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Geometric Constants in Banach Spaces Related to the Inscribed Quadrilateral of Unit Balls

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Abstract: We introduce a new geometric constant $Jin(X)$ based on a generalization of the parallelogram law, which is symmetric and related to the length of the inscribed quadrilateral side of the unit ball. We first investigate some basic properties of this new coefficient. Next, it is shown that, for a Banach space, $Jin(X)$ becomes 16 if and only if the norm is induced by an inner product. Moreover, its properties and some relations between other well-known geometric constants are studied. Finally, a sufficient condition which implies normal structure is presented.

Keywords: Banach spaces; geometric constants; geometry of normed spaces; normal structure



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1. Introduction

In recent years, many geometric constants have been defined and studied in the literature, which makes it easier for us to deal with some problems in Banach space, because it can describe the geometric properties of space quantitatively. These geometric constants have mathematical beauty, and there are countless relationships between different geometric constants. One of the best known is the von Neumann–Jordan constant $C_{NJ}(X)$ and the James constant $J(X)$. Readers interested in this field are advised to see [1–6] and the references mentioned therein. It is worth mentioning that geometric constants play a vital role as a tool for solving other problems, such as in the study of Banach–Stone theorem, Bishop– Phelps–Bollobás theorem, and Tingley’s Problem. These are important research topics in the area of functional analysis and we recommend that readers refer to the literature [7–9].

Among all normed spaces, the Hilbert spaces are generally considered to have the simplest and clearest geometric structure. Many mathematicians have found the conditions on normed spaces under which such spaces become inner product spaces. Results of this kind are of importance in functional analysis—for example, in the theory of operator algebras and certain stability problems [10,11]. In addition, since every physical system is associated with a Hilbert space, the notion of inner product space also plays a crucial role in quantum mechanics; see [12]. The rich theory of Hilbert spaces has been created by the efforts of many mathematicians; refer to [13–15] for more details. It is worth noting that the background meaning of many famous geometric constants is closely related to the description of inner product spaces.

The first norm characterization of inner product spaces was given by Fréchet [16] in 1935.

Lemma 1 ([16]). *A complex normed space $(X, \|\cdot\|)$ is an inner product space if and only if*

$$\|x_1 + x_2\|^2 + \|x_2 + x_3\|^2 + \|x_1 + x_3\|^2 = \|x_1 + x_2 + x_3\|^2 + \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2$$

for all $x_1, x_2, x_3 \in X$.

If we consider the usual Euclidean space $(R^n, \| \cdot \|)$, then the identity $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ is called the parallelogram law, and it is well known. This identity can be naturally extended to the more general case. Research on the equivalent characterization of inner product space has been attracting much attention. We refer the readers to [17,18] for more details.

M. M. Day [19] gave the following weaker characterization, referred to as “the rhombus law”.

Lemma 2 ([19]). *Let $(X, \| \cdot \|)$ be a real normed linear space. Then, $\| \cdot \|$ derives from an inner product if and only if*

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \sim 4$$

for all $x_1, x_2 \in S_X$, where \sim stands either for \leq or for \geq .

2. Preliminaries

We now give some definitions related to geometric constants. Let X be a real Banach space with $\dim X \geq 2$ and denote by S_X and B_X the unit sphere and the unit ball, respectively.

In combination with Jordan and von Neumann’s [20] brilliant work on the characterization of inner product spaces by the parallelogram law, Clarkson [21] first proposed the von Neumann–Jordan constant $C_{NJ}(X)$ of Banach space. More precisely, the von Neumann–Jordan constant of X is defined by

$$C_{NJ}(X) = \sup \left\{ \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2(\|x_1\|^2 + \|x_2\|^2)} : x_1, x_2 \in X, (x_1, x_2) \neq (0, 0) \right\}.$$

The James constant $J(X)$ of a Banach space X is introduced by Gao and Lau [22] as follows:

$$J(X) = \sup \{ \min \{ \|x_1 + x_2\|, \|x_1 - x_2\| \} : x_1, x_2 \in S_X \}.$$

Moreover, the various properties of these constants are given in [22–24]:

- (1) $\sqrt{2} \leq J(X) \leq 2$.
- (2) $J(X) = \sqrt{2}$ whenever X represents Hilbert space; the converse is not correct.
- (3) $1 \leq C_{NJ}(X) \leq 2$.
- (4) X is a Hilbert space if $C_{NJ}(X) = 1$.
- (5) X is uniformly non-square if $C_{NJ}(X) < 2$.
- (6) $C_{NJ}(X) = C_{NJ}(X^*)$.

