

Article

General Norm Inequalities of Trapezoid Type for Fréchet Differentiable Functions in Banach Spaces

Silvestru Sever Dragomir ^{1,2} 

¹ Mathematics, College of Engineering & Science, Victoria University, P.O. Box 14428, Melbourne 8001, Australia; sever.dragomir@vu.edu.au

² DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

Abstract: In this paper we establish some error bounds in approximating the integral by general trapezoid type rules for Fréchet differentiable functions with values in Banach spaces.

Keywords: Banach spaces; norm inequalities; midpoint inequalities; Banach algebras

1. Introduction

We recall some facts about differentiation of functions between normed vector spaces [1].

Let O be an open subset of a normed vector space E , f a real-valued function defined on O , $a \in O$ and u a nonzero element of E . The function f_u given by $t \mapsto f(a + tu)$ is defined on an open interval containing 0. If the derivative $\frac{df_u}{dt}(0)$ is defined, i.e., if the limit

$$\frac{df_u}{dt}(0) := \lim_{t \rightarrow 0} \frac{f(a + tu) - f(a)}{t}$$

exists, then we denote this derivative by $\nabla_a f(u)$. It is called the Gâteaux derivative (directional derivative) of f at a in the direction u . If $\nabla_a f(u)$ is defined and $\lambda \in \mathbb{R} \setminus \{0\}$, then $\nabla_a f(\lambda u)$ is defined and $\nabla_a f(\lambda u) = \lambda \nabla_a f(u)$. The function f is Gâteaux differentiable at a if $\nabla_a f(u)$ exists for all directions u .

Let E and F be normed vector spaces, and O be an open subset of E . A function $f : O \rightarrow F$ is called Fréchet differentiable at $x \in O$ if there exists a bounded linear operator $A : E \rightarrow F$ such that

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

If there exists such an operator A , it is unique, so we write $Df(x) = A$ and call it the Fréchet derivative of f at x .

A function f that is Fréchet differentiable for any point of O is said to be C^1 if the function $O \ni x \mapsto Df(x) \in \mathcal{B}(E, F)$ is continuous, where $\mathcal{B}(E, F)$ is the space of all bounded linear operators defined on E with values in F . A function Fréchet differentiable at a point is continuous at that point. Fréchet differentiation is a linear operation. If f is Fréchet differentiable at x , it is also Gâteaux differentiable there, and $\nabla_x f(u) = Df(x)(u)$ for all $u \in E$.

We say that the function $f : O \subset E \rightarrow F$ is L -Lipschitzian on O with the constant $L > 0$ if

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ for all } x, y \in O.$$

In [2] we established among others the following midpoint and trapezoid type inequalities for L -Lipschitzian functions f on an open and convex subset C in E

$$\left\| \int_0^1 f((1-t)t + ty) dt - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{4} L \|x - y\| \quad (1)$$



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and

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)t + ty) dt \right\| \leq \frac{1}{4} L \|x - y\| \tag{2}$$

for all $x, y \in C$. The constant $\frac{1}{4}$ is best possible in both inequalities (1) and (2).

For Hermite–Hadamard’s type inequalities for functions with scalar values, namely

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) du \leq \frac{f(a) + f(b)}{2}, \tag{3}$$

where $f : [a, b] \rightarrow \mathbb{R}$ is convex on $[a, b]$, see for instance [1,3–21] and the references therein. Some of the integrals used in these papers are taken in the sense of Riemann–Stieltjes while in the current paper we consider only vector valued functions with values in Banach spaces, in the first instance, and secondly, with values in Banach algebras where some applications for some fundamental functions like the exponential are also given.

In the recent paper [22] we obtained among others the following weighted version of the trapezoid inequality (2).

Theorem 1. *Let $f : C \subset E \rightarrow F$ be a function of class C^1 on the open and convex subset C of the Banach space E with values into another Banach space F and $p : [0, 1] \rightarrow [0, \infty)$ a Lebesgue integrable and symmetric function on $[0, 1]$, namely $p(1-t) = p(t)$ for all $t \in [0, 1]$. Then for all $x, y \in C$*

$$\begin{aligned} & \left\| \left(\int_0^1 p(t) dt \right) \frac{f(x) + f(y)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right) \|Df((1-t)x + ty)(y-x)\| dt \\ & + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right) \|Df((1-t)x + ty)(y-x)\| dt \\ & = \int_0^1 \left| \int_t^{1/2} p(s) ds \right| \|Df((1-t)x + ty)(y-x)\| dt \\ & =: B(f, p, x, y). \end{aligned} \tag{4}$$

Moreover, we have the upper bounds

$$\begin{aligned} B(f, p, x, y) & \leq \frac{1}{2} \left(\int_0^1 p(s) ds \right) \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \\ & \leq \frac{1}{2} \left(\int_0^1 p(s) ds \right) \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|, \end{aligned} \tag{5}$$

$$\begin{aligned} B(f, p, x, y) & \leq \frac{1}{2} \left(\int_0^1 p(s) ds - \int_0^1 \operatorname{sgn}\left(t - \frac{1}{2}\right) t p(t) dt \right) \\ & \quad \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \\ & \leq \frac{1}{2} \left(\int_0^1 p(s) ds \right) \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \end{aligned} \tag{6}$$

and

$$\begin{aligned}
 B(f, p, x, y) &\leq \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\
 &\quad \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\
 &\leq \left[\int_0^{1/2} \left(\int_t^{1/2} p(s) ds \right)^r dt + \int_{1/2}^1 \left(\int_{1/2}^t p(s) ds \right)^r dt \right]^{1/r} \\
 &\quad \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|
 \end{aligned} \tag{7}$$

for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.

