \mathfrak{i}'\mathfrak{ii}\mathfrak{so}(2, 1)$.

- When $\omega_2 = 0$, these contracted algebras are of Newton–Hooke-type [14] (see also [142,145,147–149]) with structure [150]

$$\mathfrak{so}_{\omega_1,0,\omega_3,\omega_4}(5) \equiv \mathbb{R}^6 \oplus_S (\mathfrak{so}_{\omega_1}(2) \oplus \mathfrak{so}_{\omega_3,\omega_4}(3)) \equiv \mathfrak{i}_6(\mathfrak{so}_{\omega_1}(2) \oplus \mathfrak{so}_{\omega_3,\omega_4}(3)), \quad (80)$$

where $\mathbb{R}^6 = \langle J_{02}, J_{03}, J_{04}, J_{12}, J_{13}, J_{14} \rangle$ is an abelian subalgebra, and the direct sum is between the subalgebras $\mathfrak{so}_{\omega_1}(2) = \langle J_{01} \rangle$ and $\mathfrak{so}_{\omega_3,\omega_4}(3) = \langle J_{23}, J_{24}, J_{34} \rangle$.

- The fully contracted case in the CK family corresponds to setting the four $\omega_m = 0$. This is the so-called flag algebra

$$\mathfrak{so}_{0,0,0,0}(5) \equiv \mathbb{R}^4 \oplus_S (\mathbb{R}^3 \oplus_S (\mathbb{R}^2 \oplus_S \mathbb{R})) \equiv \mathfrak{iiiiso}(1), \quad (81)$$

where $\mathbb{R}^4 = \langle J_{01}, J_{02}, J_{03}, J_{04} \rangle$, $\mathbb{R}^3 = \langle J_{12}, J_{13}, J_{14} \rangle$, $\mathbb{R}^2 = \langle J_{23}, J_{24} \rangle$ and $\mathbb{R} = \langle J_{34} \rangle \equiv \mathfrak{iso}(1)$.

Therefore, the kinematical algebras associated with different models of $(3+1)$ D spacetimes of constant curvature [14,151] belong to the CK family $\mathfrak{so}_\omega(5)$ [20,152].

It is worth stressing that the polarity \mathcal{D} (44) also remains as an automorphism of the whole family of CK algebras in such a manner that this map interchanges isomorphic Lie algebras within the family in the form

$$\mathfrak{so}_{\omega_1,\omega_2,\omega_3,\omega_4}(5) \xleftrightarrow{\mathcal{D}} \mathfrak{so}_{\omega_4,\omega_3,\omega_2,\omega_1}(5); \quad (82)$$

therefore, interchanging the contraction parameters $\omega_1 \leftrightarrow \omega_4$ and $\omega_2 \leftrightarrow \omega_3$. Consequently, the CK algebras with $\omega_4 = 0$ (76) are related, through \mathcal{D} , to those with $\omega_1 = 0$ (75) and so they are isomorphic; those with $\omega_4 = \omega_3 = 0$ are twice-inhomogeneous algebras and isomorphic to the ones with $\omega_1 = \omega_2 = 0$ (77); those with $\omega_3 = 0$ are also Newton–Hooke-type algebras isomorphic to (80); and the (single) flag algebra (81) remains unchanged under \mathcal{D} .

We also recall that all the CK algebras in $\mathfrak{so}_\omega(5)$ (even the flag algebra) have the same number of functionally independent Casimirs [18]. At this dimension, there are two (second- and fourth-order) Casimir invariants, exactly equal to the rank of the simple algebra $\mathfrak{so}(5)$; for this reason, they are also called quasisimple orthogonal algebras.

4.1. Symmetric Homogeneous Cayley–Klein Spaces

Since, by construction, the $\mathbb{Z}_2^{\otimes 4}$ -grading is preserved for the CK algebra $\mathfrak{so}_\omega(5)$, the same Cartan decompositions (34) in invariant $\mathfrak{h}_\omega^{(m)}$ and anti-invariant $\mathfrak{t}_\omega^{(m)}$ subspaces under $\Theta^{(m)}$ (33) also hold ($m = 1, 2, 3, 4$)

$$\mathfrak{so}_\omega(5) = \mathfrak{h}_\omega^{(m)} \oplus \mathfrak{t}_\omega^{(m)}. \quad (83)$$

Now, from (70), we can express the relations (35) by taking into account the contraction parameter ω_m :

$$[\mathfrak{h}_\omega^{(m)}, \mathfrak{h}_\omega^{(m)}] \subset \mathfrak{h}_\omega^{(m)}, \quad [\mathfrak{h}_\omega^{(m)}, \mathfrak{t}_\omega^{(m)}] \subset \mathfrak{t}_\omega^{(m)}, \quad [\mathfrak{t}_\omega^{(m)}, \mathfrak{t}_\omega^{(m)}] \subset \omega_m \mathfrak{h}_\omega^{(m)}. \quad (84)$$

This, in turn, means that again, for any value of ω_m , $\mathfrak{h}_\omega^{(m)}$ is always a Lie subalgebra; however, the subspace $\mathfrak{t}_\omega^{(m)}$ becomes an abelian subalgebra when $\omega_m = 0$.

Next, as in Section 3.1, we construct the homogeneous CK spaces as the coset spaces [131–133]

$$\mathbb{S}_\omega^{(m)} = \text{SO}_\omega(5) / H_\omega^{(m)}, \quad (85)$$

where $\text{SO}_\omega(5)$ is the CK Lie group with Lie algebra $\mathfrak{so}_\omega(5)$, and $H_\omega^{(m)}$ is the isotropy subgroup of $\text{SO}_\omega(5)$ with Lie algebra $\mathfrak{h}_\omega^{(m)}$. We recall that, usually, a CK geometry (6) is identified with the space of points $\mathbb{S}_\omega^{(1)}$, without taking into account other spaces. Along

this paper, a CK geometry will be understood as the full set of the four homogeneous spaces (85).

The four spaces $\mathbb{S}_\omega^{(m)}$ (85) are symmetric and reductive spaces of constant sectional curvature K equal to the graded contraction parameter ω_m . These are (see (40)–(43)):

1. 4D CK space of points:

$$\mathbb{S}_\omega^{(1)} = \text{SO}_\omega(5) / \text{SO}_{\omega_2, \omega_3, \omega_4}(4), \quad K = \omega_1. \quad (86)$$

2. 6D CK space of lines:

$$\mathbb{S}_\omega^{(2)} = \text{SO}_\omega(5) / (\text{SO}_{\omega_1}(2) \otimes \text{SO}_{\omega_3, \omega_4}(3)), \quad K = \omega_2. \quad (87)$$

3. 6D CK space of 2-planes:

$$\mathbb{S}_\omega^{(3)} = \text{SO}_\omega(5) / (\text{SO}_{\omega_1, \omega_2}(3) \otimes \text{SO}_{\omega_4}(2)), \quad K = \omega_3. \quad (88)$$

4. 4D CK space of 3-hyperplanes:

$$\mathbb{S}_\omega^{(4)} = \text{SO}_\omega(5) / \text{SO}_{\omega_1, \omega_2, \omega_3}(4), \quad K = \omega_4. \quad (89)$$

We stress that, strictly speaking, only the rank-one spaces $\mathbb{S}_\omega^{(1)}$ and $\mathbb{S}_\omega^{(4)}$ are of constant curvature in the sense that *all* their sectional curvatures are equal to ω_1 and ω_4 , respectively. However, the rank-two spaces $\mathbb{S}_\omega^{(2)}$ and $\mathbb{S}_\omega^{(3)}$ are not, in general, of constant curvature in the above sense; however, they are as close to constant curvature as a rank-two space would allow [132]. In particular, the sectional curvature K of the space of lines $\mathbb{S}_\omega^{(2)}$ along any 2-plane direction spanned by any two tangent vectors (J_{0i}, J_{0j}) , (J_{1i}, J_{1j}) and (J_{0i}, J_{1i}) ($i, j = 2, 3, 4$) is constant and equal to ω_2 ; however, the remaining sectional curvatures could be different but proportional to ω_2 . When $\omega_2 = 0$, $\mathbb{S}_\omega^{(2)}$ is a proper flat space with $K = 0$. Similarly, for $\mathbb{S}_\omega^{(3)}$.

By taking into account the above comments, we can say, roughly speaking, that the coefficients $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ that label the CK family $\mathfrak{so}_\omega(5)$ are just the constant curvatures of the four aforementioned spaces. Therefore, two isomorphic algebras in the family $\mathfrak{so}_\omega(5)$ lead to two different sets of four homogeneous spaces through their corresponding Lie groups, and such sets of spaces are those that determine each specific CK geometry amongst the 81 ones. In this respect, we remark that the polarity \mathcal{D} (44), which relates isomorphic CK algebras in the form (82), also interchanges the homogeneous CK spaces as in (46):

$$\mathbb{S}_\omega^{(1)} \xleftrightarrow{\mathcal{D}} \mathbb{S}_\omega^{(4)}, \quad \mathbb{S}_\omega^{(2)} \xleftrightarrow{\mathcal{D}} \mathbb{S}_\omega^{(3)}.$$

For instance, the 4D Euclidean algebra $\mathfrak{iso}(4)$ corresponding to take $\omega = (0, +, +, +)$ yields a flat space of points $\mathbb{S}^{(1)} = \text{ISO}(4) / \text{SO}(4)$ (86) but a positively curved space of 3-hyperplanes $\mathbb{S}^{(4)} = \text{ISO}(4) / \text{ISO}(3)$ (89). Conversely, the isomorphic algebra $\mathfrak{i}'\mathfrak{so}(4) \simeq \mathfrak{iso}(4)$ arising for $\omega = (+, +, +, 0)$, via \mathcal{D} , gives rise to a positively curved space of points $\mathbb{S}^{(1)} = \text{ISO}(4) / \text{ISO}(3)$ but a flat space of 3-hyperplanes $\mathbb{S}^{(4)} = \text{ISO}(4) / \text{SO}(4)$.

It is possible to construct other 4D and 6D symmetric homogeneous spaces from the CK group $\text{SO}_\omega(5)$, which, depending on each particular CK geometry, could be different from the four above ones (85). In particular, any composition of the automorphisms $\Theta^{(m)}$ (33), which form a basis for the $\mathbb{Z}_2^{\otimes 4}$ -grading, gives rise to another automorphism that provides another Cartan decomposition, like (83), and, from it, the corresponding coset space can be constructed. For instance, the composition $\Theta^{(1)}\Theta^{(4)}$ leads to the 6D symmetric homogeneous space

$$\text{SO}_\omega(5) / H'_\omega, \quad H'_\omega \equiv \text{SO}_{\omega_{04}}(2) \otimes \text{SO}_{\omega_2, \omega_3}(3) = \langle J_{04} \rangle \otimes \langle J_{12}, J_{13}, J_{23} \rangle \quad (90)$$

(to be compared with (41)), which can be interpreted as another 6D CK space of lines. This fact can clearly be appreciated in the Lorentzian spacetimes where there exist time-like and space-like lines. In the single case of $\mathfrak{so}(5)$ with $\omega = (+, +, +, +)$ all such possible CK spaces are equivalent to the four spaces (39). Furthermore, it is also possible to obtain generalizations of the polarity \mathcal{D} (44) relating such other homogeneous spaces belonging to different CK geometries with the same (isomorphic) CK algebra. In the 2D case shown in Table 1, the prefix “Co-” in the name of some CK geometries reminds the action of \mathcal{D} that here interchanges $\omega_1 \leftrightarrow \omega_2$; thus, keeping the three geometries in the diagonal unchanged. The full description of all the 2D CK spaces and the generalizations of the polarity \mathcal{D} can be found in [7,12].

4.2. Cayley–Klein Lie Bialgebra

Let us start with the classical r -matrix $r_{04,13}$ (47) for $\mathfrak{so}(5)$. The Lie bialgebra contraction procedure introduced in [31] shows that it is not only necessary to apply the contraction map Φ (68) to the Lie generators of $\mathfrak{so}(5)$ in order to obtain a classical r -matrix for the CK algebra $\mathfrak{so}_\omega(5)$; however, additionally a possible transformation of the quantum deformation parameter z must be considered. We recall that the idea to transform the deformation parameter in contractions of quantum groups was formerly introduced in [153,154]. In our case, the coboundary Lie bialgebra contraction that ensures a well defined limit $\omega_m \rightarrow 0$ (for any m) of both the classical r -matrix and the cocommutator for $\mathfrak{so}(5)$ is given by the transformation [31]

$$\Psi(z) = \frac{z}{\sqrt{\omega_4}} = \frac{z}{\sqrt{\omega_1\omega_2\omega_3\omega_4}}. \quad (91)$$

Then, we apply the composition of the maps (68) and (91) to $r_{04,13}$ in the form

$$r = (\Phi^{-1} \otimes \Phi^{-1}) \circ \Psi^{-1}(r_{04,13}),$$

obtaining that

$$r = z(J_{14} \wedge J_{01} + J_{24} \wedge J_{02} + J_{34} \wedge J_{03} + \sqrt{\omega_1\omega_4} J_{23} \wedge J_{12}). \quad (92)$$

Its Schouten bracket (15) turns out to be

$$\begin{aligned} [[r, r]] = & z^2 (J_{01} \wedge J_{04} \wedge J_{14} + J_{02} \wedge J_{04} \wedge J_{24} + J_{03} \wedge J_{04} \wedge J_{34} + \omega_1\omega_4 J_{12} \wedge J_{13} \wedge J_{23} \\ & + \omega_4(\omega_3 J_{01} \wedge J_{02} \wedge J_{12} + J_{01} \wedge J_{03} \wedge J_{13} + J_{02} \wedge J_{03} \wedge J_{23}) \\ & + \omega_1(J_{12} \wedge J_{14} \wedge J_{24} + J_{13} \wedge J_{14} \wedge J_{34} + \omega_2 J_{23} \wedge J_{24} \wedge J_{34})). \end{aligned} \quad (93)$$

It can be checked that r (92) is a solution of the modified classical Yang–Baxter Equation (14) for any Lie algebra within the CK family $\mathfrak{so}_\omega(5)$ (so for any value of $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$). The corresponding cocommutator can either be obtained from the Lie bialgebra contraction of (48) or through the relation (12) with (92) giving rise to the CK Lie bialgebra $(\mathfrak{so}_\omega(5), \delta(r))$; namely

$$\begin{aligned}
\delta(J_{04}) &= 0, & \delta(J_{13}) &= 0, \\
\delta(J_{12}) &= z\sqrt{\omega_1\omega_4} J_{12} \wedge J_{13}, & \delta(J_{23}) &= z\sqrt{\omega_1\omega_4} J_{23} \wedge J_{13}, \\
\delta(J_{01}) &= z(J_{01} \wedge J_{04} + \omega_1 J_{24} \wedge J_{12} + \omega_1 J_{34} \wedge J_{13} + \sqrt{\omega_1\omega_4} J_{02} \wedge J_{23}), \\
\delta(J_{02}) &= z(J_{02} \wedge J_{04} + \omega_1 J_{12} \wedge J_{14} + \omega_1 \omega_2 J_{34} \wedge J_{23} \\
&\quad + \omega_2 \sqrt{\omega_1\omega_4} J_{23} \wedge J_{01} + \sqrt{\omega_1\omega_4} J_{12} \wedge J_{03}), \\
\delta(J_{03}) &= z(J_{03} \wedge J_{04} + \omega_1 J_{13} \wedge J_{14} + \omega_1 \omega_2 J_{23} \wedge J_{24} + \omega_3 \sqrt{\omega_1\omega_4} J_{02} \wedge J_{12}), \\
\delta(J_{14}) &= z(J_{14} \wedge J_{04} + \omega_4 J_{13} \wedge J_{03} + \omega_3 \omega_4 J_{12} \wedge J_{02} + \omega_2 \sqrt{\omega_1\omega_4} J_{24} \wedge J_{23}), \\
\delta(J_{24}) &= z(J_{24} \wedge J_{04} + \omega_4 J_{23} \wedge J_{03} + \omega_3 \omega_4 J_{01} \wedge J_{12} \\
&\quad + \omega_3 \sqrt{\omega_1\omega_4} J_{12} \wedge J_{34} + \sqrt{\omega_1\omega_4} J_{23} \wedge J_{14}), \\
\delta(J_{34}) &= z(J_{34} \wedge J_{04} + \omega_4 J_{02} \wedge J_{23} + \omega_4 J_{01} \wedge J_{13} + \sqrt{\omega_1\omega_4} J_{24} \wedge J_{12}).
\end{aligned} \tag{94}$$

Now, some remarks are in order.

- The last term in the CK r -matrix (92) is also an r -matrix $r_{13} = z\sqrt{\omega_1\omega_4} J_{23} \wedge J_{12}$ generating the CK Lie bialgebra $(\mathfrak{so}_{\omega_2, \omega_3}(3), \delta(r_{13}))$ with generators $\langle J_{12}, J_{13}, J_{23} \rangle$ and primitive generator J_{13} , which is a sub-Lie bialgebra of $(\mathfrak{so}_\omega(5), \delta(r))$. Notice that, as in (48), J_{04} is the “main” primitive generator such that the product zJ_{04} is dimensionless, while J_{13} is a “secondary” primitive generator.
- The same z -polarity \mathcal{D}_z (50) is an automorphism of the whole family of CK bialgebras relating the cocommutators (94) as

$$(\mathfrak{so}_{\omega_1, \omega_2, \omega_3, \omega_4}(5), \delta(r)) \xleftrightarrow{\mathcal{D}_z} (\mathfrak{so}_{\omega_4, \omega_3, \omega_2, \omega_1}(5), \delta(r)), \tag{95}$$

so interchanging $\omega_1 \leftrightarrow \omega_4$ and $\omega_2 \leftrightarrow \omega_3$ as in (82). Note that the classical r -matrix (92) and its Schouten bracket (93) remain unchanged under (50) and (95).

- As far as the z -maps $\Theta_z^{(m)}$ (49) is concerned, it can directly be checked from the expression of the r -matrix (92) that $\Theta_z^{(2)}$ and $\Theta_z^{(3)}$ are, again, involutive automorphisms of $(\mathfrak{so}_\omega(5), \delta(r))$ for any value of the contraction parameters ω . Moreover, both $\Theta_z^{(1)}$ and $\Theta_z^{(4)}$ become involutions whenever, at least, either $\omega_1 = 0$ or $\omega_4 = 0$, that is, when the last term of r (92) vanishes. Therefore, a complete $\mathbb{Z}_2^{\otimes 4}$ -grading spanned by the four $\Theta_z^{(m)}$ (49) is kept for the 45 Lie bialgebras with contraction parameters $\omega = (0, \omega_2, \omega_3, \omega_4)$ and $\omega = (\omega_1, \omega_2, \omega_3, 0)$, which cover inhomogeneous algebras and their further contractions. The quantum algebras for the first set of 27 bialgebras with $\omega = (0, \omega_2, \omega_3, \omega_4)$ were fully constructed in [102], while the second set is related to the first one by means of \mathcal{D}_z (95); in these results, it can be appreciated that the term $\exp(zJ_{04}/2)$ always appears in the deformed coproduct Δ_z , for any value of $\omega = (0, \omega_2, \omega_3, \omega_4)$, showing that zJ_{04} is dimensionless and that, in this sense, J_{04} is the principal primitive generator.
- The coisotropy condition (22) is always satisfied by the four CK subalgebras $\mathfrak{h}_\omega^{(m)}$

$$\delta(\mathfrak{h}_\omega^{(m)}) \subset \mathfrak{h}_\omega^{(m)} \wedge \mathfrak{so}_\omega(5), \quad m = 1, \dots, 4, \tag{96}$$

but none of them fulfils the Poisson subgroup condition for any value of the contraction parameters (even for the flag algebra with all $\omega_m = 0$)

$$\delta(\mathfrak{h}_\omega^{(m)}) \not\subset \mathfrak{h}_\omega^{(m)} \wedge \mathfrak{h}_\omega^{(m)}.$$

Thus far, we have obtained, in a unified setting, a family of coboundary Lie bialgebra structures $(\mathfrak{so}_\omega(5), \delta(r))$, with quasitriangular classical r -matrix (92) and cocommutator δ (94), which covers 81 particular Lie bialgebras with the aforementioned properties.