The modified von Neumann–Jordan constant

$$C'_{NJ}(X) := \sup \left\{ \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{4} : x_1, x_2 \in S_X \right\}$$

was studied by Takahashi [25] and Alonso et al. [2].

Recall that the Banach space X is called uniformly non-square [26] if there exists a $\delta \in (0, 1)$ such that for any $x_1, x_2 \in S_X$ either $\frac{\|x_1 + x_2\|}{2} \leq 1 - \delta$ or $\frac{\|x_1 - x_2\|}{2} \leq 1 - \delta$. It is known that X is uniformly non-square if and only if $C'_{NJ}(X) < 2$.

Definition 1 ([27]). *A Banach space $(X, \| \cdot \|)$ is non-square if, for every $x_1, x_2 \in S_X$, we have*

$$\min \{ \|x_1 + x_2\|, \|x_1 - x_2\| \} < 2.$$

Baronti et al. defined the following two new constants, which are related to the perimeter of an inscribed triangle in a semicircle of normed space.

Definition 2 ([28]).

$$A_1(X) = \frac{1}{2} \inf_{x_1 \in S_X} \sup_{x_2 \in S_X} (\|x_1 - x_2\| + \|x_1 + x_2\|)$$

and

$$A_2(X) = \frac{1}{2} \sup_{x_1 \in S_X} \sup_{x_2 \in S_X} (\|x_1 - x_2\| + \|x_1 + x_2\|).$$

Javier Alonso and Enrique Llorens-Fuster [29] also introduced the constants

$$t(x) = \inf_{x_1 \in S_X} \sup_{x_2 \in S_X} \sqrt{\|x_1 + x_2\| \|x_1 - x_2\|},$$

$$T(x) = \sup_{x_1, x_2 \in S_X} \sqrt{\|x_1 + x_2\| \|x_1 - x_2\|}.$$

It is intuitive to show that the geometric meaning of $t(x)$ and $T(x)$ is the geometric mean of the diagonals of a “rhombus”.

Fu et al. also introduced the constants:

$$J_L(X) = \inf\{\|x_1 - x_2\|, \|x_2 - x_3\|, \|x_1 - x_3\| : \|x_1\| = \|x_2\| = \|x_3\| = 1, x_1 + x_2 + x_3 = 0\};$$

$$Y_J(X) = \sup\{\|x_1 - x_2\|, \|x_2 - x_3\|, \|x_1 - x_3\| : \|x_1\| = \|x_2\| = \|x_3\| = 1, x_1 + x_2 + x_3 = 0\},$$

which are related to the side lengths of the inscribed triangles of unit balls to study the geometric properties of Banach spaces. Moreover, the various properties of these constants are given in [30]:

- (1) $1 \leq J_L(X) \leq 2, \sqrt{3} \leq Y_J(X) \leq 2.$
- (2) Let X be a Hilbert space, then $J_L(X) = Y_J(X) = \sqrt{3}.$
- (3) $\frac{2}{J(X)} \leq J_L(X) \leq Y_J(X) \leq \sqrt{4J(X) - 1}.$

Inspired by the above constants, and in combination with the characterization of the inner product space, we define a new geometric constant $Jin(X)$ and some related properties are discussed. Furthermore, we attempt to relate this to other important geometric concepts and finally give an application that is closely related to normal structure.

The paper is organized as follows: we exhibit some basic properties of this new coefficient in the next section. Furthermore, the connections between Hilbert spaces are investigated. The relationship between the constant $Jin(X)$ and other well-known constants is emphasized in terms of nontrivial inequalities. In Section 4, we establish a new necessary condition for Banach spaces with normal structure in the form of $Jin(X)$.

3. Constant $Jin(X)$

It is well known that the circle and its inscribed polygon is an important research topic in Euclidean geometry. The results of this study reveal many important geometric properties of Euclidean planes. Some results have been generalized to Banach spaces, such as orthogonality, angle, circumference, and other geometric concepts. Instead of Euclidean plane geometry, we consider a more general case, in the framework of a Banach space. We can investigate the Banach space by considering its unit sphere, because the unit sphere can largely reflect the geometric properties of a space X , such as the well-known convexity. According to the result of [31], in many cases, we only need to consider the extreme points on the unit ball to study the geometric constant; for some classical Banach spaces, such as l_1, l_∞ , the two dimensional case, the number of its extreme points on the unit ball is four or more. In a way, our four variables are significant, either in terms of the four sides of an inscribed quadrilateral and its two diagonals to understand, or through four variables, which are special points. Considering these factors, we begin by introducing the following key definition:

Definition 3.