Motivated by the above results, in this paper we establish some upper bounds for the quantity

$$\left\| \left(\int_0^1 p(s) ds - \gamma \right) f(y) + \gamma f(x) - \int_0^1 p(t) f((1-t)x + ty) dt \right\|$$

in the case that $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex subset C of the Banach space E with values into another Banach space F , $x, y \in C$, $p : [0, 1] \rightarrow \mathbb{C}$ is a Lebesgue integrable function and $\gamma \in \mathbb{C}$. Some particular cases of interest for different choices of γ are given. Applications for Banach algebras are also provided.

2. Some Identities of Interest

Consider a function $f : C \subset E \rightarrow F$ that is defined on the open and convex set C . We have the following properties for the auxiliary function

$$\varphi_{(x,y)}(t) := f((1-t)x + ty), \quad t \in [0, 1],$$

where $x, y \in C$.

Lemma 1. Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Then for all $x, y \in C$ the auxiliary function $\varphi_{(x,y)}$ is differentiable on $(0, 1)$ and

$$\varphi'_{(x,y)}(t) = Df((1-t)x + ty)(y-x). \tag{8}$$

Also

$$\varphi'_{(x,y)}(0+) = Df(x)(y-x) \tag{9}$$

and

$$\varphi'_{(x,y)}(1-) = Df(y)(y-x). \tag{10}$$

Proof. Let $t \in (0, 1)$ and $h \neq 0$ small enough such that $t + h \in (0, 1)$. Then

$$\begin{aligned}
 &\frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\
 &= \frac{f((1-t-h)x + (t+h)y) - f((1-t)x + ty)}{h} \\
 &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}.
 \end{aligned} \tag{11}$$

Since f is Fréchet differentiable, hence by taking the limit over $h \rightarrow 0$ in (11) we get

$$\begin{aligned} \varphi'_{(x,y)}(t) &= \lim_{h \rightarrow 0} \frac{\varphi_{(x,y)}(t+h) - \varphi_{(x,y)}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \\ &= Df((1-t)x + ty)(y-x), \end{aligned}$$

which proves (8).

Additionally, we have

$$\begin{aligned} \varphi'_{(x,y)}(0+) &= \lim_{h \rightarrow 0+} \frac{\varphi_{(x,y)}(h) - \varphi_{(x,y)}(0)}{h} \\ &= \lim_{h \rightarrow 0+} \frac{f((1-h)x + hy) - f(x)}{h} \\ &= \lim_{h \rightarrow 0+} \frac{f(x + h(y-x)) - f(x)}{h} = Df(x)(y-x) \end{aligned}$$

since f is assumed to be Fréchet differentiable in x . This proves (9).

The equality (10) follows in a similar way. \square

We have the following identity for the Riemann–Stieltjes integral:

Lemma 2. Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Let $u : [0, 1] \rightarrow \mathbb{C}$ be of bounded variation on $[0, 1]$ and $s \in [0, 1]$. Then for all $x, y \in C$ and any $\gamma, \mu \in \mathbb{C}$,

$$\begin{aligned} &[u(1) - \mu]\varphi_{(x,y)}(1) + [\gamma - u(0)]\varphi_{(x,y)}(0) + (\mu - \gamma)\varphi_{(x,y)}(s) \\ &- \int_0^1 \varphi_{(x,y)}(t)du(t) \\ &= \int_0^s [u(t) - \gamma]\varphi'_{(x,y)}(t)dt + \int_s^1 [u(t) - \mu]\varphi'_{(x,y)}(t)dt. \end{aligned} \tag{12}$$

In particular, for $\mu = \gamma$ we have

$$\begin{aligned} &[u(1) - \gamma]\varphi_{(x,y)}(1) + [\gamma - u(0)]\varphi_{(x,y)}(0) - \int_0^1 \varphi_{(x,y)}(t)du(t) \\ &= \int_0^1 [u(t) - \gamma]\varphi'_{(x,y)}(t)dt. \end{aligned} \tag{13}$$

Proof. Using integration by parts rule for the Riemann–Stieltjes integral, we have

$$\begin{aligned} \int_0^s [u(t) - \gamma]\varphi'_{(x,y)}(t)dt &= [u(s) - \gamma]\varphi_{(x,y)}(s) - [u(0) - \gamma]\varphi_{(x,y)}(0) \\ &- \int_0^s \varphi_{(x,y)}(t)du(t) \end{aligned}$$

and

$$\begin{aligned} \int_s^1 [u(t) - \mu]\varphi'_{(x,y)}(t)dt &= [u(1) - \mu]\varphi_{(x,y)}(1) - [u(s) - \mu]\varphi_{(x,y)}(s) \\ &- \int_s^1 \varphi_{(x,y)}(t)du(t) \end{aligned}$$

for any $s \in [0, 1]$.