However, it is worth stressing that, for some values of the contraction parameters, the CK cocommutator could involve imaginary quantities due to the term $\sqrt{\omega_1\omega_4}$ in the CK r -matrix, although the CK algebras are always real ones. If we require a *real* Lie bialgebra, then

$$\omega_1\omega_4 \geq 0, \quad (97)$$

which excludes the 18 cases with the following values for $(\omega_1, \omega_2, \omega_3, \omega_4)$:

Complex Lie bialgebras: $(+, \omega_2, \omega_3, -)$ and $(-, \omega_2, \omega_3, +)$, $\forall \omega_2, \omega_3$.

Hence, there remain 63 real Lie bialgebras, which are explicitly presented in Table 2 according to the sign or zero value of the contraction parameters and following the notation (75)–(81).

Table 2. The 63 Cayley–Klein algebras with commutation relations (71) endowed with a real Lie bialgebra structure $(\mathfrak{so}_\omega(5), \delta(r))$, determined by the r -matrix (92), according to the sign of the graded contraction parameters $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ such that $\omega_1\omega_4 \geq 0$.

• Simple Lie algebras $\mathfrak{so}(p, q)$	
$\mathfrak{so}(5)$	$\omega = (+, +, +, +)$
$\mathfrak{so}(4, 1)$	$\omega = (+, -, -, +)$
$\mathfrak{so}(3, 2)$	$\omega = (+, +, -, +), (+, -, +, +), (-, +, +, -),$ $(-, +, -, -), (-, -, +, -), (-, -, -, -)$
• Inhomogeneous Lie algebras $\mathfrak{iso}(p, q) \simeq \mathfrak{i}'\mathfrak{so}(p, q)$	
$\mathfrak{iso}(4)$	$\omega = (0, +, +, +), (+, +, +, 0)$
$\mathfrak{iso}(3, 1)$	$\omega = (0, +, +, -), (0, -, +, +), (0, +, -, -), (0, -, -, +),$ $(-, +, +, 0), (+, +, -, 0), (-, -, +, 0), (+, -, -, 0)$
$\mathfrak{iso}(2, 2)$	$\omega = (0, +, -, +), (0, -, +, -), (0, -, -, -), (+, -, +, 0), (-, +, -, 0), (-, -, -, 0)$
• Newton–Hooke-type algebras $\mathfrak{i}_6(\mathfrak{so}(p, q) \oplus \mathfrak{so}(p', q'))$	
$\mathfrak{i}_6(\mathfrak{so}(2) \oplus \mathfrak{so}(3))$	$\omega = (+, 0, +, +), (+, +, 0, +)$
$\mathfrak{i}_6(\mathfrak{so}(2) \oplus \mathfrak{so}(2, 1))$	$\omega = (+, 0, -, +), (+, -, 0, +)$
$\mathfrak{i}_6(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(2, 1))$	$\omega = (-, 0, +, -), (-, 0, -, -), (-, +, 0, -), (-, -, 0, -)$
• Twice inhomogeneous Lie algebras $\mathfrak{iiiso}(p, q) \simeq \mathfrak{i}'\mathfrak{i}'\mathfrak{so}(p, q)$	
$\mathfrak{iiiso}(3)$	$\omega = (0, 0, +, +), (+, +, 0, 0)$
$\mathfrak{iiiso}(2, 1)$	$\omega = (0, 0, +, -), (0, 0, -, +), (0, 0, -, -), (-, +, 0, 0), (+, -, 0, 0), (-, -, 0, 0)$
• Carroll-type algebras $\mathfrak{ii}'\mathfrak{so}(p, q) \simeq \mathfrak{i}'\mathfrak{iso}(p, q)$	
$\mathfrak{ii}'\mathfrak{so}(3)$	$\omega = (0, +, +, 0)$
$\mathfrak{ii}'\mathfrak{so}(2, 1)$	$\omega = (0, +, -, 0), (0, -, +, 0), (0, -, -, 0)$
• $\mathfrak{i}_6(\mathfrak{so}(p, q) \oplus \mathfrak{iso}(p', q'))$	
$\mathfrak{i}_6(\mathfrak{so}(2) \oplus \mathfrak{iso}(2))$	$\omega = (+, 0, 0, +)$
$\mathfrak{i}_6(\mathfrak{so}(1, 1) \oplus \mathfrak{iso}(1, 1))$	$\omega = (-, 0, 0, -)$
• Inhomogeneous Newton–Hooke-type algebras $\mathfrak{i}_4(\mathfrak{i}_4(\mathfrak{so}(p, q) \oplus \mathfrak{so}(p', q')))$	
$\mathfrak{i}_4(\mathfrak{i}_4(\mathfrak{so}(2) \oplus \mathfrak{so}(2)))$	$\omega = (0, +, 0, +), (+, 0, +, 0)$
$\mathfrak{i}_4(\mathfrak{i}_4(\mathfrak{so}(2) \oplus \mathfrak{so}(1, 1)))$	$\omega = (0, +, 0, -), (0, -, 0, +), (-, 0, +, 0), (+, 0, -, 0)$
$\mathfrak{i}_4(\mathfrak{i}_4(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(1, 1)))$	$\omega = (0, -, 0, -), (-, 0, -, 0)$
• Thrice inhomogeneous Lie algebras $\mathfrak{iiiiso}(p, q) \simeq \mathfrak{i}'\mathfrak{i}'\mathfrak{i}'\mathfrak{so}(p, q)$	
$\mathfrak{iiiiso}(2)$	$\omega = (0, 0, 0, +), (+, 0, 0, 0)$
$\mathfrak{iiiiso}(1, 1)$	$\omega = (0, 0, 0, -), (-, 0, 0, 0)$
• Inhomogeneous Carroll-type algebras $\mathfrak{iii}'\mathfrak{so}(p, q) \simeq \mathfrak{i}'\mathfrak{i}'\mathfrak{iso}(p, q)$	
$\mathfrak{iii}'\mathfrak{so}(2)$	$\omega = (0, 0, +, 0), (0, +, 0, 0)$
$\mathfrak{iii}'\mathfrak{so}(1, 1)$	$\omega = (0, 0, -, 0), (0, -, 0, 0)$
• Flag algebra $\mathfrak{iiiso}(1) \simeq \mathfrak{i}'\mathfrak{i}'\mathfrak{i}'\mathfrak{so}(1)$	
$\mathfrak{iiiso}(1)$	$\omega = (0, 0, 0, 0)$

4.3. Dual Cayley–Klein Algebra and Noncommutative Cayley–Klein Spaces

We consider the generators \hat{x}^{ab} in $\mathfrak{so}_\omega(5)^*$ dual to J_{ab} in $\mathfrak{so}_\omega(5)$ with pairing (52) and compute the commutation rules of $\mathfrak{so}_\omega(5)^*$ (10) from the CK cocommutator (94) obtaining

$$\begin{aligned}
 [\hat{x}^{01}, \hat{x}^{02}] &= 0, & [\hat{x}^{01}, \hat{x}^{12}] &= z\omega_3\omega_4\hat{x}^{24}, & [\hat{x}^{02}, \hat{x}^{12}] &= z\omega_3(\sqrt{\omega_1\omega_4}\hat{x}^{03} - \omega_4\hat{x}^{14}), \\
 [\hat{x}^{01}, \hat{x}^{03}] &= 0, & [\hat{x}^{01}, \hat{x}^{13}] &= z\omega_4\hat{x}^{34}, & [\hat{x}^{03}, \hat{x}^{13}] &= -z\omega_4\hat{x}^{14}, \\
 [\hat{x}^{01}, \hat{x}^{04}] &= z\hat{x}^{01}, & [\hat{x}^{01}, \hat{x}^{14}] &= 0, & [\hat{x}^{04}, \hat{x}^{14}] &= -z\hat{x}^{14}, \\
 [\hat{x}^{02}, \hat{x}^{03}] &= 0, & [\hat{x}^{02}, \hat{x}^{23}] &= z(\sqrt{\omega_1\omega_4}\hat{x}^{01} + \omega_4\hat{x}^{34}), & [\hat{x}^{03}, \hat{x}^{23}] &= -z\omega_4\hat{x}^{24}, \\
 [\hat{x}^{02}, \hat{x}^{04}] &= z\hat{x}^{02}, & [\hat{x}^{02}, \hat{x}^{24}] &= 0, & [\hat{x}^{04}, \hat{x}^{24}] &= -z\hat{x}^{24}, \\
 [\hat{x}^{03}, \hat{x}^{04}] &= z\hat{x}^{03}, & [\hat{x}^{03}, \hat{x}^{34}] &= 0, & [\hat{x}^{04}, \hat{x}^{34}] &= -z\hat{x}^{34}, \\
 [\hat{x}^{12}, \hat{x}^{13}] &= z\sqrt{\omega_1\omega_4}\hat{x}^{12}, & [\hat{x}^{12}, \hat{x}^{23}] &= 0, & [\hat{x}^{13}, \hat{x}^{23}] &= -z\sqrt{\omega_1\omega_4}\hat{x}^{23}, \\
 [\hat{x}^{12}, \hat{x}^{14}] &= z\omega_1\hat{x}^{02}, & [\hat{x}^{12}, \hat{x}^{24}] &= -z(\omega_1\hat{x}^{01} + \sqrt{\omega_1\omega_4}\hat{x}^{34}), & [\hat{x}^{14}, \hat{x}^{24}] &= 0, \\
 [\hat{x}^{13}, \hat{x}^{14}] &= z\omega_1\hat{x}^{03}, & [\hat{x}^{13}, \hat{x}^{34}] &= -z\omega_1\hat{x}^{01}, & [\hat{x}^{14}, \hat{x}^{34}] &= 0, \\
 [\hat{x}^{23}, \hat{x}^{24}] &= z\omega_2(\omega_1\hat{x}^{03} - \sqrt{\omega_1\omega_4}\hat{x}^{14}), & [\hat{x}^{23}, \hat{x}^{34}] &= -z\omega_1\omega_2\hat{x}^{02}, & [\hat{x}^{24}, \hat{x}^{34}] &= 0,
 \end{aligned} \tag{98}$$

$$\begin{aligned}
 [\hat{x}^{01}, \hat{x}^{23}] &= -z\omega_2\sqrt{\omega_1\omega_4}\hat{x}^{02}, & [\hat{x}^{01}, \hat{x}^{24}] &= 0, & [\hat{x}^{01}, \hat{x}^{34}] &= 0, \\
 [\hat{x}^{02}, \hat{x}^{13}] &= 0, & [\hat{x}^{02}, \hat{x}^{14}] &= 0, & [\hat{x}^{02}, \hat{x}^{34}] &= 0, \\
 [\hat{x}^{03}, \hat{x}^{12}] &= -z\sqrt{\omega_1\omega_4}\hat{x}^{02}, & [\hat{x}^{03}, \hat{x}^{14}] &= 0, & [\hat{x}^{03}, \hat{x}^{24}] &= 0, \\
 [\hat{x}^{04}, \hat{x}^{12}] &= 0, & [\hat{x}^{04}, \hat{x}^{13}] &= 0, & [\hat{x}^{04}, \hat{x}^{23}] &= 0, \\
 [\hat{x}^{12}, \hat{x}^{34}] &= z\omega_3\sqrt{\omega_1\omega_4}\hat{x}^{24}, & [\hat{x}^{13}, \hat{x}^{24}] &= 0, & [\hat{x}^{14}, \hat{x}^{23}] &= -z\sqrt{\omega_1\omega_4}\hat{x}^{24}.
 \end{aligned} \tag{99}$$

Alternatively, the same result is achieved by applying a contraction map directly to the commutation relations of the dual algebra $\mathfrak{so}(5)^*$ (53) and (54). By taking into account the pairing (52) and the maps Φ (68) and Ψ (91), the full contraction map for $\mathfrak{so}(5)^*$ turns out to be

$$\Phi \circ \Psi(\hat{x}^{ab}, z) := (\Phi(\hat{x}^{ab}), \Psi(z)) = \left(\frac{\hat{x}^{ab}}{\sqrt{\omega_{ab}}}, \frac{z}{\sqrt{\omega_{04}}} \right).$$

We remark that the commutation relations (98) and (99) define a real dual CK algebra $\mathfrak{so}(5)^*$ under the constraint (97), thus covering the 63 cases given in Table 2. Note also that all the commutators (99) vanish for either $\omega_1 = 0$ or $\omega_4 = 0$, corresponding to the dual algebra of inhomogeneous algebras and their contractions.

Similarly to (55), we express the dual CK algebra $\mathfrak{so}_\omega(5)^*$ as the sum of two vector spaces

$$\mathfrak{so}_\omega(5)^* = \mathfrak{h}_{\perp, \omega}^{(m)} \oplus \mathfrak{t}_{\perp, \omega}^{(m)}, \quad m = 1, \dots, 4,$$

where $\mathfrak{h}_{\perp, \omega}^{(m)}$ and $\mathfrak{t}_{\perp, \omega}^{(m)}$ are the annihilators of the vector subspaces $\mathfrak{h}_\omega^{(m)}$ and $\mathfrak{t}_\omega^{(m)}$ introduced in (83) and fulfilling (84). As we already performed in Section 3.3 for $\mathfrak{so}(5)^*$, we define the first-order noncommutative CK spaces by

$$\mathbb{S}_{z, \omega}^{(m)} := \mathfrak{h}_{\perp, \omega}^{(m)}, \quad m = 1, \dots, 4; \tag{100}$$

see (56)–(59). Each linear noncommutative space $\mathbb{S}_{z, \omega}^{(m)}$ is the first-order in the quantum coordinates of the complete noncommutative space associated with the homogeneous CK space $\mathbb{S}_\omega^{(m)}$ (85). We display, in Table 3, the defining commutation relations for the four noncommutative CK spaces along with the Lie brackets among $\mathfrak{h}_{\perp, \omega}^{(m)}$ and $\mathfrak{t}_{\perp, \omega}^{(m)}$.

Table 3. The first-order noncommutative Cayley–Klein spaces $\mathbb{S}_{z,\omega}^{(m)}$ (100) and the relations among $\mathfrak{h}_{\perp,\omega}^{(m)}$ and $\mathfrak{t}_{\perp,\omega}^{(m)}$. Real commutation relations are ensured whenever $\omega_1\omega_4 \geq 0$ covering the 63 cases shown in Table 2.

<p>• Noncommutative CK space of points $\mathbb{S}_{z,\omega}^{(1)} \equiv \mathfrak{h}_{\perp,\omega}^{(1)} = \langle \hat{x}^{01}, \hat{x}^{02}, \hat{x}^{03}, \hat{x}^{04} \rangle$</p> $[\hat{x}^{0i}, \hat{x}^{04}] = z\hat{x}^{0i} \quad [\hat{x}^{0i}, \hat{x}^{0j}] = 0 \quad i, j = 1, 2, 3$ $[\mathfrak{h}_{\perp,\omega}^{(1)}, \mathfrak{h}_{\perp,\omega}^{(1)}] \subset \mathfrak{h}_{\perp,\omega}^{(1)} \quad [\mathfrak{h}_{\perp,\omega}^{(1)}, \mathfrak{t}_{\perp,\omega}^{(1)}] \subset \mathfrak{t}_{\perp,\omega}^{(1)} + \omega_1\omega_4\mathfrak{h}_{\perp,\omega}^{(1)} \quad [\mathfrak{t}_{\perp,\omega}^{(1)}, \mathfrak{t}_{\perp,\omega}^{(1)}] \subset \omega_1\mathfrak{h}_{\perp,\omega}^{(1)} + \omega_1\omega_4\mathfrak{t}_{\perp,\omega}^{(1)}$		
<p>• Noncommutative CK space of lines $\mathbb{S}_{z,\omega}^{(2)} \equiv \mathfrak{h}_{\perp,\omega}^{(2)} = \langle \hat{x}^{02}, \hat{x}^{03}, \hat{x}^{04}, \hat{x}^{12}, \hat{x}^{13}, \hat{x}^{14} \rangle$</p> $[\hat{x}^{02}, \hat{x}^{03}] = 0 \quad [\hat{x}^{02}, \hat{x}^{04}] = z\hat{x}^{02} \quad [\hat{x}^{03}, \hat{x}^{04}] = z\hat{x}^{03}$ $[\hat{x}^{12}, \hat{x}^{13}] = z\sqrt{\omega_1\omega_4}\hat{x}^{12} \quad [\hat{x}^{12}, \hat{x}^{14}] = z\omega_1\hat{x}^{02} \quad [\hat{x}^{13}, \hat{x}^{14}] = z\omega_1\hat{x}^{03}$ $[\hat{x}^{02}, \hat{x}^{12}] = z\omega_3(\sqrt{\omega_1\omega_4}\hat{x}^{03} - \omega_4\hat{x}^{14}) \quad [\hat{x}^{03}, \hat{x}^{12}] = -z\sqrt{\omega_1\omega_4}\hat{x}^{02} \quad [\hat{x}^{04}, \hat{x}^{12}] = 0$ $[\hat{x}^{02}, \hat{x}^{13}] = 0 \quad [\hat{x}^{03}, \hat{x}^{13}] = -z\omega_4\hat{x}^{14} \quad [\hat{x}^{04}, \hat{x}^{13}] = 0$ $[\hat{x}^{02}, \hat{x}^{14}] = 0 \quad [\hat{x}^{03}, \hat{x}^{14}] = 0 \quad [\hat{x}^{04}, \hat{x}^{14}] = -z\hat{x}^{14}$ $[\mathfrak{h}_{\perp,\omega}^{(2)}, \mathfrak{h}_{\perp,\omega}^{(2)}] \subset \mathfrak{h}_{\perp,\omega}^{(2)} \quad [\mathfrak{h}_{\perp,\omega}^{(2)}, \mathfrak{t}_{\perp,\omega}^{(2)}] \subset \mathfrak{t}_{\perp,\omega}^{(2)} \quad [\mathfrak{t}_{\perp,\omega}^{(2)}, \mathfrak{t}_{\perp,\omega}^{(2)}] \subset \omega_1\omega_2\mathfrak{h}_{\perp,\omega}^{(2)}$		
<p>• Noncommutative CK space of 2-planes $\mathbb{S}_{z,\omega}^{(3)} \equiv \mathfrak{h}_{\perp,\omega}^{(3)} = \langle \hat{x}^{03}, \hat{x}^{04}, \hat{x}^{13}, \hat{x}^{14}, \hat{x}^{23}, \hat{x}^{24} \rangle$</p> $[\hat{x}^{03}, \hat{x}^{13}] = -z\omega_4\hat{x}^{14} \quad [\hat{x}^{03}, \hat{x}^{23}] = -z\omega_4\hat{x}^{24} \quad [\hat{x}^{13}, \hat{x}^{23}] = -z\sqrt{\omega_1\omega_4}\hat{x}^{23}$ $[\hat{x}^{04}, \hat{x}^{14}] = -z\hat{x}^{14} \quad [\hat{x}^{04}, \hat{x}^{24}] = -z\hat{x}^{24} \quad [\hat{x}^{14}, \hat{x}^{24}] = 0$ $[\hat{x}^{03}, \hat{x}^{04}] = z\hat{x}^{03} \quad [\hat{x}^{13}, \hat{x}^{04}] = 0 \quad [\hat{x}^{23}, \hat{x}^{04}] = 0$ $[\hat{x}^{03}, \hat{x}^{14}] = 0 \quad [\hat{x}^{13}, \hat{x}^{14}] = z\omega_1\hat{x}^{03} \quad [\hat{x}^{23}, \hat{x}^{14}] = z\sqrt{\omega_1\omega_4}\hat{x}^{24}$ $[\hat{x}^{03}, \hat{x}^{24}] = 0 \quad [\hat{x}^{13}, \hat{x}^{24}] = 0 \quad [\hat{x}^{23}, \hat{x}^{24}] = z\omega_2(\omega_1\hat{x}^{03} - \sqrt{\omega_1\omega_4}\hat{x}^{14})$ $[\mathfrak{h}_{\perp,\omega}^{(3)}, \mathfrak{h}_{\perp,\omega}^{(3)}] \subset \mathfrak{h}_{\perp,\omega}^{(3)} \quad [\mathfrak{h}_{\perp,\omega}^{(3)}, \mathfrak{t}_{\perp,\omega}^{(3)}] \subset \mathfrak{t}_{\perp,\omega}^{(3)} \quad [\mathfrak{t}_{\perp,\omega}^{(3)}, \mathfrak{t}_{\perp,\omega}^{(3)}] \subset \omega_3\omega_4\mathfrak{h}_{\perp,\omega}^{(3)}$		
<p>• Noncommutative CK space of 3-hyperplanes $\mathbb{S}_{z,\omega}^{(4)} \equiv \mathfrak{h}_{\perp,\omega}^{(4)} = \langle \hat{x}^{04}, \hat{x}^{14}, \hat{x}^{24}, \hat{x}^{34} \rangle$</p> $[\hat{x}^{i4}, \hat{x}^{04}] = z\hat{x}^{i4} \quad [\hat{x}^{i4}, \hat{x}^{j4}] = 0 \quad i, j = 1, 2, 3$ $[\mathfrak{h}_{\perp,\omega}^{(4)}, \mathfrak{h}_{\perp,\omega}^{(4)}] \subset \mathfrak{h}_{\perp,\omega}^{(4)} \quad [\mathfrak{h}_{\perp,\omega}^{(4)}, \mathfrak{t}_{\perp,\omega}^{(4)}] \subset \mathfrak{t}_{\perp,\omega}^{(4)} + \omega_1\omega_4\mathfrak{h}_{\perp,\omega}^{(4)} \quad [\mathfrak{t}_{\perp,\omega}^{(4)}, \mathfrak{t}_{\perp,\omega}^{(4)}] \subset \omega_4\mathfrak{h}_{\perp,\omega}^{(4)} + \omega_1\omega_4\mathfrak{t}_{\perp,\omega}^{(4)}$		