$$\begin{aligned} Jin(X) &= \sup \left\{ \sum \|x_i - x_j\|^2 : x_i, x_j \in S_X, 1 \leq i < j \leq 4, i, j \in \{1, 2, 3, 4\}, \sum_{i=1}^4 x_i = 0 \right\} \\ &= \sup \left\{ \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 + \|x_1 - x_4\|^2 \right. \\ &\quad \left. + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 : x_1, x_2, x_3, x_4 \in S_X, x_1 + x_2 + x_3 + x_4 = 0 \right\}. \end{aligned}$$

In fact, upon observation, we find that the constant $Jin(X)$ is symmetric because the positions of x_1, x_2, x_3 , and x_4 are interchangeable, and point $x_1 + x_2$ is actually symmetric to point $x_3 + x_4$. The constant $Jin(X)$ can be understood in a very intuitive way: the sum of the squares of the lengths of the four sides of the inscribed quadrilateral plus the sum of the squares of the two diagonals.

Proposition 1. *Suppose that X is a normed space. Then,*

$$Jin(X) \geq 8C'_{NJ}(X) + 8.$$

Proof. Assume $x_1 = x, x_2 = -x, x_3 = y, x_4 = -y$ and $x, y \in S_X$; then, we have

$$\begin{aligned} &\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 \\ &+ \|x_1 - x_4\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 \\ &= \|2x\|^2 + \|x + y\|^2 + \|x - y\|^2 \\ &+ \|x + y\|^2 + \|x - y\|^2 + \|2y\|^2 \\ &= 2\|x + y\|^2 + 2\|x - y\|^2 + 8. \end{aligned}$$

Hence, we can deduce that

$$Jin(X) \geq 8C'_{NJ}(X) + 8.$$

From Proposition 1, we can obtain the following estimate. \square

Proposition 2. *Suppose that X is a normed space. Then,*

$$16 \leq Jin(X) \leq 24.$$

Proof. Using the same method as in Proposition 1, we can conclude that

$$Jin(X) \geq 2\|x + y\|^2 + 2\|x - y\|^2 + 8$$

for any $x, y \in S_X$, letting $x = -y$ and hence $Jin(X) \geq 16$.

The latter assertion can be derived from the following estimate

$$\|x_1 + x_2\|^2 \leq 2\|x_1\|^2 + 2\|x_2\|^2,$$

as desired.

Next, we give the following two examples, which illustrate the relationship between the constant $Jin(X)$ and the four variables. \square

Example 1. Let X be \mathbb{R}^2 with the norm defined by

$$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}.$$

Letting $x_1 = (1, 1), x_2 = (1, -1), x_3 = (-1, 1), x_4 = (-1, -1)$, it is easy to see that

$$\|x_1 - x_2\| = \|x_2 - x_3\| = \|x_1 - x_3\| = \|x_1 - x_4\| = \|x_2 - x_4\| = \|x_3 - x_4\| = 2$$

and $x_1 + x_2 + x_3 + x_4 = 0$. Hence, we can obtain $Jin(X) = 24$.

Example 2. Let X be \mathbb{R}^2 with the norm defined by

$$\|(x_1, x_2)\| = |x_1| + |x_2|.$$

Letting $x_1 = (1, 0), x_2 = (-1, 0), x_3 = (0, 1), x_4 = (0, -1)$, we then have

$$\|x_1 - x_2\| = \|x_2 - x_3\| = \|x_1 - x_3\| = \|x_1 - x_4\| = \|x_2 - x_4\| = \|x_3 - x_4\| = 2$$

and $x_1 + x_2 + x_3 + x_4 = 0$. Thus, $Jin(X) = 24$.

Proposition 3. Suppose that X is a normed space. Then,

$$Jin(X) \geq 4J(X)^2 + 8.$$

Proof. Using the same method as in Proposition 1 and applying the elementary inequality, we can conclude that

$$\begin{aligned} Jin(X) &\geq 2\|x_1 + x_2\|^2 + 2\|x_1 - x_2\|^2 + 8 \\ &= 4 \frac{\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2}{2} + 8 \\ &\geq 4 \min\{\|x_1 + x_2\|^2, \|x_1 - x_2\|^2\} + 8 \end{aligned}$$

for any $x_1, x_2 \in S_X$. Hence,

$$Jin(X) \geq 4J(X)^2 + 8.$$

□

Theorem 1. Let X be a Banach space. Then, $Jin(X) = 16$ if and only if X is a Hilbert space.

Proof. By Lemma 1, we can deduce that

$$\|x_1 + x_2\|^2 + \|x_2 + x_3\|^2 + \|x_1 + x_3\|^2 = 4$$

for any $x_1, x_2, x_3, x_4 \in S_X, x_1 + x_2 + x_3 + x_4 = 0$.