If we add these two equalities, then we get

$$\begin{aligned} & \int_0^s [u(t) - \gamma] \varphi'_{(x,y)}(t) dt + \int_s^1 [u(t) - \mu] \varphi'_{(x,y)}(t) dt \\ &= [u(1) - \mu] \varphi_{(x,y)}(1) + [\gamma - u(0)] \varphi_{(x,y)}(0) + [\mu - u(s)] \varphi_{(x,y)}(s) \\ &+ [u(s) - \gamma] \varphi_{(x,y)}(s) - \int_0^s \varphi_{(x,y)}(t) du(t) - \int_s^1 \varphi_{(x,y)}(t) du(t) \\ &= [u(1) - \mu] \varphi_{(x,y)}(1) + [\gamma - u(0)] \varphi_{(x,y)}(0) + (\mu - \gamma) \varphi_{(x,y)}(s) \\ &- \int_0^1 \varphi_{(x,y)}(t) du(t) \end{aligned}$$

for any $s \in [0, 1]$, which proves the desired equality (12). \square

Remark 1. From the equality (13) we have for $s \in [0, 1]$ and $\gamma = u(s)$ that

$$\begin{aligned} & [u(1) - u(s)] \varphi_{(x,y)}(1) + [u(s) - u(0)] \varphi_{(x,y)}(0) - \int_0^1 \varphi_{(x,y)}(t) du(t) \\ &= \int_0^1 [u(t) - u(s)] \varphi'_{(x,y)}(t) dt \end{aligned} \quad (14)$$

and, in particular

$$\begin{aligned} & \left[u(1) - u\left(\frac{1}{2}\right) \right] \varphi_{(x,y)}(1) + \left[u\left(\frac{1}{2}\right) - u(0) \right] \varphi_{(x,y)}(0) \\ & - \int_0^1 \varphi_{(x,y)}(t) du(t) \\ &= \int_0^1 \left[u(t) - u\left(\frac{1}{2}\right) \right] \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (15)$$

Additionally, if $m \in [0, 1]$ is such that $u(m) = \frac{u(0)+u(1)}{2}$, then from (14) we get

$$\begin{aligned} & [u(1) - u(0)] \frac{\varphi_{(x,y)}(1) + \varphi_{(x,y)}(0)}{2} - \int_0^1 \varphi_{(x,y)}(t) du(t) \\ &= \int_0^1 [u(t) - u(m)] \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (16)$$

Now, if we take $\gamma = (1 - \alpha)u(0) + \alpha u(1)$, $\alpha \in [0, 1]$ in (13), then we get

$$\begin{aligned} & [u(1) - u(0)] \left[(1 - \alpha) \varphi_{(x,y)}(1) + \alpha \varphi_{(x,y)}(0) \right] - \int_0^1 \varphi_{(x,y)}(t) du(t) \\ &= \int_0^1 [u(t) - (1 - \alpha)u(0) - \alpha u(1)] \varphi'_{(x,y)}(t) dt \end{aligned} \quad (17)$$

and, in particular

$$\begin{aligned} & [u(1) - u(0)] \frac{\varphi_{(x,y)}(1) + \varphi_{(x,y)}(0)}{2} - \int_0^1 \varphi_{(x,y)}(t) du(t) \\ &= \int_0^1 \left[u(t) - \frac{u(0) + u(1)}{2} \right] \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (18)$$

The case of weighted integrals is as follows:

Corollary 1. Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Let $p : [0, 1] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[0, 1]$ and $s \in [0, 1]$. Then for all $x, y \in C$ and any $\gamma, \mu \in \mathbb{C}$,

$$\begin{aligned} & \left(\int_0^1 p(s) ds - \mu \right) \varphi_{(x,y)}(1) + \gamma \varphi_{(x,y)}(0) + (\mu - \gamma) \varphi_{(x,y)}(s) \\ & - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ & = \int_0^s \left(\int_0^t p(\tau) d\tau - \gamma \right) \varphi'_{(x,y)}(t) dt + \int_s^1 \left(\int_0^t p(\tau) d\tau - \mu \right) \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (19)$$

In particular, for $\mu = \gamma$ we have

$$\begin{aligned} & \left(\int_0^1 p(s) ds - \gamma \right) \varphi_{(x,y)}(1) + \gamma \varphi_{(x,y)}(0) - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ & = \int_0^1 \left(\int_0^t p(\tau) d\tau - \gamma \right) \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (20)$$

The proof follows by Lemma 2 applied for the function $u : [0, 1] \rightarrow \mathbb{C}$, $u(t) = \int_0^t p(s) ds$ that is absolutely continuous on $[0, 1]$ and therefore of bounded variation and

$$\int_0^1 \varphi_{(x,y)}(t) du(t) = \int_0^1 p(t) \varphi_{(x,y)}(t) dt.$$

Remark 2. With the assumptions of Corollary 1 and by utilizing Remark 1 we get

$$\begin{aligned} & \left(\int_s^1 p(\tau) d\tau \right) \varphi_{(x,y)}(1) + \left(\int_0^s p(\tau) d\tau \right) \varphi_{(x,y)}(0) - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ & = \int_0^1 \left(\int_s^t p(\tau) d\tau \right) \varphi'_{(x,y)}(t) dt \end{aligned} \quad (21)$$

and, in particular

$$\begin{aligned} & \left(\int_{1/2}^1 p(\tau) d\tau \right) \varphi_{(x,y)}(1) + \left(\int_0^{1/2} p(\tau) d\tau \right) \varphi_{(x,y)}(0) \\ & - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ & = \int_0^1 \left(\int_{1/2}^t p(\tau) d\tau \right) \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (22)$$