Now, we analyse the structure and properties of such noncommutative CK spaces, which do strongly depend on the contraction/curvature parameters. The four noncommutative spaces close on a Lie subalgebra $\mathfrak{h}_{\perp,\omega}^{(m)}$, in agreement with the coisotropy condition (96), and the noncommutative spaces of lines and 2-planes are both reductive and symmetric as it was also the case for $\mathfrak{so}(5)^*$ (see (60)). Furthermore, the explicit presence of the curvature parameters allows us to highlight some properties for the contracted noncommutative spaces straightforwardly. In particular, if we set $\omega_4 = 0$ in the noncommutative space of points, we find that

$$\omega_4 = 0: \quad [\mathfrak{h}_{\perp,\omega}^{(1)}, \mathfrak{h}_{\perp,\omega}^{(1)}] \subset \mathfrak{h}_{\perp,\omega}^{(1)}, \quad [\mathfrak{h}_{\perp,\omega}^{(1)}, \mathfrak{t}_{\perp,\omega}^{(1)}] \subset \mathfrak{t}_{\perp,\omega}^{(1)}, \quad [\mathfrak{t}_{\perp,\omega}^{(1)}, \mathfrak{t}_{\perp,\omega}^{(1)}] \subset \omega_1\mathfrak{h}_{\perp,\omega}^{(1)}.$$

Likewise, taking $\omega_1 = 0$ in the noncommutative space of 3-hyperplanes, we obtain that

$$\omega_1 = 0: \quad [\mathfrak{h}_{\perp,\omega}^{(4)}, \mathfrak{h}_{\perp,\omega}^{(4)}] \subset \mathfrak{h}_{\perp,\omega}^{(4)}, \quad [\mathfrak{h}_{\perp,\omega}^{(4)}, \mathfrak{t}_{\perp,\omega}^{(4)}] \subset \mathfrak{t}_{\perp,\omega}^{(4)}, \quad [\mathfrak{t}_{\perp,\omega}^{(4)}, \mathfrak{t}_{\perp,\omega}^{(4)}] \subset \omega_4\mathfrak{h}_{\perp,\omega}^{(4)}.$$

Thus, both contracted noncommutative spaces are reductive and symmetric (to be compared with (84)). A remarkable common property for the four noncommutative spaces is that, when $\omega_m = 0$, the subspace $\mathfrak{t}_{\perp, \omega}^{(m)}$ becomes an abelian subalgebra ($m = 1, 2, 3, 4$):

$$\omega_m = 0: \quad [\mathfrak{h}_{\perp, \omega}^{(m)}, \mathfrak{h}_{\perp, \omega}^{(m)}] \subset \mathfrak{h}_{\perp, \omega}^{(m)}, \quad [\mathfrak{h}_{\perp, \omega}^{(m)}, \mathfrak{t}_{\perp, \omega}^{(m)}] \subset \mathfrak{t}_{\perp, \omega}^{(m)}, \quad [\mathfrak{t}_{\perp, \omega}^{(m)}, \mathfrak{t}_{\perp, \omega}^{(m)}] = 0.$$

Such relations can be applied, for instance, to the Poincaré and Euclidean algebras with $\omega_1 = 0$ (75) for the noncommutative space of points, to the Newton–Hooke-type algebras with $\omega_2 = 0$ (80) for the noncommutative space of lines, to the twice inhomogeneous algebras (Galilei) with $\omega_1 = \omega_2 = 0$ (77) for the noncommutative spaces of points and lines, and so on up to the flag algebra (81) for the four noncommutative spaces.

The dual polarity d_z (62) also holds for $\mathfrak{so}(5)_\omega^*$ interchanging the noncommutative CK spaces as in (63) and the curvature parameters in the form $\omega_1 \leftrightarrow \omega_4$ and $\omega_2 \leftrightarrow \omega_3$. Moreover, if we consider the z -maps $\theta_z^{(m)}$ (61) in the dual CK algebra $\mathfrak{so}_\omega(5)^*$, we again find that only $\theta_z^{(2)}$ and $\theta_z^{(3)}$ are always involutive automorphisms of the commutation rules (98) and (99) (as for $\mathfrak{so}(5)^*$). However both $\theta_z^{(1)}$ and $\theta_z^{(4)}$ become involutions whenever the product $\omega_1 \omega_4 = 0$. Consequently, when at least either $\omega_1 = 0$ or $\omega_4 = 0$, the four maps $\theta_z^{(m)}$ (61) span a $\mathbb{Z}_2^{\otimes 4}$ -grading for $\mathfrak{so}_\omega(5)^*$, and the four noncommutative CK spaces (100) are all reductive and symmetric; recall that these 45 cases correspond to the inhomogeneous algebras $\mathfrak{iso}(p, q)$ with $p + q = 4$ with curvature coefficients $(0, \omega_2, \omega_3, \omega_4)$ (75) or $(\omega_1, \omega_2, \omega_3, 0)$ (76) and their further contractions.

Finally, as we advanced at the end of Section 2.2, it is worth stressing that the structure of the first-order noncommutative CK space of points $\mathbb{S}_{z, \omega}^{(1)}$, shown in Table 3, is shared by the 63 CK real Lie bialgebras with $\omega_1 \omega_4 \geq 0$ of Table 2 since no ω_m appears within $\mathbb{S}_{z, \omega}^{(1)}$, and this is formally similar to the κ -Minkowski spacetime (25). Furthermore, the commutation relations of $\mathbb{S}_{z, \omega}^{(1)}$ are kept linear under full quantization for the 27 CK bialgebras with the parameters $(0, \omega_2, \omega_3, \omega_4)$, while higher-order terms in the quantum coordinates are expected for the CK bialgebras with $\omega_1 \neq 0$.

Similar properties hold for the first-order noncommutative CK space of 3-hyperplanes $\mathbb{S}_{z, \omega}^{(4)}$, which remains as a linear full noncommutative space for the 27 CK bialgebras with parameters $(\omega_1, \omega_2, \omega_3, 0)$. Nevertheless, if one looks at the *four* first-order noncommutative CK spaces in Table 3 *altogether*, then one finds that the four contraction parameters appear explicitly. Thus, the set of four noncommutative spaces $\mathbb{S}_{z, \omega}^{(m)}$ is *different* for each specific CK bialgebra except for the nine cases with $\omega_1 = \omega_4 = 0$, for which all the terms involving any ω_m vanish.

Consequently, this observation suggests the necessity of constructing other noncommutative spaces beyond the usual noncommutative spacetime for a given quantum deformation. To the best of our knowledge, there are very scarce results that concern noncommutative spaces of lines in this research direction [99,100].

A physical (kinematical) analysis on the noncommutative CK spaces of points and lines will be addressed in Section 6.

4.4. Drinfel'd Double Structures for Cayley–Klein Algebras

Let us consider the classical r -matrix \tilde{r}_D (66) coming from the Drinfel'd double structure of $\mathfrak{so}(5)$. We apply the composition of the contraction maps (68) and (91) in the form

$$r_D = (\Phi^{-1} \otimes \Phi^{-1}) \circ \Psi^{-1}(\tilde{r}_D), \quad \forall \omega_m \neq 0,$$

obtaining that

$$\begin{aligned} r_D &= z \left(J_{14} \wedge J_{01} + J_{24} \wedge J_{02} + J_{34} \wedge J_{03} + \sqrt{\omega_1 \omega_4} J_{23} \wedge J_{12} + \frac{1}{\sqrt{\omega_2 \omega_3}} J_{13} \wedge J_{04} \right) \\ &= r + \frac{z}{\sqrt{\omega_2 \omega_3}} J_{13} \wedge J_{04}, \quad \forall \omega_m \neq 0, \end{aligned} \quad (101)$$

which is a superposition of the classical CK r -matrix r (92) with a Reshetikhin twist $J_{13} \wedge J_{04}$ formed by the two commuting primitive generators. We remark that, by construction, r_D is a classical r -matrix coming from the Drinfel'd double structure for the simple Lie algebras contained in the CK family $\mathfrak{so}_\omega(5)$. Moreover, r_D leads to the same Schouten bracket as for r (93) (there are no twist contributions) so that it is a solution of the modified classical Yang–Baxter Equation (14).

If we now require r_D (101) to define a real Lie bialgebra, $(\mathfrak{so}_\omega(5), \delta_D(r_D))$, we have to impose the restriction corresponding to r (97) together with the new one determined by the twist:

$$\omega_1 \omega_4 > 0 \quad \text{and} \quad \omega_2 \omega_3 > 0, \quad (102)$$

which leads to four possible cases as shown in Table 4, where we have named them according with their kinematical interpretation that we shall show in Section 6.

Table 4. Simple Lie algebras with a real r -matrix r_D (101) coming from a Drinfel'd double structure according to the sign of the graded contraction parameters $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ and bilinear form \mathbf{I}_ω (74), along with their contractions to non-simple Lie algebras endowed with a real Lie bialgebra $(\mathfrak{so}_\omega(5), \delta_D(r_D))$.

Simple Lie algebras with a Drinfel'd double real structure				
(I)	Spherical	$\mathfrak{so}(5)$	$\omega = (+, +, +, +)$	$\mathbf{I}_\omega = (+, +, +, +, +)$
(II)	De Sitter	$\mathfrak{so}(4, 1)$	$\omega = (+, -, -, +)$	$\mathbf{I}_\omega = (+, +, -, +, +)$
(III)	Anti-de Sitter	$\mathfrak{so}(3, 2)$	$\omega = (-, +, +, -)$	$\mathbf{I}_\omega = (+, -, -, -, +)$
(IV)	Anti-de Sitter	$\mathfrak{so}(3, 2)$	$\omega = (-, -, -, -)$	$\mathbf{I}_\omega = (+, -, +, -, +)$
Non-simple Lie algebras with a real Lie bialgebra via contraction				
(Ia)	Euclidean	$\mathfrak{iso}(4)$	$\omega = (0, +, +, +)$	
(Ia')	Para-Euclidean	$\mathfrak{i}'\mathfrak{so}(4)$	$\omega = (+, +, +, 0)$	
(IIa)	Poincaré	$\mathfrak{iso}(3, 1)$	$\omega = (0, -, -, +)$	
(IIa')		$\mathfrak{i}'\mathfrak{so}(3, 1)$	$\omega = (+, -, -, 0)$	
(IIIa)	Poincaré	$\mathfrak{iso}(3, 1)$	$\omega = (0, +, +, -)$	
(IIIa')	Para-Poincaré	$\mathfrak{i}'\mathfrak{so}(3, 1)$	$\omega = (-, +, +, 0)$	
(IVa)		$\mathfrak{iso}(2, 2)$	$\omega = (0, -, -, -)$	
(IVa')		$\mathfrak{i}'\mathfrak{so}(2, 2)$	$\omega = (-, -, -, 0)$	
(Ib)	Carroll	$\mathfrak{ii}'\mathfrak{so}(3)$	$\omega = (0, +, +, 0)$	$(\text{Ib}) \equiv (\text{Ib}') \equiv (\text{IIIb}) \equiv (\text{IIIb}')$
(IIb)		$\mathfrak{ii}'\mathfrak{so}(2, 1)$	$\omega = (0, -, -, 0)$	$(\text{IIb}) \equiv (\text{IIb}') \equiv (\text{IVb}) \equiv (\text{IVb}')$

In order to obtain the possible graded contractions of r_D (101), we first indicate that this diverges under the limits $\omega_2 \rightarrow 0$ and $\omega_3 \rightarrow 0$, so that the restriction $\omega_2 \omega_3 > 0$ must be kept. Secondly, both contractions $\omega_1 \rightarrow 0$ and $\omega_4 \rightarrow 0$ are well-defined and consistent with the condition (97). This, in turn, means that there are ten possible contractions for r_D that provide an r -matrix generating a real Lie bialgebra for non-simple Lie algebras (see Table 2).

These are also displayed in Table 4 (with the kinematical terminology for the cases that will appear in Section 6), where the notation indicates the sequence of contractions

of the real Lie bialgebra $(\mathfrak{so}_\omega(5), \delta_D(r_D))$, with the “prime” corresponding to $\omega_4 = 0$. For instance:

$$\begin{array}{lll} \text{(I)} \xrightarrow{\omega_1=0} \text{(Ia)} \xrightarrow{\omega_4=0} \text{(Ib')} & \text{Spherical} \rightarrow \text{Euclidean} \rightarrow \text{Carroll} \\ \mathfrak{so}(5) \longrightarrow \mathfrak{iso}(4) \longrightarrow \mathfrak{ii}'\mathfrak{so}(3) & (+, +, +, +) \rightarrow (0, +, +, +) \rightarrow (0, +, +, 0) \\ \text{(III)} \xrightarrow{\omega_4=0} \text{(IIIa')} \xrightarrow{\omega_1=0} \text{(IIIb)} & \text{Anti-de Sitter} \rightarrow \text{Para-Poincaré} \rightarrow \text{Carroll} \\ \mathfrak{so}(3, 2) \longrightarrow \mathfrak{i}'\mathfrak{so}(3, 1) \longrightarrow \mathfrak{ii}'\mathfrak{so}(3) & (-, +, +, -) \rightarrow (-, +, +, 0) \rightarrow (0, +, +, 0) \end{array}$$

Recall the Lie algebra isomorphisms provided by \mathcal{D} (44): $\mathfrak{i}'\mathfrak{so}(4) \simeq \mathfrak{iso}(4)$, $\mathfrak{i}'\mathfrak{so}(3, 1) \simeq \mathfrak{iso}(3, 1)$ and $\mathfrak{i}'\mathfrak{so}(2, 2) \simeq \mathfrak{iso}(2, 2)$. Hence, any contraction sequence ends on either the Carroll bialgebra (cases (I) and (III)) or on the $\mathfrak{ii}'\mathfrak{so}(2, 1)$ one (cases (II) and (IV)).

The effect of the twist $J_{13} \wedge J_{04}$ in r_D (101) with respect to the CK r -matrix (92) can be highlighted by associating it with a second deformation parameter ϑ in a similar form to (67), that is,

$$r_D = r + \vartheta J_{13} \wedge J_{04}, \quad (103)$$

such that the r -matrix coming from a Drinfel'd double structure corresponds to the one-parametric deformation with

$$\vartheta \equiv \frac{z}{\sqrt{\omega_2 \omega_3}}, \quad \omega_2 \omega_3 > 0. \quad (104)$$

The cocommutator δ_D , obtained with (12), is just the CK cocommutator δ (94) plus new terms coming from the twist, which are denoted δ_ϑ . Hence, $\delta_D = \delta + \delta_\vartheta$ with δ_ϑ given by

$$\begin{aligned} \delta_\vartheta(J_{04}) &= 0, & \delta_\vartheta(J_{13}) &= 0, \\ \delta_\vartheta(J_{12}) &= \vartheta \omega_2 J_{23} \wedge J_{04}, & \delta(J_{23}) &= \vartheta \omega_3 J_{04} \wedge J_{12}, \\ \delta(J_{01}) &= \vartheta(J_{04} \wedge J_{03} + \omega_1 J_{13} \wedge J_{14}), \\ \delta(J_{02}) &= \vartheta \omega_1 \omega_2 J_{13} \wedge J_{24}, \\ \delta(J_{03}) &= \vartheta \omega_2 \omega_3 (J_{01} \wedge J_{04} + \omega_1 J_{13} \wedge J_{34}), \\ \delta(J_{14}) &= \vartheta \omega_2 \omega_3 (J_{04} \wedge J_{34} + \omega_4 J_{01} \wedge J_{13}), \\ \delta(J_{24}) &= \vartheta \omega_3 \omega_4 J_{02} \wedge J_{13}, \\ \delta(J_{34}) &= \vartheta (J_{14} \wedge J_{04} + \omega_4 J_{03} \wedge J_{13}). \end{aligned} \quad (105)$$

Consequently, $(\mathfrak{so}_{\omega_2, \omega_3}(3), \delta_D)$ with generators $\langle J_{12}, J_{13}, J_{23} \rangle$ does not remain as a Lie sub-bialgebra of $(\mathfrak{so}_\omega(5), \delta_D(r_D))$. However, if this is enlarged with the primitive generator J_{04} , then it provides the Lie sub-bialgebra $(\mathfrak{so}_{\omega_2, \omega_3}(3) \oplus \mathfrak{so}_{\omega_4}(2), \delta_D)$.

The polarity \mathcal{D}_z (50) and the involutions $\Theta_z^{(2)}$ and $\Theta_z^{(3)}$ (49) also hold for the two-parametric deformation determined by (103) provided that ϑ is unchanged. Nevertheless, in the proper Drinfel'd double case with a single deformation parameter z , with the identification (104), the above maps do not remain since $z \rightarrow -z$.

As far as the first-order noncommutative spaces associated to $(\mathfrak{so}_\omega(5), \delta_D(r_D))$ is concerned, it is straightforward to prove that the coisotropy condition (22) is only fulfilled for the subalgebras $\mathfrak{h}_\omega^{(1)}$ and $\mathfrak{h}_\omega^{(4)}$ (see (105)):

$$\delta_D(\mathfrak{h}_\omega^{(l)}) \subset \mathfrak{h}_\omega^{(l)} \wedge \mathfrak{so}_\omega(5), \quad l = 1, 4, \quad \delta_D(\mathfrak{h}_\omega^{(k)}) \not\subset \mathfrak{h}_\omega^{(k)} \wedge \mathfrak{so}_\omega(5), \quad k = 2, 3,$$

for any of the 14 Lie algebras displayed in Table 4. Therefore, only the twisted noncommutative CK spaces of points and 3-hyperplanes $\mathbb{S}_{z, \vartheta, \omega}^{(l)}$ ($l = 1, 4$) can consistently be constructed. In particular, from (105) and applying the quantum duality (10) with pairing (52), we

directly obtain the defining commutation relations for the twisted noncommutative CK space of points $\mathbb{S}_{z,\vartheta,\omega}^{(1)}$:

$$\begin{aligned} [\hat{x}^{01}, \hat{x}^{04}] &= z\hat{x}^{01} + \vartheta\omega_2\omega_3\hat{x}^{03}, & [\hat{x}^{02}, \hat{x}^{04}] &= z\hat{x}^{02}, \\ [\hat{x}^{03}, \hat{x}^{04}] &= z\hat{x}^{03} - \vartheta\hat{x}^{01}, & [\hat{x}^{0i}, \hat{x}^{0j}] &= 0, \quad i, j = 1, 2, 3. \end{aligned} \quad (106)$$

In the same way, $\mathbb{S}_{z,\vartheta,\omega}^{(4)}$ can also be obtained. Clearly $\mathbb{S}_{z,\vartheta,\omega}^{(1)}$ (106) is not isomorphic to $\mathbb{S}_{z,\omega}^{(1)}$ given in Table 3. Moreover, since $\omega_2\omega_3 > 0$, this factor can be scaled to +1 within the commutators (106) via the scalings

$$\hat{x}^{03} \rightarrow \sqrt{\omega_2\omega_3} \hat{x}^{03}, \quad \vartheta \rightarrow \sqrt{\omega_2\omega_3} \vartheta,$$

which shows that $\mathbb{S}_{z,\vartheta,\omega}^{(1)}$ is the *common* first-order twisted noncommutative CK space of points for the 14 Lie bialgebras shown in Table 4; clearly, higher-order terms in the quantum coordinates may arise for each specific case.

Finally, we stress that it is not ensured at all that a given contracted Drinfel'd double r_D gives rise to a Drinfel'd double structure for a non-semisimple Lie algebra and, in fact, this problem should be studied case by case. Nevertheless, we can answer negatively to this question for the contracted r -matrices of Table 4. It was established in [127], from the results given in [145], that there does not exist any Drinfel'd double structure for Poincaré, Euclidean and Carroll algebras at this dimension.

In contrast, as we commented at the end of Section 2.3, in lower dimensions, such structures do exist and the classification of Drinfel'd doubles was recently performed for the (2 + 1)D Poincaré [127] and 3D Euclidean algebras [128]. Moreover, to the best of our knowledge, the classification of Drinfel'd doubles for the (anti-)de Sitter algebras has only been carried out in (2 + 1) dimensions [115].

In (3 + 1) dimensions, there is no such classification for the simple Lie algebras $\mathfrak{so}(p, q)$, and there has only been constructed the Drinfel'd double structure here considered for $\mathfrak{so}(5)$, reviewed in Section 3.4, and from it a Drinfel'd double for the anti-de Sitter algebra $\mathfrak{so}(3, 2)$ [129], that we advance, which is just the case (III) in Table 4. Therefore, we have obtained two new r -matrices coming from Drinfel'd doubles, one for the de Sitter $\mathfrak{so}(4, 1)$ and another for the anti-de Sitter $\mathfrak{so}(3, 2)$ (cases (II) and (IV)), although our results do not convey a complete classification.

The physical (kinematical) interpretation of the CK r -matrices r (92) and r_D (101) along with their associated first-order noncommutative spaces will be described in detail in Section 6.

5. Kinematical Algebras and Homogeneous Spaces

As we already mentioned in the previous section, the kinematical algebras introduced in [14] arise as particular cases of graded contractions of $\mathfrak{so}(5)$ [20,152], and thus they appear within the CK family $\mathfrak{so}_\omega(5)$ for some specific values of the contraction parameters $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ (except for the static algebra, which does not belong to the CK family). These kinematical algebras have recently been derived from deformation theory in [145,155]; in this respect, recall that Lie algebra deformations [156] can be regarded as the opposite processes to Lie algebra contractions [13,138,157,158].

In order to deal with kinematical algebras, let us introduce a physical basis denoting by P_0 , $\mathbf{P} = (P_1, P_2, P_3)$, $\mathbf{K} = (K_1, K_2, K_3)$ and $\mathbf{J} = (J_1, J_2, J_3)$ the generators of time translations, spatial translations, boosts and rotations, respectively. These ten generators are isometries of a (3 + 1)D spacetime of constant curvature. The 11 kinematical algebras [14] are contained within a three-parametric Lie algebra, here denoted $\mathfrak{so}_{\Lambda, c, \lambda}(5)$, with commutation relations given by

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, P_j] = \epsilon_{ijk} P_k, \quad [J_i, K_j] = \epsilon_{ijk} K_k, \quad [J_i, P_0] = 0, \quad (107)$$

$$\begin{aligned}
[K_i, P_0] &= \lambda P_i, & [K_i, P_j] &= \frac{1}{c^2} \delta_{ij} P_0, & [K_i, K_j] &= -\lambda \frac{1}{c^2} \epsilon_{ijk} J_k, \\
[P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= \Lambda \frac{1}{c^2} \epsilon_{ijk} J_k,
\end{aligned} \tag{108}$$

where, from now on, the indices $i, j, k = 1, 2, 3$ and sum over repeated indices will be understood. Recall also that the commutators (107) are a consequence of 3-space isotropy [14], and they are shared by any Lie algebra in $\mathfrak{so}_{\Lambda, c, \lambda}(5)$, while the Lie brackets (108) distinguish the specific kinematical algebra according to the values of the real parameters Λ , c and λ .

The family $\mathfrak{so}_{\Lambda, c, \lambda}(5)$ has two Casimir operators: a quadratic one, coming from the Killing–Cartan form, which is given by

$$C = \frac{1}{c^2} P_0^2 - \lambda \mathbf{P}^2 + \Lambda \left(\mathbf{K}^2 - \lambda \frac{1}{c^2} \mathbf{J}^2 \right), \tag{109}$$

and a fourth-order Casimir [18] (with the exception of the static algebra [14] corresponding to set $\Lambda = \lambda = 0$ and $c \rightarrow \infty$ in (108), so all of these brackets vanish).

Moreover, $\mathfrak{so}_{\Lambda, c, \lambda}(5)$ is endowed with the parity \mathcal{P} and the time-reversal \mathcal{T} involutive automorphisms defined by [14]

$$\begin{aligned}
\mathcal{P}(P_0, \mathbf{P}, \mathbf{K}, \mathbf{J}) &= (P_0, -\mathbf{P}, -\mathbf{K}, \mathbf{J}), \\
\mathcal{T}(P_0, \mathbf{P}, \mathbf{K}, \mathbf{J}) &= (-P_0, \mathbf{P}, -\mathbf{K}, \mathbf{J}), \\
\mathcal{PT}(P_0, \mathbf{P}, \mathbf{K}, \mathbf{J}) &= (-P_0, -\mathbf{P}, \mathbf{K}, \mathbf{J}).
\end{aligned} \tag{110}$$

Each of them provides a type of contraction: the composition \mathcal{PT} corresponds to the (flat) spacetime contraction ($\Lambda \rightarrow 0$), the parity \mathcal{P} to the speed-space contraction ($c \rightarrow \infty$) and the time-reversal \mathcal{T} to the speed-time contraction ($\lambda \rightarrow 0$) (see [135] for the (2 + 1)D kinematical algebras and contractions within the CK family $\mathfrak{so}_\omega(4)$ and their Drinfel’d–Jimbo quantum deformation). In other words, the quantities Λ , $1/c^2$ and λ behave as graded contraction parameters, each of them corresponding to the \mathbb{Z}_2 -grading of $\mathfrak{so}_{\Lambda, c, \lambda}(5)$ determined by \mathcal{PT} , \mathcal{P} and \mathcal{T} , respectively.

From the Lie group $\text{SO}_{\Lambda, c, \lambda}(5)$ of $\mathfrak{so}_{\Lambda, c, \lambda}(5)$, we construct the (3 + 1)D spacetime and the 6D space of lines as the coset spaces

$$\begin{aligned}
\mathbf{ST}^{3+1} &= \text{SO}_{\Lambda, c, \lambda}(5) / H_{\text{st}}, & H_{\text{st}} &= \langle \mathbf{K}, \mathbf{J} \rangle, \\
\mathbf{L}^6 &= \text{SO}_{\Lambda, c, \lambda}(5) / H_{\text{line}}, & H_{\text{line}} &= \langle P_0 \rangle \otimes \langle \mathbf{J} \rangle = \langle P_0 \rangle \otimes \text{SO}(3),
\end{aligned} \tag{111}$$

such that H_{st} and H_{line} are the isotropy subgroups of an event and a line, respectively. Thus, these are symmetric homogeneous spaces associated, in this order, with the composition \mathcal{PT} and parity \mathcal{P} involutions.

Similarly to the discussion on the curvature of the CK spaces (86)–(89) in Section 4.1, we remark that the (3 + 1)D spacetime \mathbf{ST}^{3+1} is a rank-one homogeneous space such that all their sectional curvatures K are equal and constant. However, the 6D space of lines \mathbf{L}^6 is of rank-two, and only the sectional curvatures K of any two-plane direction spanned by any two tangent vectors (P_i, P_j) , (K_i, K_j) and (P_i, K_i) ($i, j = 1, 2, 3$) are equal among themselves and constant, with the remaining ones, (P_i, K_j) with $i \neq j$, as generically non-constant (or zero).

Furthermore, when $\text{SO}_{\Lambda, c, \lambda}(5)$ is a non-simple Lie group, the metric on either space (111) could be degenerated, and, in this case, an invariant foliation arises so that an additional metric defined on each leaf of the foliation is necessary to determine completely the metric structure of the space [132]. Moreover, it is important to take into account that, in principle, the (3 + 1)D spacetime \mathbf{ST}^{3+1} does not necessarily coincide with the CK space of points $\mathbb{S}_\omega^{(1)}$ (86) (in most cases it does) and, likewise, with the 6D space of lines \mathbf{L}^6 with respect to the CK space of lines $\mathbb{S}_\omega^{(2)}$ (87). Nevertheless, they can always be identified with another CK space, as for instance (90) for the space of lines. This fact will depend on the kinematical assignation of the geometrical CK generators that we shall study next in Section 6.

In what follows, we describe the 11 kinematical algebras although we shall only focus on the homogeneous spaces (111) for nine of them: the Lorentzian, Newtonian and Carrollian cases. Additionally, we shall show how the three classical Riemannian algebras (and their homogenous spaces) can also be recovered from the family $\mathfrak{so}_{\Lambda,c,\lambda}(5)$ (5). These nine kinematical algebras/spaces plus the three Riemannian ones are those that will appear in Section 6, and they are summarized in Table 5.

Table 5. Kinematical algebras with commutation relations (107) and (108) together with their corresponding symmetric homogeneous (3 + 1)D spacetimes and 6D spaces of lines (111) of sectional curvature K according to the values of the graded contraction parameters (Λ, c, λ) . The same results for the three Riemannian cases are similarly displayed.

Lorentzian algebras and homogeneous spaces		
<ul style="list-style-type: none"> • De Sitter $\mathfrak{l}_+ = \mathfrak{so}(4, 1): \Lambda > 0, c \text{ finite}, \lambda = 1$ $\mathbf{dS}^{3+1} = \text{SO}(4, 1)/\text{SO}(3, 1),$ $K = -\Lambda < 0$ $\mathbf{LdS}^6 = \text{SO}(4, 1)/(\text{SO}(1, 1) \otimes \text{SO}(3)),$ $K = -\frac{1}{c^2}$	<ul style="list-style-type: none"> • Poincaré $\mathfrak{l}_0 = \mathfrak{iso}(3, 1): \Lambda = 0, c \text{ finite}, \lambda = 1$ $\mathbf{M}^{3+1} = \text{ISO}(3, 1)/\text{SO}(3, 1), K = 0$ $\mathbf{LM}^6 = \text{ISO}(3, 1)/(\mathbb{R} \otimes \text{SO}(3)),$ $K = -\frac{1}{c^2}$	<ul style="list-style-type: none"> • Anti-de Sitter $\mathfrak{l}_- = \mathfrak{so}(3, 2): \Lambda < 0, c \text{ finite}, \lambda = 1$ $\mathbf{AdS}^{3+1} = \text{SO}(3, 2)/\text{SO}(3, 1),$ $K = -\Lambda > 0$ $\mathbf{LAdS}^6 = \text{SO}(3, 2)/(\text{SO}(2) \otimes \text{SO}(3)),$ $K = -\frac{1}{c^2}$
Newtonian algebras and homogeneous spaces		
<ul style="list-style-type: none"> • Expanding Newton–Hooke $\mathfrak{n}_+ = \mathfrak{ie}(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(3)): \Lambda > 0,$ $c = \infty, \lambda = 1$ $\mathbf{N}_+^{3+1} = \text{N}_+/\text{ISO}(3), K = -\Lambda < 0$ $\mathbf{LN}_+^6 = \text{N}_+/(\text{SO}(1, 1) \otimes \text{SO}(3)),$ $K = 0$	<ul style="list-style-type: none"> • Galilei $\mathfrak{n}_0 = \mathfrak{iiiso}(3): \Lambda = 0, c = \infty, \lambda = 1$ $\mathbf{N}_0^{3+1} \equiv \mathbf{G}^{3+1} = \mathbf{IISO}(3)/\text{ISO}(3),$ $K = 0$ $\mathbf{LG}^6 = \mathbf{IISO}(3)/(\mathbb{R} \otimes \text{SO}(3)), K = 0$	<ul style="list-style-type: none"> • Oscillating Newton–Hooke $\mathfrak{n}_- = \mathfrak{ie}(\mathfrak{so}(2) \oplus \mathfrak{so}(3)): \Lambda < 0,$ $c = \infty, \lambda = 1$ $\mathbf{N}_-^{3+1} = \text{N}_-/\text{ISO}(3), K = -\Lambda > 0$ $\mathbf{LN}_-^6 = \text{N}_-/(\text{SO}(2) \otimes \text{SO}(3)), K = 0$
Carrollian algebras and homogeneous spaces		
<ul style="list-style-type: none"> • Para-Euclidean $\mathfrak{c}_+ = \mathfrak{i}'\mathfrak{so}(4): \Lambda > 0, c = 1, \lambda = 0$ $\mathbf{C}_+^{3+1} = \mathbf{I}'\text{SO}(4)/\text{ISO}(3), K = \Lambda > 0$ $\mathbf{LC}_+^6 = \mathbf{I}'\text{SO}(4)/(\mathbb{R} \otimes \text{SO}(3)),$ $K = \Lambda > 0$	<ul style="list-style-type: none"> • Carroll $\mathfrak{c}_0 = \mathfrak{ii}'\mathfrak{so}(3): \Lambda = 0, c = 1, \lambda = 0$ $\mathbf{C}_0^{3+1} \equiv \mathbf{C}^{3+1} = \mathbf{II}'\text{SO}(3)/\text{ISO}(3),$ $K = 0$ $\mathbf{LC}^6 = \mathbf{II}'\text{SO}(3)/(\mathbb{R} \otimes \text{SO}(3)), K = 0$	<ul style="list-style-type: none"> • Para-Poincaré $\mathfrak{c}_- = \mathfrak{i}'\mathfrak{so}(3, 1): \Lambda < 0, c = 1, \lambda = 0$ $\mathbf{C}_-^{3+1} = \mathbf{I}'\text{SO}(3, 1)/\text{ISO}(3), K = \Lambda < 0$ $\mathbf{LC}_-^6 = \mathbf{I}'\text{SO}(3, 1)/(\mathbb{R} \otimes \text{SO}(3)),$ $K = \Lambda < 0$
Riemannian algebras and homogeneous spaces		
<ul style="list-style-type: none"> • Hyperbolic $\mathfrak{so}(4, 1): \Lambda > 0, c = i, \lambda = 1 \text{ or}$ $\Lambda < 0, c = 1, \lambda = -1$ $\mathbf{H}^4 = \text{SO}(4, 1)/\text{SO}(4), K < 0$ $\mathbf{LH}^6 = \text{SO}(4, 1)/(\text{SO}(1, 1) \otimes \text{SO}(3)),$ $K = +1$	<ul style="list-style-type: none"> • Euclidean $\mathfrak{iso}(4): \Lambda = 0, c = i, \lambda = 1 \text{ or}$ $\Lambda = 0, c = 1, \lambda = -1$ $\mathbf{E}^4 = \text{ISO}(4)/\text{SO}(4), K = 0$ $\mathbf{LE}^6 = \text{ISO}(4)/(\mathbb{R} \otimes \text{SO}(3)), K = +1$	<ul style="list-style-type: none"> • Spherical $\mathfrak{so}(5): \Lambda < 0, c = i, \lambda = 1 \text{ or}$ $\Lambda > 0, c = 1, \lambda = -1$ $\mathbf{S}^4 = \text{SO}(5)/\text{SO}(4), K > 0$ $\mathbf{LS}^6 = \text{SO}(5)/(\text{SO}(2) \otimes \text{SO}(3)),$ $K = +1$

5.1. Lorentzian Algebras

If we set the parameter $\lambda = 1$ and consider c finite, we find that $\mathfrak{so}_{\Lambda,c,\lambda}(5)$ covers the three Lorentzian algebras \mathfrak{l}_Λ of relativistic (3 + 1)D spacetimes such that the Lie brackets (108) now read

$$\begin{aligned}
 [K_i, P_0] &= P_i, & [K_i, P_j] &= \frac{1}{c^2} \delta_{ij} P_0, & [K_i, K_j] &= -\frac{1}{c^2} \epsilon_{ijk} J_k, \\
 [P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= \Lambda \frac{1}{c^2} \epsilon_{ijk} J_k,
 \end{aligned} \tag{112}$$

where c is the speed of light and Λ is the cosmological constant. Then, we obtain the de Sitter (dS) $\mathfrak{l}_+ = \mathfrak{so}(4, 1)$, anti-de Sitter (AdS) $\mathfrak{l}_- = \mathfrak{so}(3, 2)$ and Poincaré $\mathfrak{l}_0 = \mathfrak{iso}(3, 1)$ algebras. The quadratic Casimir (109) for \mathfrak{l}_Λ reads

$$C = \frac{1}{c^2} P_0^2 - \mathbf{P}^2 + \Lambda \left(\mathbf{K}^2 - \frac{1}{c^2} \mathbf{J}^2 \right), \tag{113}$$

and the fourth-order Casimir, related to the Pauli–Lubanski 4-vector, can be found in [18].