Additionally, if $m \in [0, 1]$ is such that $\int_0^m p(\tau) d\tau = \frac{1}{2} \int_0^1 p(\tau) d\tau$, then

$$\begin{aligned} & \frac{\varphi_{(x,y)}(1) + \varphi_{(x,y)}(0)}{2} \int_0^1 p(\tau) d\tau - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ & = \int_0^1 \left(\int_m^t p(\tau) d\tau \right) \varphi'_{(x,y)}(t) dt. \end{aligned} \quad (23)$$

Now, for $\alpha \in [0, 1]$ we get

$$\begin{aligned} & \left[(1 - \alpha) \varphi_{(x,y)}(1) + \alpha \varphi_{(x,y)}(0) \right] \int_0^1 p(\tau) d\tau - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ & = \int_0^1 \left(\int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right) \varphi'_{(x,y)}(t) dt \end{aligned} \quad (24)$$

and, in particular

$$\begin{aligned} & \frac{\varphi_{(x,y)}(1) + \varphi_{(x,y)}(0)}{2} \int_0^1 p(\tau) d\tau - \int_0^1 p(t) \varphi_{(x,y)}(t) dt \\ &= \int_0^1 \left(\int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right) \varphi'_{(x,y)}(t) dt. \end{aligned} \tag{25}$$

3. Main Results

We have:

Theorem 2. Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Let $p : [0, 1] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[0, 1]$ and $s \in [0, 1]$. Then for all $x, y \in C$ and any $\gamma, \mu \in \mathbb{C}$,

$$\begin{aligned} & \left\| \left(\int_0^1 p(s) ds - \mu \right) f(y) + \gamma f(x) + (\mu - \gamma) f((1-s)x + sy) \right. \\ & \quad \left. - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| \|Df((1-t)x + ty)(y-x)\| dt \\ & \quad + \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| \|Df((1-t)x + ty)(y-x)\| dt \\ & =: B(f, p, x, y, \gamma, \mu). \end{aligned} \tag{26}$$

In particular, for $\mu = \gamma$ we have

$$\begin{aligned} & \left\| \left(\int_0^1 p(s) ds - \gamma \right) f(y) + \gamma f(x) - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \int_0^1 \left| \int_0^t p(\tau) d\tau - \gamma \right| \|Df((1-t)x + ty)(y-x)\| dt \\ & =: B(f, p, x, y, \gamma). \end{aligned} \tag{27}$$

Moreover, we have the upper bounds

$$\begin{aligned} & B(f, p, x, y, \gamma, \mu) \\ & \leq \left\{ \begin{aligned} & \max \left\{ \sup_{t \in [0,s]} \left| \int_0^t p(\tau) d\tau - \gamma \right|, \sup_{t \in [s,1]} \left| \int_0^t p(\tau) d\tau - \mu \right| \right\} \\ & \quad \times \left[\int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \right] \\ & \left[\left(\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right|^r dt \right) + \left(\int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right|^r dt \right) \right]^{1/r} \\ & \quad \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ & r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ & \left[\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| dt + \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| dt \right] \\ & \quad \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \end{aligned} \right\} \end{aligned} \tag{28}$$

and

$$\begin{aligned}
 & B(f, p, x, y, \gamma) \tag{29} \\
 & \leq \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \gamma \right| \left[\int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \right] \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \gamma \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}, \quad r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \gamma \right| dt \right] \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|. \end{cases}
 \end{aligned}$$

Proof. We have by (19) that

$$\begin{aligned}
 & \left(\int_0^1 p(s) ds - \mu \right) f(y) + \gamma f(x) + (\mu - \gamma) f((1-s)x + sy) \tag{30} \\
 & - \int_0^1 p(t) f((1-t)x + ty) dt \\
 & = \int_0^s \left(\int_0^t p(\tau) d\tau - \gamma \right) Df((1-t)x + ty)(y-x) dt \\
 & + \int_s^1 \left(\int_0^t p(\tau) d\tau - \mu \right) Df((1-t)x + ty)(y-x) dt
 \end{aligned}$$

for all $x, y \in \mathbb{C}$ and any $\gamma, \mu \in \mathbb{C}$.