The Lorentz subalgebra corresponds to $\mathfrak{so}(3,1) = \langle \mathbf{K}, \mathbf{J} \rangle$, which is the Lie algebra of the isotropy subgroup $H_{\text{st}} = \text{SO}(3,1)$ (111). The constant sectional curvature of the $(3+1)\text{D}$ spacetime is $K = -\Lambda$. Notice that the cosmological constant can be expressed in terms of a *time* universe radius τ through $\Lambda = \pm 1/\tau^2$, so that to take $\Lambda = 0$ corresponds to the limit $\tau \rightarrow \infty$, which is simply the spacetime contraction providing the flat Minkowskian spacetime \mathbf{M}^{3+1} from the $(3+1)\text{D}$ (A)dS spacetimes. The isotropy subgroup of a line is $H_{\text{line}} = \text{SO}_{-\Lambda}(2) \otimes \text{SO}(3)$, and the homogeneous space of (time-like) lines (111) is, in the three cases, of negative constant sectional curvature $K = -1/c^2$ [132]. Recall that the notation $\text{SO}_{-\Lambda}(2)$ means that $\text{SO}_+(2) = \text{SO}(2)$, $\text{SO}_-(2) = \text{SO}(1,1)$ and $\text{SO}_0(2) = \mathbb{R}$.

5.2. Newtonian Algebras

The non-relativistic limit $c \rightarrow \infty$ (or speed-space contraction) of \mathfrak{l}_Λ (112) gives rise to three Newtonian algebras \mathfrak{n}_Λ with Lie brackets

$$\begin{aligned} [K_i, P_0] &= P_i, & [K_i, P_j] &= 0, & [K_i, K_j] &= 0, \\ [P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= 0, \end{aligned}$$

where $\Lambda = \pm 1/\tau^2$ and τ is again the time universe radius. The second-order Casimir (113) reduces to

$$\mathcal{C} = -\mathbf{P}^2 + \Lambda \mathbf{K}^2,$$

and the corresponding fourth-order Casimir can be found in [18]. This non-relativistic limit is obtained by setting $\lambda = 1$ and $c \rightarrow \infty$ in (108) and (109). In this way, we find the expanding Newton–Hooke (NH) \mathfrak{n}_+ , oscillating NH \mathfrak{n}_- and the Galilei $\mathfrak{n}_0 \equiv \mathfrak{iso}(3)$ algebras, which have the following structure (see (80) and (77), respectively):

$$\begin{aligned} \mathfrak{n}_+ &\equiv \mathfrak{i}_6(\mathfrak{so}(1,1) \oplus \mathfrak{so}(3)) \\ &\equiv \mathbb{R}^6 \oplus_S (\mathfrak{so}(1,1) \oplus \mathfrak{so}(3)): \quad \mathbb{R}^6 = \langle \mathbf{P}, \mathbf{K} \rangle, \quad \mathfrak{so}(1,1) = \langle P_0 \rangle, \quad \mathfrak{so}(3) = \langle \mathbf{J} \rangle. \\ \mathfrak{n}_- &\equiv \mathfrak{i}_6(\mathfrak{so}(2) \oplus \mathfrak{so}(3)) \\ &\equiv \mathbb{R}^6 \oplus_S (\mathfrak{so}(2) \oplus \mathfrak{so}(3)): \quad \mathbb{R}^6 = \langle \mathbf{P}, \mathbf{K} \rangle, \quad \mathfrak{so}(2) = \langle P_0 \rangle, \quad \mathfrak{so}(3) = \langle \mathbf{J} \rangle. \\ \mathfrak{n}_0 &\equiv \mathfrak{iso}(3) \equiv \mathbb{R}^4 \oplus_S (\mathbb{R}^3 \oplus_S \mathfrak{so}(3)): \quad \mathbb{R}^4 = \langle P_0, \mathbf{P} \rangle, \quad \mathbb{R}^3 = \langle \mathbf{K} \rangle, \quad \mathfrak{so}(3) = \langle \mathbf{J} \rangle. \end{aligned}$$

The isotropy subgroup H_{st} (111) is now the 3D Euclidean subgroup $\text{ISO}(3)$ spanned by rotations and (commuting) Newtonian boosts, and the $(3+1)\text{D}$ spacetime has the same sectional curvature as in the Lorentzian spacetimes: $K = -\Lambda$. The metric is degenerate and corresponds to an “absolute-time”, so that there exists an invariant foliation under the action of the Newtonian group N_Λ , whose leaves are defined by a constant time, which is determined by a 3D non-degenerate Euclidean spatial metric restricted to each leaf of the foliation [132,147]. The isotropy subgroup of a line is again $H_{\text{line}} = \text{SO}_{-\Lambda}(2) \otimes \text{SO}(3)$, but the homogeneous space of lines (111) is flat, i.e., $K = 0$ [132].

5.3. Carrollian Algebras

We set $\lambda = 0$ and $c = 1$ in the commutators (108) yielding the three Carrollian algebras \mathfrak{c}_Λ with Lie brackets and second-order Casimir given by

$$\begin{aligned} [K_i, P_0] &= 0, & [K_i, P_j] &= \delta_{ij} P_0, & [K_i, K_j] &= 0, \\ [P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= \Lambda \epsilon_{ijk} J_k, \end{aligned} \quad (114)$$

$$\mathcal{C} = P_0^2 + \Lambda \mathbf{K}^2.$$

Notice that now the parameter Λ has dimensions of length^{-2} instead of time^{-2} , and the Carrollian boosts have dimensions of speed instead of speed^{-1} (which were the cases in the Lorentzian and Newtonian algebras).

The algebra c_Λ comprises the para-Euclidean algebra $c_+ \equiv i'so(4)$ (isomorphic to the Euclidean $iso(4)$), the para-Poincaré algebra $c_- \equiv i'so(3,1)$ (isomorphic to the Poincaré $iso(3,1)$) and the proper Carroll algebra $c_0 \equiv ii'so(3) \equiv i'iso(3)$, which have the following structure [147] (see (76), (78) and (79)):

$$\begin{aligned} c_+ &\equiv i'so(4) = \mathbb{R}^4 \oplus_S so(4): & \mathbb{R}^4 &= \langle P_0, \mathbf{K} \rangle, & so(4) &= \langle \mathbf{P}, \mathbf{J} \rangle. \\ c_- &\equiv i'so(3,1) = \mathbb{R}^4 \oplus_S so(3,1): & \mathbb{R}^4 &= \langle P_0, \mathbf{K} \rangle, & so(3,1) &= \langle \mathbf{P}, \mathbf{J} \rangle. \\ c_0 &\equiv i'iso(3) = \mathbb{R}^4 \oplus_S (\mathbb{R}^3 \oplus_S so(3)): & \mathbb{R}^4 &= \langle P_0, \mathbf{K} \rangle, & \mathbb{R}^3 &= \langle \mathbf{P} \rangle, & so(3) &= \langle \mathbf{J} \rangle, \\ c_0 &\equiv ii'so(3) = \mathbb{R}^4 \oplus_S (\mathbb{R}^3 \oplus_S so(3)): & \mathbb{R}^4 &= \langle P_0, \mathbf{P} \rangle, & \mathbb{R}^3 &= \langle \mathbf{K} \rangle, & so(3) &= \langle \mathbf{J} \rangle. \end{aligned}$$

The isotropy subgroup H_{st} (111) is, again, the 3D Euclidean subgroup $ISO(3)$ spanned by rotations and (commuting) Carrollian boosts, but the $(3+1)$ D spacetime has sectional curvature $K = +\Lambda$ (instead of $K = -\Lambda$ as in the Lorentzian and Newtonian spacetimes). The metric is degenerate corresponding to an “absolute-space”, and there exists an invariant foliation under the action of the Carrollian group characterized by a 1D time metric restricted to each leaf of the foliation [147]. The isotropy subgroup of a line is $H_{line} = \mathbb{R} \otimes SO(3)$ and the homogeneous space of lines (111) has the same curvature as the spacetimes, thus, equal to $+\Lambda$.

5.4. The Two Remaining Kinematical Algebras

For the sake of completeness, we also mention that the para-Galilei algebra [14] arises for $\lambda = 0$ and $c \rightarrow \infty$, that is, the commutators (108) reduce to

$$\begin{aligned} [K_i, P_0] &= 0, & [K_i, P_j] &= 0, & [K_i, K_j] &= 0, \\ [P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= 0, \end{aligned}$$

for any value of $\Lambda \neq 0$ (apply the map $P_0 \rightarrow \pm P_0/\Lambda$), while the second-order Casimir (109) simply reads $\mathcal{C} = \Lambda \mathbf{K}^2$. The static algebra [14] corresponds to the most contracted algebra within the kinematical family for $\Lambda = \lambda = 0$ and $c \rightarrow \infty$,

$$\begin{aligned} [K_i, P_0] &= 0, & [K_i, P_j] &= 0, & [K_i, K_j] &= 0, \\ [P_0, P_i] &= 0, & [P_i, P_j] &= 0, \end{aligned} \quad (115)$$

with trivial second-order Casimir $\mathcal{C} = 0$. In fact, the static algebra is the only kinematical one that does not appear within the CK family $so_\omega(5)$ [17]; however, it can be obtained from the general solution of the grading equations for $so(5)$ [20,152]. Observe that the static algebra is not a quasisimple Lie algebra in the sense that it does not have the same number of Casimir invariants as the simple Lie algebra $so(5)$.

When one compares the commutation relations of the static algebra (115) with those for the Carroll one (114) with $\Lambda = 0$, one finds that the Carroll algebra can be regarded as a centrally extended algebra, with non-trivial central extension P_0 , from the static algebra, and there cannot be added any other central extension to the Carroll algebra [14] (see [19] for the central extensions of the CK algebras in any dimension).

In this respect, we remark that, in [145], the $(3+1)$ D kinematical algebras were constructed from the static algebra through deformation theory (see [155] for higher dimensions). We also recall that twist deformations for the para-Galilei, static and Carroll algebras were obtained in [146].

Neither the para-Galilei nor the static algebra will appear within the deformations that we shall describe in Section 6; therefore, they are omitted in Table 5.

5.5. Riemannian Algebras

Additionally, but not kinematically, we can set $\lambda = 1$ and the speed of light equal to the imaginary unit $c = i$ in (108) finding the commutators

$$\begin{aligned} [K_i, P_0] &= P_i, & [K_i, P_j] &= -\delta_{ij}P_0, & [K_i, K_j] &= \epsilon_{ijk}J_k, \\ [P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= -\Lambda \epsilon_{ijk}J_k, \end{aligned} \quad (116)$$

with second-order Casimir (109) given by

$$\mathcal{C} = -P_0^2 - \mathbf{P}^2 + \Lambda(\mathbf{K}^2 + \mathbf{J}^2).$$

In this way, we obtain $\mathfrak{so}(5)$ for $\Lambda < 0$, $\mathfrak{iso}(4)$ for $\Lambda = 0$ and $\mathfrak{so}(4, 1)$ for $\Lambda > 0$. The generator P_0 now behaves as another space translation, while the generators \mathbf{K} are no longer boosts but rotations. The isotropy subgroup H_{st} (111) is $\text{SO}(4) = \langle \mathbf{K}, \mathbf{J} \rangle$, such that we recover the three classical 4D Riemannian spaces of constant sectional curvature $K = -\Lambda$: spherical ($K > 0$), Euclidean ($K = 0$) and hyperbolic ($K < 0$) spaces. The isotropy subgroup of a line is $H_{\text{line}} = \text{SO}_{-\Lambda}(2) \otimes \text{SO}(3)$, and the corresponding 6D space of lines has positive curvature $K = +1$ for any value of Λ [132].

Alternatively, we can set $\lambda = -1$ and $c = 1$ in (108) obtaining that

$$\begin{aligned} [K_i, P_0] &= -P_i, & [K_i, P_j] &= \delta_{ij}P_0, & [K_i, K_j] &= \epsilon_{ijk}J_k, \\ [P_0, P_i] &= -\Lambda K_i, & [P_i, P_j] &= \Lambda \epsilon_{ijk}J_k, \end{aligned}$$

$$\mathcal{C} = P_0^2 + \mathbf{P}^2 + \Lambda(\mathbf{K}^2 + \mathbf{J}^2),$$

which are equivalent to the Lie brackets (116) by means of the maps $P_0 \rightarrow -P_0$ and $\Lambda \rightarrow -\Lambda$. Therefore, we again obtain the same Riemannian algebras (and homogeneous spaces), but now $\mathfrak{so}(5)$ for $\Lambda > 0$, $\mathfrak{iso}(4)$ for $\Lambda = 0$ and $\mathfrak{so}(4, 1)$ for $\Lambda < 0$.

6. Kinematical Lie Bialgebras and Noncommutative Spaces

Our aim now is to interpret in the kinematical framework the Lie bialgebras coming from the classical CK r -matrix r (92) and the Drinfel'd doubles further provided by r_D (101) together with the corresponding first-order noncommutative spaces of points and lines displayed in Table 3 and the twisted one (106). With this in mind, we perform different identifications between the “geometrical” generators J_{ab} of $\mathfrak{so}_\omega(5)$ (71) and the kinematical ones of $\mathfrak{so}_{\Lambda, c, \lambda}(5)$ (107) and (108), which will convey physical correspondences between the contraction/curvature CK parameters ω and Λ, c, λ .

According to [102], the main idea is to start with the main primitive generator J_{04} and to identify it either with a spatial translation P_i or with the time translation P_0 . Since the product zJ_{04} must be dimensionless, we shall obtain the so-called [135] “space-like” deformations $J_{04} \equiv P_i$, with the deformation parameter z being a fundamental *length scale*, and the “time-like” deformations $J_{04} \equiv P_0$, with z being a fundamental *time scale*.

In particular, we shall study first three classes of kinematical deformations, called A, B and C, such that their properties are determined by the two primitive (undeformed) generators (J_{04}, J_{13}) corresponding to (P_3, K_2) , (P_2, J_2) and (P_0, J_2) , respectively. We remark that, in these three classes, the time translation generator $P_0 \equiv J_{0l}$ for $l = 1, 2, 4$, and the remaining case $P_0 \equiv J_{03}$ would provide results that are equivalent, under certain Lie algebra automorphisms, to those already contained in the class A, and thus we omit it.

Therefore, the classes A and B will give rise to space-like deformations, while the class C will lead to time-like ones. Additionally, we shall construct an AdS Lie bialgebra for which z is dimensionless with primitive generators (J_2, P_0) , and it will correspond to the new Drinfel'd double of case (IV) in Table 4; we shall call it class D. We point out that such four classes of kinematical deformations will contain the four Drinfel'd doubles for the simple Lie algebras of Table 4.

The main results that will be obtained along Sections 6.1–6.4 concerning the kinematical r -matrices are presented in Table 6. From them, their corresponding first-order noncommutative spacetimes and spaces of lines will be computed as summarized in Table 7. Comments on these results will be presented in Sections 6.5 and 6.6.

Table 6. Four classes of real classical r -matrices for the kinematical and Riemannian algebras with commutation relations (107) and (108). For each class, we display the dimensions of the quantum deformation parameter z and the primitive generators (determined by (J_{04}, J_{13})), the r -matrix, the specific Lie algebras according to the values of the graded contraction parameters (Λ, c, λ) as in Table 5, the CK parameters ω and the Drinfel'd double r_D together with the corresponding case given in Table 4.

Class	z & (J_{04}, J_{13})	Kinematical Real r -Matrices and Lie Algebras			$\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$	Drinfel'd Double r_D
A	Length (P_3, K_2)	$r = z(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1 + \sqrt{-\Lambda} J_3 \wedge K_1)$				$r_D = r + zK_2 \wedge P_3$
		AdS	$\mathfrak{so}(3, 2)$	$\Lambda < 0, c, \lambda = 1$	$\omega = (+, -, +, +)$	No
		Poincaré	$\mathfrak{iso}(3, 1)$	$\Lambda = 0, c, \lambda = 1$	$\omega = (0, -, +, +)$	No
		Oscillating NH	\mathfrak{n}_-	$\Lambda < 0, c = \infty, \lambda = 1$	$\omega = (+, 0, +, +)$	No
		Galilei	$\mathfrak{ii}\mathfrak{so}(3)$	$\Lambda = 0, c = \infty, \lambda = 1$	$\omega = (0, 0, +, +)$	No
		Spherical	$\mathfrak{so}(5)$	$\Lambda < 0, c = i, \lambda = 1$	$\omega = (+, +, +, +)$	(I)
		Euclidean	$\mathfrak{iso}(4)$	$\Lambda = 0, c = i, \lambda = 1$	$\omega = (0, +, +, +)$	(Ia)
B	Length (P_2, J_2)	$r = z(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3 + \sqrt{\Lambda} K_3 \wedge K_1)$				$r_D = r - zJ_2 \wedge P_2$
		dS	$\mathfrak{so}(4, 1)$	$\Lambda > 0, c = 1, \lambda = 1$	$\omega = (+, -, -, +)$	(II)
		Poincaré	$\mathfrak{iso}(3, 1)$	$\Lambda = 0, c = 1, \lambda = 1$	$\omega = (0, -, -, +)$	(IIa)
C	Time (P_0, J_2)	$r = z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{-\lambda} \Lambda J_1 \wedge J_3)$				$r_D = r - zJ_2 \wedge P_0$
		AdS	$\mathfrak{so}(3, 2)$	$\Lambda < 0, c = 1, \lambda = 1$	$\omega = (-, +, +, -)$	(III)
		Poincaré	$\mathfrak{iso}(3, 1)$	$\Lambda = 0, c = 1, \lambda = 1$	$\omega = (0, +, +, -)$	(IIIa)
		Para-Euclidean	$\mathfrak{i}'\mathfrak{so}(4)$	$\Lambda > 0, c = 1, \lambda = 0$	$\omega = (+, +, +, 0)$	(Ia')
		Para-Poincaré	$\mathfrak{i}'\mathfrak{so}(3, 1)$	$\Lambda < 0, c = 1, \lambda = 0$	$\omega = (-, +, +, 0)$	(IIIa')
		Carroll	$\mathfrak{ii}'\mathfrak{so}(3)$	$\Lambda = 0, c = 1, \lambda = 0$	$\omega = (0, +, +, 0)$	(Ib) \equiv (IIIb)
		Spherical	$\mathfrak{so}(5)$	$\Lambda > 0, c = 1, \lambda = -1$	$\omega = (+, +, +, +)$	(I)
		Euclidean	$\mathfrak{iso}(4)$	$\Lambda = 0, c = 1, \lambda = -1$	$\omega = (0, +, +, +)$	(Ia)
D	None (J_2, P_0)	$r = z(K_1 \wedge K_3 + K_2 \wedge P_2 + P_1 \wedge P_3 + J_3 \wedge J_1)$				$r_D = r + zP_0 \wedge J_2$
		AdS	$\mathfrak{so}(3, 2)$	$\Lambda = -1, c = 1, \lambda = 1$	$\omega = (-, -, -, -)$	(IV)

Table 7. The first-order noncommutative spacetimes $\mathbf{ST}_z^{3+1} = \langle \hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3 \rangle$ and spaces of lines $\mathbf{L}_z^6 = \langle \hat{x}^1, \hat{x}^2, \hat{x}^3, \hat{\xi}^1, \hat{\xi}^2, \hat{\xi}^3 \rangle$ for the classes A, B and C of kinematical and Riemannian Lie bialgebras shown in Table 6 with the notation of Table 5; the case D has no associated noncommutative space. First-order twisted noncommutative spacetimes coming from the Drinfel'd double structures and their contractions presented in Table 4 are also written in terms of the twist deformation parameter ϑ such that $\vartheta \equiv z$ corresponds to the proper (or contracted) Drinfel'd double.