By taking the norm in (30), we get

$$\begin{aligned}
 & \left\| \left(\int_0^1 p(s) ds - \mu \right) f(y) + \gamma f(x) + (\mu - \gamma) f((1-s)x + sy) \right. \tag{31} \\
 & \left. - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\
 & \leq \left\| \int_0^s \left(\int_0^t p(\tau) d\tau - \gamma \right) Df((1-t)x + ty)(y-x) dt \right\| \\
 & + \left\| \int_s^1 \left(\int_0^t p(\tau) d\tau - \mu \right) Df((1-t)x + ty)(y-x) dt \right\| \\
 & \leq \int_0^s \left\| \left(\int_0^t p(\tau) d\tau - \gamma \right) Df((1-t)x + ty)(y-x) \right\| dt \\
 & + \int_s^1 \left\| \left(\int_0^t p(\tau) d\tau - \mu \right) Df((1-t)x + ty)(y-x) \right\| dt \\
 & = \int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| \|Df((1-t)x + ty)(y-x)\| dt \\
 & + \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| \|Df((1-t)x + ty)(y-x)\| dt \\
 & = B(f, p, x, y, \gamma, \mu).
 \end{aligned}$$

By using Hölder’s inequality we have

$$\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| \|Df((1-t)x + ty)(y-x)\| dt$$

$$\leq \begin{cases} \sup_{t \in [0,s]} \left| \int_0^t p(\tau) d\tau - \gamma \right| \int_0^s \|Df((1-t)x + ty)(y-x)\| dt \\ \left(\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right|^r dt \right)^{1/r} \left(\int_0^s \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \sup_{t \in [0,s]} \|Df((1-t)x + ty)(y-x)\| \int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| dt \end{cases}$$

and

$$\int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| \|Df((1-t)x + ty)(y-x)\| dt$$

$$\leq \begin{cases} \sup_{t \in [s,1]} \left| \int_0^t p(\tau) d\tau - \mu \right| \int_s^1 \|Df((1-t)x + ty)(y-x)\| dt \\ \left(\int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right|^r dt \right)^{1/r} \left(\int_s^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \sup_{t \in [s,1]} \|Df((1-t)x + ty)(y-x)\| \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| dt. \end{cases}$$

Therefore

$$B(f, p, x, y, \gamma, \mu)$$

$$\leq \begin{cases} \sup_{t \in [0,s]} \left| \int_0^t p(\tau) d\tau - \gamma \right| \int_0^s \|Df((1-t)x + ty)(y-x)\| dt \\ \left(\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right|^r dt \right)^{1/r} \left(\int_0^s \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \sup_{t \in [0,s]} \|Df((1-t)x + ty)(y-x)\| \int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| dt \\ \sup_{t \in [s,1]} \left| \int_0^t p(\tau) d\tau - \mu \right| \int_s^1 \|Df((1-t)x + ty)(y-x)\| dt \\ \left(\int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right|^r dt \right)^{1/r} \left(\int_s^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \sup_{t \in [s,1]} \|Df((1-t)x + ty)(y-x)\| \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| dt. \end{cases}$$

$$\begin{aligned}
 & \left\{ \begin{aligned}
 & \max \left\{ \sup_{t \in [0,s]} \left| \int_0^t p(\tau) d\tau - \gamma \right|, \sup_{t \in [s,1]} \left| \int_0^t p(\tau) d\tau - \mu \right| \right\} \\
 & \times \left[\int_0^s \|Df((1-t)x + ty)(y-x)\| dt + \int_s^1 \|Df((1-t)x + ty)(y-x)\| dt \right] \\
 & \left[\left(\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right|^p dt \right) + \left(\int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right|^r dt \right) \right]^{1/r} \\
 & \times \left(\int_0^s \|Df((1-t)x + ty)(y-x)\|^q dt + \int_s^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\
 & r, q > 1, \frac{1}{r} + \frac{1}{q} = 1
 \end{aligned} \right. \\
 & \leq \left\{ \begin{aligned}
 & \max \left\{ \sup_{t \in [0,s]} \|Df((1-t)x + ty)(y-x)\|, \right. \\
 & \left. \sup_{t \in [s,1]} \|Df((1-t)x + ty)(y-x)\| \right\} \\
 & \times \left[\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| dt + \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| dt \right] \\
 & \left\{ \begin{aligned}
 & \max \left\{ \sup_{t \in [0,s]} \left| \int_0^t p(\tau) d\tau - \gamma \right|, \sup_{t \in [s,1]} \left| \int_0^t p(\tau) d\tau - \mu \right| \right\} \\
 & \times \left[\int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \right] \\
 & \left[\left(\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right|^r dt \right) + \left(\int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right|^r dt \right) \right]^{1/r} \\
 & \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} \\
 & r, q > 1, \frac{1}{r} + \frac{1}{q} = 1
 \end{aligned} \right. \\
 & \left[\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| dt + \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| dt \right] \\
 & \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|,
 \end{aligned} \right. \\
 & = \left\{ \begin{aligned}
 & \left[\int_0^s \left| \int_0^t p(\tau) d\tau - \gamma \right| dt + \int_s^1 \left| \int_0^t p(\tau) d\tau - \mu \right| dt \right] \\
 & \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|,
 \end{aligned} \right.
 \end{aligned}$$

which proves the inequality (27). \square

Corollary 2. Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Let $p : [0, 1] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[0, 1]$ and $s \in [0, 1]$. Then for all $x, y \in C$ we have

$$\begin{aligned}
 & \left\| \left(\int_s^1 p(\tau) d\tau \right) f(y) + \left(\int_0^s p(\tau) d\tau \right) f(x) - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \tag{32} \\
 & \leq \left\{ \begin{aligned}
 & \sup_{t \in [0,1]} \left| \int_s^t p(\tau) d\tau \right| \left[\int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \right] \\
 & \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right|^r dt \right]^{1/r} \\
 & \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q} r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\
 & \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right| dt \right] \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|.
 \end{aligned} \right.
 \end{aligned}$$

In particular,

$$\begin{aligned} & \left\| \left(\int_{\frac{1}{2}}^1 p(\tau) d\tau \right) f(y) + \left(\int_0^{\frac{1}{2}} p(\tau) d\tau \right) f(x) \right. \\ & \left. - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \begin{cases} \sup_{t \in [0,1]} \left| \int_{1/2}^t p(\tau) d\tau \right| \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \\ \left[\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}, r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left(\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right| dt \right) \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|. \end{cases} \end{aligned} \tag{33}$$

The proof follows by Theorem 2 on choosing $\gamma = \int_0^s p(\tau) d\tau, s \in [0, 1]$.