Class A	• Noncommutative spacetimes with $\Lambda \leq 0, \lambda = 1$: $\mathbf{AdS}_z^{3+1}, \mathbf{M}_z^{3+1}, \mathbf{N}_{-z}^{3+1}, \mathbf{G}_z^{3+1}, \mathbf{S}_z^4, \mathbf{E}_z^4$			
	$[\hat{x}^0, \hat{x}^3] = z\hat{x}^0$	$[\hat{x}^1, \hat{x}^3] = z\hat{x}^1$	$[\hat{x}^2, \hat{x}^3] = z\hat{x}^2$	$[\hat{x}^0, \hat{x}^1] = [\hat{x}^0, \hat{x}^2] = [\hat{x}^1, \hat{x}^2] = 0$
	• Twisted noncommutative spaces of points with $\Lambda \leq 0, \lambda = 1, c = i$: $\mathbf{S}_{z,\vartheta}^4, \mathbf{E}_{z,\vartheta}^4$			
	$[\hat{x}^0, \hat{x}^3] = z\hat{x}^0 + \vartheta\hat{x}^2$	$[\hat{x}^1, \hat{x}^3] = z\hat{x}^1$	$[\hat{x}^2, \hat{x}^3] = z\hat{x}^2 - \vartheta\hat{x}^0$	$[\hat{x}^0, \hat{x}^1] = [\hat{x}^0, \hat{x}^2] = [\hat{x}^1, \hat{x}^2] = 0$
	• Noncommutative space of lines with $\Lambda < 0, \lambda = 1$: $\mathbf{LAdS}_z^6, \mathbf{LN}_{-z}^6, \mathbf{LS}_z^6$			
	$[\hat{x}^1, \hat{x}^2] = 0$	$[\hat{x}^1, \hat{x}^3] = z\hat{x}^1$	$[\hat{x}^2, \hat{x}^3] = z\hat{x}^2$	
	$[\hat{\xi}^1, \hat{\xi}^2] = z\sqrt{-\Lambda}\hat{\xi}^1$	$[\hat{\xi}^1, \hat{\xi}^3] = -z\Lambda\hat{x}^1$	$[\hat{\xi}^2, \hat{\xi}^3] = -z\Lambda\hat{x}^2$	
	$[\hat{x}^1, \hat{\xi}^1] = z(\sqrt{-\Lambda}\hat{x}^2 - \hat{\xi}^3)$	$[\hat{x}^2, \hat{\xi}^1] = -z\sqrt{-\Lambda}\hat{x}^1$	$[\hat{x}^3, \hat{\xi}^1] = 0$	
	$[\hat{x}^1, \hat{\xi}^2] = 0$	$[\hat{x}^2, \hat{\xi}^2] = -z\hat{\xi}^3$	$[\hat{x}^3, \hat{\xi}^2] = 0$	
	$[\hat{x}^1, \hat{\xi}^3] = 0$	$[\hat{x}^2, \hat{\xi}^3] = 0$	$[\hat{x}^3, \hat{\xi}^3] = -z\hat{\xi}^3$	
	• Noncommutative space of lines with $\Lambda = 0, \lambda = 1$: $\mathbf{LM}_z^6, \mathbf{LG}_z^6, \mathbf{LE}_z^6$			
	$[\hat{x}^1, \hat{x}^2] = 0$	$[\hat{x}^1, \hat{x}^3] = z\hat{x}^1$	$[\hat{x}^2, \hat{x}^3] = z\hat{x}^2$	
	$[\hat{\xi}^i, \hat{\xi}^j] = 0$	$[\hat{\xi}^i, \hat{\xi}^j] = -z\delta_{ij}\hat{\xi}^3$		
Class B	• Noncommutative spacetimes with $\Lambda \geq 0, c = 1, \lambda = 1$: $\mathbf{dS}_z^{3+1}, \mathbf{M}_z^{3+1}$			
	$[\hat{x}^0, \hat{x}^2] = z\hat{x}^0$	$[\hat{x}^1, \hat{x}^2] = z\hat{x}^1$	$[\hat{x}^3, \hat{x}^2] = z\hat{x}^3$	$[\hat{x}^0, \hat{x}^1] = [\hat{x}^0, \hat{x}^3] = [\hat{x}^1, \hat{x}^3] = 0$
	• Twisted noncommutative spacetimes with $\Lambda \geq 0, c = 1, \lambda = 1$: $\mathbf{dS}_{z,\vartheta}^{3+1}, \mathbf{M}_{z,\vartheta}^{3+1}$			
	$[\hat{x}^0, \hat{x}^2] = z\hat{x}^0$	$[\hat{x}^1, \hat{x}^2] = z\hat{x}^1 + \vartheta\hat{x}^3$	$[\hat{x}^3, \hat{x}^2] = z\hat{x}^3 - \vartheta\hat{x}^1$	$[\hat{x}^0, \hat{x}^1] = [\hat{x}^0, \hat{x}^3] = [\hat{x}^1, \hat{x}^3] = 0$
	• Noncommutative space of lines with $\Lambda = 0, c = 1, \lambda = 1$: \mathbf{LM}_z^6			
	$[\hat{x}^1, \hat{x}^2] = z\hat{x}^1$	$[\hat{x}^1, \hat{x}^3] = 0$	$[\hat{x}^3, \hat{x}^2] = z\hat{x}^3$	
	$[\hat{\xi}^i, \hat{\xi}^j] = 0$	$[\hat{\xi}^i, \hat{\xi}^j] = -z\delta_{ij}\hat{\xi}^2$		
Class C	• Noncommutative spacetimes with $-\lambda\Lambda \geq 0, c = 1$: $\mathbf{AdS}_z^{3+1}, \mathbf{M}_z^{3+1}, \mathbf{C}_{+z}^{3+1}, \mathbf{C}_{-z}^{3+1}, \mathbf{C}_z^{3+1}, \mathbf{S}_z^4, \mathbf{E}_z^4$			
	$[\hat{x}^1, \hat{x}^0] = z\hat{x}^1$	$[\hat{x}^2, \hat{x}^0] = z\hat{x}^2$	$[\hat{x}^3, \hat{x}^0] = z\hat{x}^3$	$[\hat{x}^1, \hat{x}^2] = [\hat{x}^1, \hat{x}^3] = [\hat{x}^2, \hat{x}^3] = 0$
	• Twisted noncommutative spacetimes with $-\lambda\Lambda \geq 0, c = 1$: $\mathbf{AdS}_{z,\vartheta}^{3+1}, \mathbf{M}_{z,\vartheta}^{3+1}, \mathbf{C}_{+z,\vartheta}^{3+1}, \mathbf{C}_{-z,\vartheta}^{3+1}, \mathbf{C}_{z,\vartheta}^{3+1}, \mathbf{S}_{z,\vartheta}^4, \mathbf{E}_{z,\vartheta}^4$			
	$[\hat{x}^1, \hat{x}^0] = z\hat{x}^1 + \vartheta\hat{x}^3$	$[\hat{x}^2, \hat{x}^0] = z\hat{x}^2$	$[\hat{x}^3, \hat{x}^0] = z\hat{x}^3 - \vartheta\hat{x}^1$	$[\hat{x}^0, \hat{x}^1] = [\hat{x}^0, \hat{x}^3] = [\hat{x}^1, \hat{x}^3] = 0$
	• Noncommutative space of lines with $-\lambda\Lambda \geq 0, c = 1$: $\mathbf{LAdS}_z^6, \mathbf{LM}_z^6, \mathbf{LC}_{+z}^6, \mathbf{LC}_{-z}^6, \mathbf{LC}_z^6, \mathbf{LS}_z^6, \mathbf{LE}_z^6$			
	$[\hat{x}^i, \hat{x}^j] = 0$	$[\hat{\xi}^i, \hat{\xi}^j] = 0$	$[\hat{x}^i, \hat{\xi}^j] = 0$	

6.1. Class A: Space-Like Deformations with Primitive Generators (P_3, K_2)

We consider the following kinematical assignment [102]

$$P_0 = J_{01}, \quad \mathbf{P} = (J_{02}, J_{03}, J_{04}), \quad \mathbf{K} = (J_{12}, J_{13}, J_{14}), \quad \mathbf{J} = (J_{34}, -J_{24}, J_{23}), \quad (117)$$

which, in the array form used in Section 3.1, gives

$$\begin{array}{ccccccc} J_{01} & J_{02} & J_{03} & J_{04} & & P_0 & P_1 & P_2 & P_3 \\ & J_{12} & J_{13} & J_{14} & & & K_1 & K_2 & K_3 \\ & & J_{23} & J_{24} & \equiv & & & J_3 & -J_2 \\ & & & J_{34} & & & & & J_1 \end{array} \quad (118)$$

which shows that the spaces $\mathbb{S}_\omega^{(1)}$ (86) and $\mathbb{S}_\omega^{(2)}$ (87) coincide with the spacetime \mathbf{ST}^{3+1} and the space of (time-like) lines \mathbf{L}^6 (111), respectively (see (40) and (41)).

When we impose, under the identification (117), that the commutation rules of $\mathfrak{so}_\omega(5)$ (71) fulfil the common Lie brackets of any kinematical algebra (107), we find that $\omega_3 = \omega_4 = +1$. Next, the remaining specific kinematical commutation relations (108) imply that $\omega_1 = -\Lambda$, $\omega_2 = -1/c^2$ and $\lambda = 1$. In this way, we find a set of six kinematical algebras in $\mathfrak{so}_\omega(5)$ with graded contraction parameters

$$(\omega_1, \omega_2, \omega_3, \omega_4) = (-\Lambda, -1/c^2, +1, +1), \quad \lambda = 1, \quad (119)$$

and bilinear form \mathbf{I}_ω (74) given by

$$\mathbf{I}_\omega = \left(+1, -\Lambda, \frac{\Lambda}{c^2}, \frac{\Lambda}{c^2}, \frac{\Lambda}{c^2} \right).$$

These are the three Lorentzian (c finite) and the three Newtonian ($c = \infty$) algebras described in Sections 5.1 and 5.2. Moreover, the three Riemannian algebras of Section 5.5 also appear for $c = i$, that is, $\omega_2 = +1$. Thus, this class A covers nine of the Lie algebras shown in Table 5. Recall that, in the Lorentzian cases, the sectional curvature of the $(3+1)$ D spacetime \mathbf{ST}^{3+1} is minus the cosmological constant $\omega_1 = -\Lambda$, while the 6D space of (time-like) lines \mathbf{L}^6 is of negative curvature $\omega_2 = -1/c^2$. In the Newtonian cases, \mathbf{L}^6 is a flat space with $\omega_2 = 0$ ($c = \infty$). The kinematical automorphisms (110) are related to the CK ones $\Theta^{(m)}$ (33) through

$$\mathcal{P} = \Theta^{(2)}, \quad \mathcal{T} = \Theta^{(1)}\Theta^{(2)}, \quad \mathcal{PT} = \Theta^{(1)}.$$

Now, we apply the geometrical-kinematical identification (118) to the CK r -matrix r (92) and to the Drinfel'd double one r_D (101), thus, obtaining the following kinematical r -matrices

$$r = z(K_3 \wedge P_0 + J_1 \wedge P_2 - J_2 \wedge P_1 + \sqrt{-\Lambda} J_3 \wedge K_1), \quad r_D = r + z\sqrt{-c^2} K_2 \wedge P_3, \quad (120)$$

with primitive generators $P_3 \equiv J_{04}$ and $K_2 \equiv J_{13}$, so that z has dimensions of a *length* with dimensionless product zP_3 . Next, the constraint (97) $\omega_1\omega_4 = -\Lambda \geq 0$ excludes three cases in order to deal with real bialgebras: dS, expanding NH and hyperbolic algebras, all of them with $\Lambda > 0$ (first column in Table 5). The Drinfel'd double r -matrix r_D subjected to the additional condition (102) $\omega_2\omega_3 = -1/c^2 > 0$ only holds for $c = i$, that is, for the spherical and Euclidean algebras, thereby, recovering the cases (I) and (Ia) in Table 4. Thus, these results finally comprise *six* real Lie bialgebras as shown in Table 6.

The cocommutator δ for r (120) can then be deduced by applying (12), or by introducing directly the kinematical assignments (118) and (119) into the CK cocommutator (94). It can be checked that the isotropy subalgebras of an event \mathfrak{h}_{st} and a line $\mathfrak{h}_{\text{line}}$ given by (see (111))

$$\mathfrak{h}_{\text{st}} = \langle \mathbf{K}, \mathbf{J} \rangle, \quad \mathfrak{h}_{\text{line}} = \langle P_0 \rangle \oplus \langle \mathbf{J} \rangle, \quad (121)$$

both satisfy the coisotropy condition (22).

Now, we proceed to obtain the corresponding first-order noncommutative spacetimes \mathbf{ST}_z^{3+1} and spaces of (time-like) lines \mathbf{L}_z^6 associated with the homogeneous spaces (111). For this purpose, we introduce the quantum coordinates $(\hat{x}^0, \hat{x}^i, \hat{\xi}^i, \hat{\theta}^i)$ dual, in this order, to the generators (P_0, P_i, K_i, J_i) ($i = 1, 2, 3$) via the canonical pairing (11), so with non-zero entries:

$$\langle \hat{x}^0, P_0 \rangle = \langle \hat{x}^i, P_i \rangle = \langle \hat{\xi}^i, K_i \rangle = \langle \hat{\theta}^i, J_i \rangle = 1. \quad (122)$$

Hence, the first-order noncommutative spaces are defined as the annihilators of the vector subspaces \mathfrak{h}_{st} and $\mathfrak{h}_{\text{line}}$ (121):

$$\mathbf{ST}_z^{3+1} = \langle \hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3 \rangle, \quad \mathbf{L}_z^6 = \langle \hat{x}^1, \hat{x}^2, \hat{x}^3, \hat{\xi}^1, \hat{\xi}^2, \hat{\xi}^3 \rangle. \quad (123)$$

Notice that, for the Riemannian cases, the quantum time coordinate \hat{x}^0 corresponds to a spatial one, while the noncommutative rapidities $\hat{\zeta}^i$ become quantum angular coordinates. The corresponding defining commutation relations for these noncommutative spaces can be deduced either from the dual of the cocommutator δ for r (120), or by introducing the following identification between the quantum CK coordinates \hat{x}^{ab} and the kinematical ones (122) in $\mathbb{S}_{z,\omega}^{(1)}$ and $\mathbb{S}_{z,\omega}^{(2)}$ given in Table 3:

$$\begin{array}{cccc} \hat{x}^{01} & \hat{x}^{02} & \hat{x}^{03} & \hat{x}^{04} \\ & \hat{x}^{12} & \hat{x}^{13} & \hat{x}^{14} \\ & & \hat{x}^{23} & \hat{x}^{24} \\ & & & \hat{x}^{34} \end{array} \equiv \begin{array}{cccc} \hat{x}^0 & \hat{x}^1 & \hat{x}^2 & \hat{x}^3 \\ & \hat{\zeta}^1 & \hat{\zeta}^2 & \hat{\zeta}^3 \\ & & \hat{\theta}^3 & -\hat{\theta}^2 \\ & & & \hat{\theta}^1 \end{array}$$

(which is the dual counterpart of (118)), together with (119). Likewise, the first-order twisted noncommutative spaces of points can be obtained from $\mathbb{S}_{z,\vartheta,\omega}^{(1)}$ (106) by taking into account that it only covers the spherical and Euclidean spaces with $c = i$ ($\omega_2\omega_3 = +1$) so that \hat{x}^0 is another quantum spatial coordinate; recall that the proper Drinfel'd double structure corresponds to set $\vartheta \equiv z$ (104). All of these noncommutative spaces are explicitly presented in Table 7.

6.2. Class B: Space-Like Deformations with Primitive Generators (P_2, J_2)

We perform the identification [102]

$$P_0 = J_{02}, \quad \mathbf{P} = (J_{01}, J_{04}, J_{03}), \quad \mathbf{K} = (J_{12}, J_{24}, J_{23}), \quad \mathbf{J} = (-J_{34}, -J_{13}, J_{14}), \quad (124)$$

that is,

$$\begin{array}{cccc} J_{01} & J_{02} & J_{03} & J_{04} \\ & J_{12} & J_{13} & J_{14} \\ & & J_{23} & J_{24} \\ & & & J_{34} \end{array} \equiv \begin{array}{cccc} P_1 & P_0 & P_3 & P_2 \\ & K_1 & -J_2 & J_3 \\ & & K_3 & K_2 \\ & & & -J_1 \end{array} \quad (125)$$

The fulfilment of the kinematical commutators (107) from the CK ones (71) requires fixing $\omega_2 = \omega_3 = -1$ and $\omega_4 = +1$. The remaining commutation relations (108) lead to $\lambda = 1$, $c = 1$ and $\omega_1 = \Lambda$. Hence, we obtain the three Lorentzian algebras of Section 5.1 within $\mathfrak{so}_\omega(5)$ with the contraction parameters

$$(\omega_1, \omega_2, \omega_3, \omega_4) = (\Lambda, -1, -1, +1), \quad \lambda = 1, \quad c = 1, \quad (126)$$

and bilinear form \mathbf{I}_ω (74) given by

$$\mathbf{I}_\omega = (+1, \Lambda, -\Lambda, \Lambda, \Lambda).$$

In terms of $\Theta^{(m)}$ (33), the kinematical automorphisms (110) read

$$\mathcal{P} = \Theta^{(1)}\Theta^{(2)}\Theta^{(3)}, \quad \mathcal{T} = \Theta^{(2)}\Theta^{(3)}, \quad \mathcal{PT} = \Theta^{(1)}.$$

With the assignments (124) and (126), we find that the space $\mathbb{S}_\omega^{(1)}$ (86) is related to the spacetime \mathbf{ST}^{3+1} (since $\mathcal{PT} = \Theta^{(1)}$); however, the former has curvature $K = \omega_1$, while the latter has $K = -\omega_1 = -\Lambda$. The space of (time-like) lines \mathbf{L}^6 (111) cannot be identified with $\mathbb{S}_\omega^{(2)}$ (87) (now $\mathcal{P} \neq \Theta^{(2)}$), but it can be with the rank-2 CK space associated with the composition of involutions $\mathcal{P} = \Theta^{(1)}\Theta^{(2)}\Theta^{(3)}$ (see the comments at the end of Section 4.1).

The r -matrices (92) and (101) turn out to be

$$r = z(K_2 \wedge P_0 + J_3 \wedge P_1 - J_1 \wedge P_3 + \sqrt{\Lambda} K_3 \wedge K_1), \quad r_D = r - zJ_2 \wedge P_2. \quad (127)$$

The primitive generators are $P_2 \equiv J_{04}$ and $J_2 \equiv -J_{13}$, and z has dimensions of a *length* since the product zP_2 is dimensionless; notice that (127) is written in units with $c = 1$. The constraint $\omega_1\omega_4 = \Lambda \geq 0$ excludes the AdS algebra. Moreover, since $\omega_2\omega_3 = +1$, the r -matrix r_D is well defined for the dS and Poincaré cases, which correspond to the cases (II) and (IIa) in Table 4, as shown in Table 6. The cocommutator for r (127) can be obtained straightforwardly showing that the coisotropy condition (22) is satisfied for \mathfrak{h}_{st} (121) in both cases, but only for $\mathfrak{h}_{\text{line}}$ for the Poincaré bialgebra, thus, precluding the construction of the noncommutative dS space of (time-like) lines \mathbf{L}_z^6 (123).

We stress that to set $c = 1$ in (126) implies that the r -matrices (127) are neither well-defined for the non-relativistic algebras with $c \rightarrow \infty$, nor for the Riemannian ones with $c = i$ ($\lambda = 1$). In fact, the speed of light can be introduced explicitly in r and r_D (127) providing the commutation rules (108) by means of the scalings

$$\tilde{\mathbf{P}} = \frac{1}{c} \mathbf{P}, \quad \tilde{\mathbf{K}} = \frac{1}{c} \mathbf{K}, \quad \tilde{z} = cz, \quad (128)$$

(that preserve the product $zP_2 = \tilde{z}\tilde{P}_2$) yielding the classical r -matrices

$$\tilde{r} = \tilde{z}(\tilde{K}_2 \wedge P_0 + J_3 \wedge \tilde{P}_1 - J_1 \wedge \tilde{P}_3 + c\sqrt{\Lambda} \tilde{K}_3 \wedge \tilde{K}_1), \quad \tilde{r}_D = \tilde{r} - \tilde{z}J_2 \wedge \tilde{P}_2,$$

showing the above exclusions. It is possible to transform the deformation parameter as $\tilde{z} = c^2 z$ (without preserving zP_2) obtaining that

$$\tilde{r} = \frac{\tilde{z}}{c}(\tilde{K}_2 \wedge P_0 + J_3 \wedge \tilde{P}_1 - J_1 \wedge \tilde{P}_3) + \tilde{z}\sqrt{\Lambda} \tilde{K}_3 \wedge \tilde{K}_1, \quad \tilde{r}_D = \tilde{r} - \frac{\tilde{z}}{c}J_2 \wedge \tilde{P}_2,$$

which is not real for $c = i$; however, both of them have a well-defined limit $c \rightarrow \infty$ reducing to a Reshetikhin twist $\tilde{r} \equiv \tilde{r}_D = \tilde{z}\sqrt{\Lambda} \tilde{K}_3 \wedge \tilde{K}_1$.

Next, we construct the two first-order noncommutative spacetimes \mathbf{ST}_z^{3+1} and the noncommutative Minkowskian space of (time-like) lines \mathbf{L}_z^6 (123) by means of the dual of the cocommutator for r (127) or, alternatively, by introducing the kinematical identification

$$\begin{array}{cccc} \hat{x}^{01} & \hat{x}^{02} & \hat{x}^{03} & \hat{x}^{04} \\ & \hat{x}^{12} & \hat{x}^{13} & \hat{x}^{14} \\ & & \hat{x}^{23} & \hat{x}^{24} \\ & & & \hat{x}^{34} \end{array} \equiv \begin{array}{cccc} \hat{x}^1 & \hat{x}^0 & \hat{x}^3 & \hat{x}^2 \\ \hat{\xi}^1 & -\hat{\theta}^2 & \hat{\theta}^3 & \\ & \hat{\xi}^3 & \hat{\xi}^2 & \\ & & & -\hat{\theta}^1 \end{array}$$

dual to (125), together with the contraction parameters (126) in the commutation relations of the dual CK algebra (98) and (99). Similarly, the first-order twisted noncommutative dS and Minkowskian spacetimes can be deduced from $\mathbb{S}_{z,\theta,\omega}^{(1)}$ (106) ($\omega_2\omega_3 = +1$). All of these structures are presented in Table 7.

6.3. Class C: Time-Like Deformations with Primitive Generators (P_0, J_2)

We consider the kinematical assignment [102]

$$P_0 = J_{04}, \quad \mathbf{P} = (J_{01}, J_{02}, J_{03}), \quad \mathbf{K} = (J_{14}, J_{24}, J_{34}), \quad \mathbf{J} = (J_{23}, -J_{13}, J_{12}),$$

that is,

$$\begin{array}{cccc} J_{01} & J_{02} & J_{03} & J_{04} \\ & J_{12} & J_{13} & J_{14} \\ & & J_{23} & J_{24} \\ & & & J_{34} \end{array} \equiv \begin{array}{cccc} P_1 & P_2 & P_3 & P_0 \\ & J_3 & -J_2 & K_1 \\ & & J_1 & K_2 \\ & & & K_3 \end{array} \quad (129)$$

Starting from the CK algebra (71), we find that the Lie brackets (107) imply that $\omega_2 = \omega_3 = +1$, while the commutators (108) give rise to $\omega_1 = \Lambda$, $\omega_4 = -\lambda$ and $c = 1$. Thus, the contraction parameters and the bilinear form \mathbf{I}_ω (74) are given by

$$(\omega_1, \omega_2, \omega_3, \omega_4) = (\Lambda, +1, +1, -\lambda), \quad c = 1, \quad (130)$$

$$\mathbf{I}_\omega = (+1, \Lambda, \Lambda, \Lambda, -\lambda\Lambda).$$

Therefore, we find nine algebras in the CK family $\mathfrak{so}_\omega(5)$: the three Lorentzian algebras of Section 5.1 for $\lambda = 1$, the three Carrollian algebras of Section 5.3 for $\lambda = 0$ and the three Riemannian ones of Section 5.5 for $\lambda = -1$ (see Table 5). The kinematical automorphisms (110) turn out to be

$$\mathcal{P} = \Theta^{(1)}\Theta^{(4)}, \quad \mathcal{T} = \Theta^{(4)}, \quad \mathcal{PT} = \Theta^{(1)}.$$

In this case, the spacetime \mathbf{ST}^{3+1} is again related to the CK space $\mathbb{S}_\omega^{(1)}$ (86) (since $\mathcal{PT} = \Theta^{(1)}$), while the space of lines \mathbf{L}^6 cannot be associated with $\mathbb{S}_\omega^{(2)}$, but it can with the rank-2 symmetric CK space (90) with automorphism $\mathcal{P} = \Theta^{(1)}\Theta^{(4)}$.

The CK r -matrices (92) and (101) now become

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{-\lambda\Lambda} J_1 \wedge J_3), \quad r_D = r - zJ_2 \wedge P_0, \quad (131)$$

which lead to primitive generators $P_0 \equiv J_{04}$ and $J_2 \equiv -J_{13}$, thus, with z having dimensions of a time (recall that $c = 1$) provided that the product zP_0 is dimensionless. The constraint $\omega_1\omega_4 = -\lambda\Lambda \geq 0$ excludes the dS algebra ($\lambda = 1, \Lambda > 0$) and the hyperbolic one ($\lambda = -1, \Lambda < 0$). Since $\omega_2\omega_3 = +1$, any real r -matrix r always provides a real r_D in this class.

Consequently, the resulting *seven* real Lie bialgebras given in Table 6 also appear in Table 4 with the simple algebra AdS corresponding to the case (III). The spherical and Euclidean algebras (cases (I) and (Ia)) are again recovered as in class A, but here for different values for the parameters (Λ, c, λ) . Once the cocommutator for r (131) has been computed, it can be checked that the coisotropy condition (22) is fulfilled for both subalgebras \mathfrak{h}_{st} and $\mathfrak{h}_{\text{line}}$ (121) allowing the construction of the two noncommutative spaces (123).

It is worth stressing that this class of time-like deformations cover the so-called kappa-deformations such that the deformation parameters z and κ are related through $z \sim 1/\kappa$. Hence, the r -matrix for the Poincaré algebra of case (IIIa) in Table 4 underlies the well known κ -Poincaré deformation [64–68] and that, for the AdS algebra of case (III), provides the κ -AdS algebra [129,136].

As far as the non-relativistic limit $c \rightarrow \infty$ is concerned, we remark that the condition $c = 1$, in principle, precludes it. To be precise, if we apply the same scalings (128) to the Lie generators keeping z unchanged and so the product zP_0 as well, then c appears explicitly in the commutators (108), and the r -matrices (131) now read

$$\tilde{r} = z c^2(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3) + z\sqrt{-\lambda\Lambda} J_1 \wedge J_3, \quad \tilde{r}_D = \tilde{r} - zJ_2 \wedge P_0,$$

which diverge under the limit $c \rightarrow \infty$. Nevertheless, we can introduce the scalings (128) but with the transformed deformation parameter $\tilde{z} = c^2 z$ (not preserving zP_0) finding that [71]

$$\tilde{r} = \tilde{z}(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3) + \frac{\tilde{z}}{c^2}\sqrt{-\lambda\Lambda} J_1 \wedge J_3, \quad \tilde{r}_D = \tilde{r} - \frac{\tilde{z}}{c^2}J_2 \wedge P_0,$$

thus, allowing one to apply the limit $c \rightarrow \infty$ obtaining that

$$\tilde{r} \equiv \tilde{r}_D = \tilde{z}(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3),$$

which coincides with the κ -Poincaré r -matrix. The remarkable point is that the corresponding cocommutator is trivial, that is, $\delta(X) = 0$ for all X , so that there is no deformation for the (contracted) Newtonian algebras. In other words, the scheme of contractions for the CK

r -matrix (92) and cocommutator δ (94) ensures to always obtain both a non-trivial r -matrix and cocommutator.

In this respect, it is worth stressing that it is possible to apply a Lie bialgebra contraction in such a manner that the initial r -matrix diverges but the initial cocommutator gives a non-trivial result [31] (this is called fundamental but non-coboundary Lie bialgebra contraction). This contraction process was applied in [65,159] in order to obtain the κ -Galilei algebra by contracting κ -Poincaré and finding a non-coboundary quantum Galilei algebra, which explains its absence in our approach. In fact, such a non-coboundary Lie bialgebra contraction was recently applied in [71] in order to deduce the κ -Newtonian algebras containing both κ -NH algebras ($\Lambda \neq 0$) together with the above κ -Galilei one ($\Lambda = 0$).

The first-order noncommutative spacetimes and spaces of lines for the seven Lie bialgebras contained in this class can be obtained by introducing the identification dual to (129) given by

$$\begin{array}{ccccccc} \hat{x}^{01} & \hat{x}^{02} & \hat{x}^{03} & \hat{x}^{04} & & \hat{x}^1 & \hat{x}^2 & \hat{x}^3 & \hat{x}^0 \\ & \hat{x}^{12} & \hat{x}^{13} & \hat{x}^{14} & & & \hat{\theta}^3 & -\hat{\theta}^2 & \hat{\xi}^1 \\ & & \hat{x}^{23} & \hat{x}^{24} & \equiv & & & \hat{\theta}^1 & \hat{\xi}^2 \\ & & & \hat{x}^{34} & & & & & \hat{\xi}^3 \end{array}$$

together with the contraction parameters (130) in the commutation rules (98) and (99). In the same way, the first-order twisted noncommutative spacetimes are deduced from $\mathbb{S}_{z,\theta,\omega}^{(1)}$ (106) for the seven cases since, for all of them, $\omega_2\omega_3 = +1$. The explicit expressions for all of these noncommutative spaces can be found in Table 7.

6.4. Class D: Dimensionless Deformation with Primitive Generators (J_2, P_0)

As the last class, we study how to obtain the AdS deformation of case (IV) in Table 4 in a kinematical basis. With this aim we consider the identification (not considered in [102]) given by

$$P_0 = J_{13}, \quad \mathbf{P} = (J_{14}, -J_{12}, J_{01}), \quad \mathbf{K} = (J_{34}, -J_{23}, J_{03}), \quad \mathbf{J} = (J_{02}, J_{04}, J_{24}). \quad (132)$$

Then, the Lie brackets (107) gives that $\omega_1 = \omega_2 = \omega_3 = \omega_4 = -1$, and the commutators (108) lead to set $\Lambda = -1$, $c = 1$ and $\lambda = 1$. Therefore, we obtain a *single* Lie algebra in this class, $\text{AdS} \simeq \mathfrak{so}(3, 2)$, such that

$$(\omega_1, \omega_2, \omega_3, \omega_4) = (-1, -1, -1, -1), \quad \Lambda = -1, \quad c = 1, \quad \lambda = 1,$$

$$\mathbf{I}_\omega = (+1, -1, +1, -1, +1).$$

The kinematical automorphisms (110) read

$$\mathcal{P} = \Theta^{(1)}\Theta^{(2)}\Theta^{(3)}\Theta^{(4)}, \quad \mathcal{T} = \Theta^{(3)}\Theta^{(4)}, \quad \mathcal{PT} = \Theta^{(1)}\Theta^{(2)}.$$

The CK r -matrices (92) and (101) turn out to be

$$r = z(K_1 \wedge K_3 + K_2 \wedge P_2 + P_1 \wedge P_3 + J_3 \wedge J_1), \quad r_D = r + zP_0 \wedge J_2. \quad (133)$$

The primitive generators are $J_2 \equiv J_{04}$ and $P_0 \equiv J_{13}$, while z is dimensionless like the product zJ_2 (we are working with units with $\Lambda = -1$ and $c = 1$). The constraints $\omega_1\omega_4 = +1$ and $\omega_2\omega_3 = +1$ are automatically satisfied, so that r_D is the kinematical expression of the new Drinfel'd double r -matrix of case (IV) in Table 4.

By computing the cocommutator for r (133) (or from (94) with the identification (132)), it can be checked that the coisotropy condition (22) is not satisfied for any subalgebra (121), so that there do not exist noncommutative spacetime and space of lines (123) associated with this bialgebra.

It is rather natural to analyse whether there may exist some possible contraction from this AdS deformation, although, by following our approach, the answer is negative whenever one requires to keep a non-trivial r -matrix and cocommutator. Starting from the AdS commutation relations (108) with $\Lambda = -1$, $c = 1$ and $\lambda = 1$, it is possible to explicitly introduce such parameters by means of the scalings (coming from the automorphisms (110))

$$\tilde{P}_0 = \sqrt{-\Lambda}\sqrt{\lambda} P_0, \quad \tilde{\mathbf{P}} = \sqrt{-\Lambda} \frac{1}{c} \mathbf{P}, \quad \tilde{\mathbf{K}} = \sqrt{\lambda} \frac{1}{c} \mathbf{K}, \quad \tilde{\mathbf{J}} = \mathbf{J}, \quad (134)$$

keeping the dimensionless parameter z . By introducing (134) in r (133), we obtain that

$$\tilde{r} = -\frac{z c^2}{\Lambda \lambda} \left(-\Lambda \tilde{K}_1 \wedge \tilde{K}_3 + \sqrt{-\Lambda}\sqrt{\lambda} \tilde{K}_2 \wedge \tilde{P}_2 + \lambda \tilde{P}_1 \wedge \tilde{P}_3 - \frac{\Lambda \lambda}{c^2} \tilde{J}_3 \wedge \tilde{J}_1 \right),$$

so that this diverges under the contractions $\Lambda \rightarrow 0$, $c \rightarrow \infty$ and $\lambda \rightarrow 0$. If we transform the deformation parameter as

$$\tilde{z} = -\frac{z c^2}{\Lambda \lambda},$$

then the above contractions are well defined but only provide twisted r -matrices

$$\begin{aligned} \lim_{\Lambda \rightarrow 0} \tilde{r} &= \tilde{z} \lambda \tilde{P}_1 \wedge \tilde{P}_3, & \lim_{\lambda \rightarrow 0} \tilde{r} &= -\tilde{z} \Lambda \tilde{K}_1 \wedge \tilde{K}_3, \\ \lim_{c \rightarrow \infty} \tilde{r} &= \tilde{z} \left(-\Lambda \tilde{K}_1 \wedge \tilde{K}_3 + \sqrt{-\Lambda}\sqrt{\lambda} \tilde{K}_2 \wedge \tilde{P}_2 + \lambda \tilde{P}_1 \wedge \tilde{P}_3 \right), \end{aligned}$$

whose terms are all formed by commuting generators.

6.5. Quantum Kinematical Algebras

Tables 6 and 7 highlight the main results so far obtained from a global kinematical viewpoint; the former covers all the information of kinematical bialgebras, while the latter shows their corresponding first-order noncommutative spacetimes and spaces of lines. We now make some observations and also comment on known results as well as on some open problems concerning the kinematical bialgebras and their complete quantum deformation. Similarly, some remarks for the full noncommutative spaces will be addressed in Section 6.6.

All the kinematical r -matrices presented in Table 6 underlie quantum kinematical algebras $\mathcal{U}_z(\mathfrak{so}_{\Lambda,c,\lambda}(5))$ with real Lie bialgebras $(\mathfrak{so}_{\Lambda,c,\lambda}(5), \delta(r))$ of quasitriangular or standard type as it was described in Section 2.1. If we focus on the Poincaré bialgebra, we obtain three sequence of coboundary contractions:

Class A: space-like (No Drinfel'd double)	AdS $\xrightarrow{\Lambda \rightarrow 0}$ $\mathfrak{so}(3,2)$	Poincaré $\xrightarrow{c \rightarrow \infty}$ $\mathfrak{iso}(3,1)$	Galilei $\mathfrak{ii}\mathfrak{so}(3)$	
Class B: space-like (With Drinfel'd double)	dS $\xrightarrow{\Lambda \rightarrow 0}$ $\mathfrak{so}(4,1)$	Poincaré $\mathfrak{iso}(3,1)$		(135)
Class C: time-like (With Drinfel'd double)	AdS $\xrightarrow{\Lambda \rightarrow 0}$ $\mathfrak{so}(3,2)$	Poincaré $\xrightarrow{\lambda \rightarrow 0}$ $\mathfrak{iso}(3,1)$	Carroll $\mathfrak{ii}'\mathfrak{so}(3)$	

Recall that the class D only contains an isolated AdS bialgebra. From this approach, the non-relativistic contraction leading to a Galilei bialgebra can only be performed within the space-like deformations belonging to the class A; however, none of the three bialgebras in this sequence can be endowed with a (contracted) Drinfel'd double structure. The only possibility to obtain a space-like Poincaré bialgebra with associated contracted Drinfel'd double structure is provided by the class B, but now coming from a dS bialgebra, instead of the AdS one of class A.

Notice that the difference between the classical r -matrices r_D of classes A and B can clearly be appreciated in the kinematical basis. In class A, there appears the term $zK_2 \wedge P_3$, which only holds for the Riemannian bialgebras (with $c = i$ and K_2 becoming a rotation generator), while in class B, the Drinfel'd double structure requires adding the term $-zJ_2 \wedge P_2$ with a proper rotation generator. The sequence for the class B corresponds to perform (II) \rightarrow (IIa) in Table 4.