Corollary 3. Assume that the function $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex set C . Let $p : [0, 1] \rightarrow \mathbb{C}$ be Lebesgue integrable on $[0, 1]$. Then for all $x, y \in C$

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) [(1-\alpha)f(y) + \alpha f(x)] - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| \\ \times \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}, r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| dt \\ \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\| \end{cases} \end{aligned} \tag{34}$$

for all $\alpha \in [0, 1]$.

In particular, we have the trapezoid type inequalities

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) \frac{f(y) + f(x)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| \\ \times \int_0^1 \|Df((1-t)x + ty)(y-x)\| dt \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)(y-x)\|^q dt \right)^{1/q}, r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| dt \\ \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)(y-x)\|. \end{cases} \end{aligned} \tag{35}$$

Remark 3. Since the Fréchet derivative satisfies the condition

$$\|Df(a)(b)\| \leq \|Df(a)\| \|b\|$$

for $a \in C$ and $b \in E$, then for all $x, y \in C$ we also have the chain of inequalities

$$\begin{aligned} & \left\| \left(\int_s^1 p(\tau) d\tau \right) f(y) + \left(\int_0^s p(\tau) d\tau \right) f(x) \right. \\ & \left. - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_s^t p(\tau) d\tau \right| \left[\int_0^1 \|Df((1-t)x + ty)\| dt \right] \\ \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)\|^q dt \right)^{1/q}, r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right| dt \right] \sup_{t \in [0,1]} \|Df((1-t)x + ty)\|. \end{cases} \end{aligned} \quad (36)$$

In particular,

$$\begin{aligned} & \left\| \left(\int_{\frac{1}{2}}^1 p(\tau) d\tau \right) f(y) + \left(\int_0^{\frac{1}{2}} p(\tau) d\tau \right) f(x) \right. \\ & \left. - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_{1/2}^t p(\tau) d\tau \right| \int_0^1 \|Df((1-t)x + ty)\| dt \\ \left[\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)\|^q dt \right)^{1/q}, r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left(\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right| dt \right) \sup_{t \in [0,1]} \|Df((1-t)x + ty)\|. \end{cases} \end{aligned} \quad (37)$$

If $\alpha \in [0, 1]$, then for all $x, y \in C$ we also have the chain of inequalities

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) [(1-\alpha)f(y) + \alpha f(x)] - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ & \leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| \\ \times \int_0^1 \|Df((1-t)x + ty)\| dt \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)\|^q dt \right)^{1/q}, r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| dt \\ \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)\|. \end{cases} \end{aligned} \quad (38)$$

In particular, we have the trapezoid type inequalities

$$\left\| \left(\int_0^1 p(\tau) d\tau \right) \frac{f(y) + f(x)}{2} - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \quad (39)$$

$$\leq \|y - x\| \left\{ \begin{array}{l} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| \\ \times \int_0^1 \|Df((1-t)x + ty)\| dt \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 \|Df((1-t)x + ty)\|^q dt \right)^{1/q} \quad r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| dt \\ \times \sup_{t \in [0,1]} \|Df((1-t)x + ty)\|. \end{array} \right.$$

4. Some Examples for Banach Algebras

Let \mathcal{B} be an algebra. An algebra norm on \mathcal{B} is a map $\|\cdot\| : \mathcal{B} \rightarrow [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a Banach algebra if $\|\cdot\|$ is a complete norm.

We assume that the Banach algebra is unital, this means that \mathcal{B} has an identity 1 and that $\|1\| = 1$.

Let \mathcal{B} be a unital algebra. An element $a \in \mathcal{B}$ is invertible if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the inverse of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of \mathcal{B} is denoted by $\text{Inv}\mathcal{B}$. If $a, b \in \text{Inv}\mathcal{B}$ then $ab \in \text{Inv}\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now, by the help of power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |\alpha_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

The following result holds [23].

Lemma 3. Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have

$$\|f(y) - f(x)\| \leq \|y - x\| \int_0^1 f'_a(\|(1-s)x + sy\|) ds. \tag{40}$$

We also have:

Lemma 4. Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have

$$\|Df((1-t)x + ty)(y - x)\| \leq \|y - x\| f'_a(\|(1-t)x + ty\|) \tag{41}$$

for all $t \in [0, 1]$.