The class C deserves a special mention since it corresponds to the kappa-deformation. The sequence (135) starts with the κ -AdS bialgebra [129,136], continues with the κ -Poincaré bialgebra [64–68] and ends in the κ -Carroll one [71,102] while keeping a (contracted) Drinfel'd double structure through the process, which is denoted (III) \rightarrow (IIIa) \rightarrow (IIIb') in Table 4.

The complete Hopf algebra structure for the quantum inhomogeneous kinematical algebras and their further contractions can be found in [102], which belong to the quantum CK family $\mathcal{U}_z(\mathfrak{so}_\omega(5))$ with arbitrary $\omega = (0, \omega_2, \omega_3, \omega_4)$. Thus, such results comprise the quantum deformations of the Poincaré, Galilei and Euclidean bialgebras of class A, the Poincaré bialgebra of class B, along with the κ -Poincaré and κ -Carroll ones of class C; observe that the Euclidean bialgebra of class C is equivalent to that of class A, being just case (Ia) in Table 4.

The κ -deformation for the curved Carrollian bialgebras (with $\Lambda \neq 0$) of class C, κ -para-Euclidean and κ -para-Poincaré, can be deduced from the results given in [102] by applying the z -polarity \mathcal{D}_z (50) since this map interchanges the CK bialgebras as in (95), $(0, \omega_2, \omega_3, \omega_4) \leftrightarrow (\omega_4, \omega_3, \omega_2, 0)$, providing their explicit expressions, which are given in [71]. Generalized results on twisted (space- and time-like) Poincaré algebras can be found in [87]. For twist deformations of κ -Poincaré and their contractions to κ -Galilei algebras, we refer to [86], and, for twist deformations of the Carroll algebra, see [146].

Nevertheless, quantum deformations for the simple AdS and dS bialgebras have only been achieved for the κ -AdS of class C in [136], as a Poisson–Hopf algebra, showing the hard difficulties of this task. Consequently, the obtention of the quantum algebras for AdS of class A, dS of class B and AdS of class D remain as open problems.

In contrast to this $(3 + 1)$ D case, quantum Drinfel'd–Jimbo deformations for the semisimple (A)dS algebras, $\mathfrak{so}(3, 1)$ and $\mathfrak{so}(2, 2)$, are well known, and their space- and time-like deformations were formerly obtained in [135] within a CK framework. Later on, the $(2 + 1)$ D κ -(A)dS algebras were considered in a quantum gravity context in [160], and their twisted deformations, with underlying Drinfel'd double structures [115], were studied in [161].

By taking into account the above comments, it is worth comparing the $(2 + 1)$ D case with the $(3 + 1)$ D one with more detail, since they are quite different. In fact, the former is somewhat “special”, as it is very well known in quantum gravity. In particular, the six generators $\{J_{ab}\}$ ($a < b$; $a, b = 0, 1, \dots, 3$) span the CK family $\mathfrak{so}_{\omega_1, \omega_2, \omega_3}(4)$, which turns out to be a Lie subalgebra of $\mathfrak{so}_\omega(5)$ with non-vanishing Lie brackets included in (71). It was proven in [31] that the Drinfel'd–Jimbo r -matrix for the family $\mathfrak{so}_{\omega_1, \omega_2, \omega_3}(4)$ simply reads

$$r = z(J_{13} \wedge J_{01} + J_{23} \wedge J_{02}), \quad (136)$$

which is ω -independent in contrast to (92). Moreover, the r -matrix (136) gives rise to a cocommutator δ through (12) determining a real Lie bialgebra for any value of the three contraction parameters $(\omega_1, \omega_2, \omega_3)$. The pair of main and secondary primitive generators is (J_{03}, J_{12}) . Therefore, the family of CK bialgebras $(\mathfrak{so}_{\omega_1, \omega_2, \omega_3}(4), \delta(r))$ contains all the $3^3 = 27$ possibilities at this dimension [135].

Next, we consider the family of kinematical algebras $\mathfrak{so}_{\Lambda, c, \lambda}(4)$ spanned by the six generators $\{P_0, P_1, P_2, K_1, K_2, J_3\}$ in such a manner that the commutation rules are given by (107) and (108) setting the indices $i, j = 1, 2$ and fixing $k = 3$. Starting with the CK r -matrix (136), we look for space- and time-like $\mathfrak{so}_{\Lambda, c, \lambda}(4)$ bialgebras, which would be the $(2 + 1)$ D counterparts of the classes A, B and C in Table 6. Clearly, the class B, with commuting

primitive generators (P_2, J_2) , has no $(2 + 1)$ D counterpart since, at this dimension, there does not exist a spatial generator P_i commuting with J_3 .

Hence, we are led to the two classes A and C for $\mathfrak{so}_{\Lambda, c, \lambda}(4)$ such that the former requires setting $\lambda = 1$, while the latter obliges fixing $c = 1$. Each of them covers nine real Lie bialgebras, which are displayed in Table 8.

Table 8. Space- and time-like real classical r -matrices for the $(2 + 1)$ D kinematical and 3D Riemannian algebras with commutation relations (107) and (108) with indices $i, j = 1, 2$ and $k = 3$.

Space-like class A: $\lambda = 1$			
	Primitive (P_2, K_1)	$r = z(K_2 \wedge P_0 + J_3 \wedge P_1)$	
• Lorentzian (c finite):	dS $\mathfrak{so}(3, 1)$ $\Lambda > 0$	Poincaré $\mathfrak{iso}(2, 1)$ $\Lambda = 0$	AdS $\mathfrak{so}(2, 2)$ $\Lambda < 0$
• Newtonian ($c = \infty$):	Expanding NH \mathfrak{n}_+ $\Lambda > 0$	Galilei $\mathfrak{ii}\mathfrak{so}(2)$ $\Lambda = 0$	Oscillating NH \mathfrak{n}_- $\Lambda < 0$
• Riemannian ($c = i$):	Hyperbolic $\mathfrak{so}(3, 1)$ $\Lambda > 0$	Euclidean $\mathfrak{iso}(3)$ $\Lambda = 0$	Spherical $\mathfrak{so}(4)$ $\Lambda < 0$
Time-like class C: $c = 1$			
	Primitive (P_0, J_3)	$r = z(K_1 \wedge P_1 + K_2 \wedge P_2)$	
• Lorentzian ($\lambda = 1$):	dS $\mathfrak{so}(3, 1)$ $\Lambda > 0$	Poincaré $\mathfrak{iso}(2, 1)$ $\Lambda = 0$	AdS $\mathfrak{so}(2, 2)$ $\Lambda < 0$
• Carrollian ($\lambda = 0$):	Para-Euclidean $\mathfrak{i}'\mathfrak{so}(3)$ $\Lambda > 0$	Carroll $\mathfrak{ii}'\mathfrak{so}(2)$ $\Lambda = 0$	Para-Poincaré $\mathfrak{i}'\mathfrak{so}(2, 1)$ $\Lambda < 0$
• Riemannian ($\lambda = -1$):	Hyperbolic $\mathfrak{so}(3, 1)$ $\Lambda < 0$	Euclidean $\mathfrak{iso}(3)$ $\Lambda = 0$	Spherical $\mathfrak{so}(4)$ $\Lambda > 0$

The $(2 + 1)$ D NH algebras are $\mathfrak{n}_+ = \mathfrak{i}_4(\mathfrak{so}(1, 1) \oplus \mathfrak{so}(2))$ and $\mathfrak{n}_- = \mathfrak{i}_4(\mathfrak{so}(2) \oplus \mathfrak{so}(2))$. Clearly, there exists a second class of space-like deformations with primitive generators (P_1, K_2) ; however, this leads to equivalent results already contained within the class A [135]. The time-like class C, corresponding to the kappa-deformation, shows not only the known fact that the $(2 + 1)$ D κ -Poincaré r -matrix is shared by its curved neighbours (see [161] and references therein) but also that it holds for the three Carrollian and Riemannian deformations. The strong differences between the $(2 + 1)$ D and $(3 + 1)$ D deformations become evident when comparing the expressions of Table 8 with those given in Table 6.

Finally, we would like to point out that the CK approach to kinematical deformations may appear to be rather restrictive since this starts with the specific Drinfel'd–Jimbo r -matrix (47) for $\mathfrak{so}(5)$ and, from it, the CK r -matrix (92) is introduced, which, together with r_D (101), become the cornerstones of this work. However, we stress that this is not the case provided that one searches for deformations with a fundamental scale determined by the quantum deformation parameter. In particular, the problem of finding time-like classical r -matrices for $(3 + 1)$ D (A)dS algebras was addressed in [36].

There, it was initially considered the Lorentzian algebras, with commutation relations (107) and (108) (set $\lambda = 1$), together with the most generic classical r -matrix depending on 45 deformation parameters. Then, it was required to keep underformed the time translation generator P_0 and another commuting generator, which was chosen J_3 , that is, $\delta(P_0) = \delta(J_3) = 0$ (recall that P_0 only commutes with generators of rotations). Under these conditions, the solution of the modified classical Yang–Baxter equation (14) gave rise to two two-parametric classical r -matrices.

One of them was formed by the superposition of the κ -AdS r -matrix with the twist $P_0 \wedge J_3$, which turns out to be just r_D of the class C in Table 6 by identifying the two deformation parameters in [36] and interchanging the indices $3 \leftrightarrow 2$ in the generators (so $J_3 \leftrightarrow J_2$) through the appropriate Lie algebra automorphism. Likewise, the second solution can be identified with r_D of the class D by identifying again the two deformation parameters and applying the permutation of indices $3 \rightarrow 2 \rightarrow 1 \rightarrow 3$ with another algebra automorphism. We remark that no analysis on real Lie bialgebras and Drinfel'd doubles was carried out in [36].

Moreover, although it was claimed that the second solution (r_D of class D) was determined by a dimensionful deformation parameter, this is not exactly correct if one

requires a dimensionless classical r -matrix. In fact, from a dimensionless deformation parameter (like z in class D), a dimensionful one can be introduced by trivially multiplying it by a global factor.

6.6. Noncommutative Spacetimes and Spaces of Lines

The first-order noncommutative spaces for the kinematical bialgebras of Table 6 are shown in Table 7. There are several different situations among the four classes, ranging from class D, where there is no noncommutative space (and which is, thus, omitted), to the classes A and C, for which there exist both noncommutative spacetimes and spaces of lines for all the bialgebras. In this sense, the latter classes can be regarded as the prototypes for space- and time-like noncommutative spaces. However, when Drinfel'd double structures are taken into account, the classes B and C become the relevant ones, also providing twisted noncommutative spacetimes.

Although these results do not convey, in general, the full noncommutative spaces, for which all orders in the quantum coordinates must be considered, in some cases they do. Concerning the $(3 + 1)$ D noncommutative spacetimes, which are those commonly studied in the literature, the first-order noncommutative spacetimes in Table 7 are the complete ones for all the cases associated with a flat classical spacetime \mathbf{ST}^{3+1} (111), so with vanishing sectional curvature K ; these are just the four spaces displayed in the middle column of Table 5.

Consequently, Table 7 comprises the following full $(3 + 1)$ D (linear) noncommutative spacetimes: the space-like Minkowskian and Galilean ones together with the 4D Euclidean space in the class A; another space-like Minkowskian spacetime of class B, which is equivalent to that of class A under the automorphism corresponding to the permutation of indices $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$; and the (time-like) κ -Minkowski and κ -Carroll spacetimes of class C (the Euclidean case is equivalent to that of class A).

Additionally, complete twisted noncommutative spacetimes coming from contracted Drinfel'd doubles cover the 4D Euclidean space of class A, the space-like Minkowskian one of class B and the time-like Minkowskian and Carroll spaces of class C (again the Euclidean space here is equivalent to that of class A). We remark that more general results on twisted space- and time-like Minkowskian noncommutative spacetimes can be found in [87].

To the best of our knowledge, results for $(3 + 1)$ D noncommutative spacetimes related to curved spacetimes \mathbf{ST}^{3+1} (111) (so with $\Lambda \neq 0$) only comprise the (nonlinear) κ -AdS space (and its twisted version) [70], as well as the κ -para-Euclidean and κ -para-Poincaré [71] of class C. In such noncommutative spaces, there appear higher-order terms in the quantum coordinates governed by the cosmological constant/curvature parameter Λ . This fact allows one to distinguish them from the linear (flat) κ -Minkowski and κ -Carroll spaces with $\Lambda = 0$; however, the latter share the same linear structure.

Hence, the construction of the space-like noncommutative AdS (class A) and dS (class B) spacetimes remain as open problems, which could be faced by computing their Poisson–Lie structure by means of the Sklyanin bracket (20) and, next, studying their quantization (similarly to the κ -AdS spacetime [70]).

Noncommutative spaces of lines have scarcely been explored, and they have only been constructed for the κ -Minkowski of class C in [100] and in lower dimensions for the 4D (A)dS noncommutative spaces of worldlines in [99]; recall that the three classical Lorentzian spaces of worldlines are of non-zero curvature equal to $-1/c^2$, while both NH and Galilean spaces of lines are flat. Although the first-order noncommutative Minkowskian space of lines of class C has vanishing commutators, we stress that the brackets defining its full quantum space are not trivial at all, and, in fact, it can be endowed with a symplectic structure everywhere but in the origin.

By contrast, observe that the structure of the first-order noncommutative spaces of lines of class A is not trivial (see Table 7). From this viewpoint, noncommutative spaces of lines deserve a deeper study, and moreover it would be necessary to construct more noncommutative spaces of lines, which, when read altogether with their corresponding

noncommutative spacetimes, could allow for a deeper insight into the structure of each precise quantum deformation.

7. Conclusions and Outlook

This paper can be seen as a two-fold work with two interlinked parts that we comment on separately.

In the first part of the work (Sections 3 and 4), we considered the CK formalism for quasiorthogonal Lie algebras and their associated symmetric homogeneous spaces in order to next study their Drinfel'd–Jimbo quantum deformations. The CK approach conveys a built-in scheme of Lie algebra contractions in terms of explicit graded contraction/curvature parameters ω , in such a manner that semisimple together with non-semisimple Lie algebras and their homogeneous spaces can be described in a unified setting, which ranges from the semisimple $\mathfrak{so}(p, q)$ algebras (providing curved spaces) to the most contracted case in the CK family, the flag algebra (with associated flat spaces).

In all the contraction sequences, the same number of Casimir invariants (two, in our case) is preserved which, in turn, implies that these CK algebras share many structural properties as we have shown in the paper. As a novelty, we stress that we did not only consider the usual space of points (i.e., spacetimes) but also the symmetric homogeneous CK spaces of lines, 2-planes and 3-hyperplanes. In this global framework, Drinfel'd–Jimbo CK bialgebras were obtained from the one corresponding to $\mathfrak{so}(5)$ in a rotational basis, by always requiring the condition of obtaining a real Lie bialgebra, which finally led to the 63 real Lie bialgebras shown in Table 2.

From these results, their dual quantum counterparts were also deduced giving rise to their corresponding first-order noncommutative spaces of points, lines, etc., for which the coisotropy condition was imposed, thus, ensuring always obtaining a noncommutative space as a subalgebra of the dual Lie bialgebra; the final results are summarized in Table 3. Furthermore, r -matrices coming from Drinfel'd double structures were studied in detail as well. In particular, starting with the one corresponding to the real compact form $\mathfrak{so}(5)$ in the rotational CK basis, three classical r -matrices for the $\mathfrak{so}(p, q)$ algebras together with ten contracted r -matrices were explicitly achieved and are displayed in Table 4.

New results correspond to the dS $\mathfrak{so}(4, 1)$ algebra of case (II) and the AdS $\mathfrak{so}(3, 2)$ one of case (IV), along with the contractions from the four classical r -matrices r_D for the simple Lie algebras. We remark that such r -matrices, coming from Drinfel'd doubles, have provided, in a natural way, first-order twisted noncommutative CK spaces of points and of 3-hyperplanes for the 14 real Lie bialgebras given in Table 4.

Concerning this first part of the paper, there are, at least, two research lines that we plan to face in the future:

1. To construct new dual homogeneous CK spaces with isotropy subalgebras corresponding to the first-order noncommutative CK spaces $\mathbb{S}_{z,\omega}^{(m)} \equiv \mathfrak{h}_{\perp,\omega}^{(m)}$ ($m = 1, \dots, 4$) (100) in a similar form to that followed in [60], but moreover considering their symmetric character according to the z -involutions $\theta_z^{(m)}$ (61), which, in some cases, would provide a $\mathbb{Z}_2^{\otimes 4}$ -grading in this dual framework as well as to study their mathematical/physical properties.
2. To perform a similar construction to the one here developed for the Drinfel'd–Jimbo quantum deformations of quasiorthogonal CK algebras for other families of CK algebras [162]. Among them, we remark the quasiunitary CK algebras [163,164] (starting with the $\mathfrak{su}(p, q)$ algebras) since they are naturally related to the physical quantum space of states for any quantum system [165,166].

In the second part of the work, we focused on the kinematical algebras together with their associated symmetric homogeneous spacetimes and spaces of lines displayed in Table 5. Then, we applied the previous CK approach in order to deduce their corresponding classical r -matrices, r and r_D , given in Table 6, thus, providing kinematical bialgebras,

and, from them, we constructed the first-order noncommutative spacetimes and spaces of lines shown in Table 7.

A detailed physical discussion on the known results and open problems concerning their full quantum algebra deformation and complete quantum spaces was carried out in Sections 6.5 and 6.6, respectively. Therefore, to end with, we summarize the main conclusions and open lines of research on this issue:

1. In this paper, we only considered *coboundary* Lie bialgebra contractions—that is, those Lie bialgebras coming from a contracted classical r -matrix. However, there also exist *fundamental* Lie bialgebra contractions, under which the r -matrix diverges but the cocommutator δ is well defined [31] (thus ensuring the existence of the well-defined coproduct Δ_z).

Hence, a systematic study of all the possible fundamental but non-coboundary Lie bialgebra contractions starting with the four classical r -matrices for $\mathfrak{so}(3,2)$ and $\mathfrak{so}(4,1)$ in Table 6 is still lacking. These could give rise to new quantum deformations for non-simple kinematical algebras as was the case for the κ -Newtonian ones already obtained in [71].

2. The quantum algebra deformations for the simple algebra AdS $\mathfrak{so}(3,2)$ of the classes A and D, and for dS $\mathfrak{so}(4,1)$ of the class B are still unknown. Such structures would be useful in order to obtain the corresponding contracted quantum algebras for the kinematical algebras with $\Lambda \neq 0$ covering the NH, para-Euclidean and para-Poincaré algebras. In this contraction process, both coboundary and fundamental non-coboundary Lie bialgebra contractions may be applied.

3. From Table 7, it directly follows that quantum deformations for different kinematical algebras share the same underlying first-order noncommutative spacetime structure. When dealing with the curved cases with $\Lambda \neq 0$, differences among them could arise when higher-orders in the quantum coordinates are taken into account (as happened for the κ -spacetimes of class C obtained in [70,71]). Nevertheless, the linear noncommutative spacetime structure remains the same for the flat cases with $\Lambda = 0$, and this fact holds for the Minkowskian, Galilean and Carroll spaces.

Consequently, the construction of the “accompanying” noncommutative spaces of lines may be of interest in order to distinguish them. New physical consequences could be extracted from such new structures. In this respect, we would like to emphasize that the noncommutative space of worldlines already constructed for the κ -Poincaré algebra in [100] constitutes a prototype example in this direction.

Work on the above lines is currently in progress.

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