Proof. Let $\|u\|, \|v\| < R$. Then there exists $\delta > 0$ such that $\|u + \varepsilon v\| < R$ for all $\varepsilon \in (-\delta, \delta)$ and by (40) we get

$$\|f(u + \varepsilon v) - f(u)\| \leq \|u + \varepsilon v - u\| \int_0^1 f'_a(\|(1-s)u + s(u + \varepsilon v)\|) ds$$

namely

$$\begin{aligned} \left\| \frac{f(u + \varepsilon v) - f(u)}{\varepsilon} \right\| &\leq \|v\| \int_0^1 f'_a(\|(1-s)u + s(u + \varepsilon v)\|) ds, \quad \varepsilon \neq 0 \\ &= \|v\| \int_0^1 f'_a(\|u + s\varepsilon v\|) ds \end{aligned}$$

and by taking $\varepsilon \rightarrow 0$ we get, by the property of integral, that

$$\|Df(u)(v)\| \leq \|v\| f'_a(\|u\|). \quad (42)$$

Now, if we take in (42) $u = (1-t)x + ty$ and $v = y - x$, then we get (41). \square

We have the following result:

Theorem 3. Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. Also, let $p : [0, 1] \rightarrow \mathbb{C}$ be a Lebesgue integrable function on $[0, 1]$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| < R$ we have

$$\begin{aligned} &\left\| \left(\int_s^1 p(\tau) d\tau \right) f(y) + \left(\int_0^s p(\tau) d\tau \right) f(x) - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ &\leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_s^t p(\tau) d\tau \right| \int_0^1 f'_a(\|(1-t)x + ty\|) dt \\ \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 [f'_a(\|(1-t)x + ty\|)]^q dt \right)^{1/q}, \quad r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right| dt \right] \sup_{t \in [0,1]} f'_a(\|(1-t)x + ty\|). \end{cases} \end{aligned} \quad (43)$$

In particular,

$$\begin{aligned} &\left\| \left(\int_{\frac{1}{2}}^1 p(\tau) d\tau \right) f(y) + \left(\int_0^{\frac{1}{2}} p(\tau) d\tau \right) f(x) \right. \\ &\quad \left. - \int_0^1 p(t) f((1-t)x + ty) dt \right\| \\ &\leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_{1/2}^t p(\tau) d\tau \right| \int_0^1 f'_a(\|(1-t)x + ty\|) dt \\ \left[\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \times \left(\int_0^1 [f'_a(\|(1-t)x + ty\|)]^q dt \right)^{1/q}, \quad r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left(\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right| dt \right) \sup_{t \in [0,1]} f'_a(\|(1-t)x + ty\|). \end{cases} \end{aligned} \quad (44)$$

Also,

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) [(1-\alpha)f(y) + \alpha f(x)] - \int_0^1 p(t)f((1-t)x + ty) dt \right\| \\ & \leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| \\ \quad \times \int_0^1 f'_a(\|(1-t)x + ty\|) dt \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \quad \times \left(\int_0^1 [f'_a(\|(1-t)x + ty\|)]^q dt \right)^{1/q} \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| dt \\ \quad \times \sup_{t \in [0,1]} f'_a(\|(1-t)x + ty\|) \end{cases} \end{aligned} \tag{45}$$

for all $\alpha \in [0, 1]$.

In particular, we have the trapezoid type inequalities

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) \frac{f(y) + f(x)}{2} - \int_0^1 p(t)f((1-t)x + ty) dt \right\| \\ & \leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| \\ \quad \times \int_0^1 f'_a(\|(1-t)x + ty\|) dt \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} \\ \quad \times \left(\int_0^1 [f'_a(\|(1-t)x + ty\|)]^q dt \right)^{1/q} \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| dt \\ \quad \times \sup_{t \in [0,1]} f'_a(\|(1-t)x + ty\|). \end{cases} \end{aligned} \tag{46}$$

The proof follows by Theorem 2 and Lemma 4.

As some natural examples that are useful for applications, we can point out that if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \quad \lambda \in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \quad \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \quad \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \quad \lambda \in D(0,1); \end{aligned} \tag{47}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_a(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \quad \lambda \in D(0,1); \\ g_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \quad \lambda \in \mathbb{C}; \\ h_a(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \quad \lambda \in \mathbb{C}; \\ l_a(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \quad \lambda \in D(0,1). \end{aligned} \quad (48)$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\begin{aligned} \exp(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n, \quad \lambda \in \mathbb{C}, \\ \frac{1}{2} \ln \left(\frac{1+\lambda}{1-\lambda} \right) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0,1); \\ \sin^{-1}(\lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)n!} \lambda^{2n+1}, \quad \lambda \in D(0,1); \\ \tanh^{-1}(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0,1) \\ {}_2F_1(\alpha, \beta, \gamma, \lambda) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} \lambda^n, \quad \alpha, \beta, \gamma > 0, \\ &\lambda \in D(0,1). \end{aligned} \quad (49)$$

If we consider the exponential function $f(x) = \exp x$, then for $x, y \in \mathcal{B}$

$$\begin{aligned} \int_0^1 f'_a(\|(1-t)x + ty\|) dt &= \int_0^1 \exp(\|(1-t)x + ty\|) dt \\ &\leq \int_0^1 \exp((1-t)\|x\| + t\|y\|) dt \\ &= \int_0^1 \exp(\|x\| + t(\|y\| - \|x\|)) dt \\ &= \begin{cases} \frac{\exp\|y\| - \exp\|x\|}{\|y\| - \|x\|} & \text{if } \|y\| \neq \|x\| \\ \exp\|x\| & \text{if } \|y\| = \|x\|. \end{cases} \\ &=: E_1(x, y). \end{aligned} \quad (50)$$

Also

$$\begin{aligned}
 \int_0^1 [f'_a(\|(1-t)x + ty\|)]^q dt &= \int_0^1 \exp[q(\|(1-t)x + ty\|)] dt & (51) \\
 &\leq \int_0^1 \exp[q((1-t)\|x\| + t\|y\|)] dt \\
 &= \begin{cases} \frac{\exp(q\|y\|) - \exp(q\|x\|)}{q(\|y\| - \|x\|)} & \text{if } \|y\| \neq \|x\| \\ \exp(q\|x\|) & \text{if } \|y\| = \|x\| \end{cases} \\
 &=: E_q(x, y), \quad q > 1.
 \end{aligned}$$

Moreover, for $x, y \in \mathcal{B}$

$$\begin{aligned}
 \sup_{t \in [0,1]} f'_a(\|(1-t)x + ty\|) &\leq \sup_{t \in [0,1]} \exp((1-t)\|x\| + t\|y\|) & (52) \\
 &= \max\{\|x\|, \|y\|\}.
 \end{aligned}$$

By making use of Theorem 3 and (50)–(52), we have for $p : [0, 1] \rightarrow \mathbb{C}$, a Lebesgue integrable function on $[0, 1]$ and any $x, y \in \mathcal{B}$ and $s \in [0, 1]$, that

$$\begin{aligned}
 &\left\| \left(\int_s^1 p(\tau) d\tau \right) \exp y + \left(\int_0^s p(\tau) d\tau \right) \exp x \right. & (53) \\
 &\quad \left. - \int_0^1 p(t) \exp((1-t)x + ty) dt \right\| \\
 &\leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_s^t p(\tau) d\tau \right| E_1(x, y) \\ \left[\int_0^1 \left| \int_s^t p(\tau) d\tau \right|^p dt \right]^{1/r} [E_q(x, y)]^{1/q}, \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left(\int_0^1 \left| \int_s^t p(\tau) d\tau \right| dt \right) \max\{\|x\|, \|y\|\}. \end{cases}
 \end{aligned}$$

In particular,

$$\begin{aligned}
 &\left\| \left(\int_{\frac{1}{2}}^1 p(\tau) d\tau \right) f(y) + \left(\int_0^{\frac{1}{2}} p(\tau) d\tau \right) f(x) \right. & (54) \\
 &\quad \left. - \int_0^1 p(t) \exp((1-t)x + ty) dt \right\| \\
 &\leq \|y - x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_{1/2}^t p(\tau) d\tau \right| E_1(x, y) \\ \left[\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right|^p dt \right]^{1/r} [E_q(x, y)]^{1/q}, \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \left(\int_0^1 \left| \int_{1/2}^t p(\tau) d\tau \right| dt \right) \max\{\|x\|, \|y\|\}. \end{cases}
 \end{aligned}$$

Also,

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) [(1-\alpha)\exp(y) + \alpha\exp(x)] \right. \\ & \left. - \int_0^1 p(t) \exp((1-t)x + ty) dt \right\| \\ & \leq \|y-x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| E_1(x, y) \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} (E_q(x, y))^{1/q}, \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \alpha \int_0^1 p(\tau) d\tau \right| dt \max\{\|x\|, \|y\|\}, \end{cases} \end{aligned} \quad (55)$$

for all $\alpha \in [0, 1]$.

In particular, we have the trapezoid type inequalities

$$\begin{aligned} & \left\| \left(\int_0^1 p(\tau) d\tau \right) \frac{\exp(y) + \exp(x)}{2} - \int_0^1 p(t) \exp((1-t)x + ty) dt \right\| \\ & \leq \|y-x\| \times \begin{cases} \sup_{t \in [0,1]} \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| E_1(x, y) \\ \left[\int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right|^r dt \right]^{1/r} [E_q(x, y)]^{1/q}, \\ r, q > 1, \frac{1}{r} + \frac{1}{q} = 1 \\ \int_0^1 \left| \int_0^t p(\tau) d\tau - \frac{1}{2} \int_0^1 p(\tau) d\tau \right| dt \max\{\|x\|, \|y\|\}. \end{cases} \end{aligned} \quad (56)$$

The interested reader may apply the above inequalities for the other functions listed in (47)–(49). The details are omitted.

5. Conclusions

In this paper we established some upper bounds for the quantity

$$\left\| \left(\int_0^1 p(s) ds - \gamma \right) f(y) + \gamma f(x) - \int_0^1 p(t) f((1-t)x + ty) dt \right\|$$

in the case that $f : C \subset E \rightarrow F$ is Fréchet differentiable on the open and convex subset C of the Banach space E with values into another Banach space F , $x, y \in C$, $p : [0, 1] \rightarrow \mathbb{C}$ is a Lebesgue integrable function and $\gamma \in \mathbb{C}$. Some particular cases of interest for different choices of γ are given. The symmetric case for the coefficients of f in the above inequality is analysed. Applications for Banach algebras are also provided.

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