Applying the parallelogram law, we obtain that

$$\begin{aligned} &\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \\ &\quad + \|x_2 + x_3\|^2 + \|x_2 - x_3\|^2 \\ &\quad + \|x_1 + x_3\|^2 + \|x_1 - x_3\|^2 \\ &= 12. \end{aligned}$$

Thus, we have

$$\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 = 8.$$

Analogously, we can prove that

$$\|x_1 - x_2\|^2 + \|x_2 - x_4\|^2 + \|x_1 - x_4\|^2 = 8;$$

$$\|x_2 - x_3\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 = 8;$$

$$\|x_1 - x_3\|^2 + \|x_1 - x_4\|^2 + \|x_3 - x_4\|^2 = 8.$$

From the above, it follows that

$$\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 + \|x_1 - x_4\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 = 16.$$

For the second part of the proof, we suppose $Jin(X) = 16$. Then, we have

$$\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 + \|x_1 - x_4\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 \leq 16$$

for any $x_1, x_2, x_3, x_4 \in S_X, x_1 + x_2 + x_3 + x_4 = 0$.

Letting $x_1 = x, x_2 = -x, x_3 = y, x_4 = -y$ and $x, y \in S_X$, we obtain

$$\begin{aligned} & \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 \\ & + \|x_1 - x_4\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 \\ & = 2\|x + y\|^2 + 2\|x - y\|^2 + \|2y\|^2 + \|2y\|^2 \\ & \leq 16. \end{aligned}$$

Consequently,

$$\|x + y\|^2 + \|x - y\|^2 \leq 4$$

for any $x, y \in S_X$. Thus, X is a Hilbert space directly from Lemma 2. \square

Proposition 4. *Let X be a Banach space; if X is not non-square, then $Jin(X) = 24$.*

Proof. First note that X is not non-square and therefore there exist $x_n, y_n \in S_X$ for which

$$\|x_n + y_n\| \rightarrow 2, \|x_n - y_n\| \rightarrow 2 \quad (n \rightarrow \infty).$$

Letting $x_1 = x_n, x_2 = -x_n, x_3 = y_n, x_4 = -y_n$ and $x_n, y_n \in S_X$, we obtain

$$\begin{aligned} & \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_1 - x_3\|^2 \\ & + \|x_1 - x_4\|^2 + \|x_2 - x_4\|^2 + \|x_3 - x_4\|^2 \\ & = 2\|x_n + y_n\|^2 + 2\|x_n - y_n\|^2 + 8, \end{aligned}$$

which implies that $Jin(X) = 24$.

Proposition 4 is thereby proven.

Contractive mappings are closely related to the fixed point property, which plays an important role in the application of Banach space geometry. It is worth noting that contractive mappings have attracted more interest in other scientific branches; refer to [32–34] for more details. Garca-Falset et al. [35] proved that uniformly non-square Banach space has a fixed point property. Next, we establish a necessary condition for Banach spaces that have a fixed point property in the form of $Jin(X)$. \square

Proposition 5. *Assume X is a Banach space with $Jin(X) < 24$; then, X has the fixed point property.*

Proof. By Proposition 1, we obtain that

$$C'_{NJ}(X) \leq \frac{Jin(X) - 8}{8}.$$

This implies that $C'_{NJ}(X) < 2$ and hence X is uniformly non-square. Thus, we can deduce that X has the fixed point property (see [35]), as desired. \square

4. The Coefficient of Weak Orthogonality

Definition 4 ([36]). A nonempty bounded and convex subset K of a Banach space X is said to have normal structure if, for every convex subset H of K that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - x_1\| : x_1 \in H\} < \sup\{\|x_1 - x_2\| : x_1, x_2 \in H\}.$$

A Banach space X is said to have weak normal structure if each weakly compact convex set K in X that contains more than one point has normal structure. Obviously, for a reflexive Banach space, weak normal structure and normal structure coincide. Let Y be a subset of a Banach space X . A mapping $g : Y \rightarrow Y$ is called a nonexpansive mapping if $\|gx_1 - gx_2\| \leq \|x_1 - x_2\|$ for any $x_1, x_2 \in Y$. A large body of literature shows that normal structure plays an important role in the fixed point theory of nonexpansive mappings (see [37]).

Definition 5 ([38]). A Banach space X is said to have the property WORTH whenever

$$\limsup_{n \rightarrow \infty} \|\|z_n + z\| - \|z_n - z\|\| = 0$$

for all weakly null sequences z_n in X and all the elements z of X .

Years later, Sims introduced a parameter [39] on the original basis:

$$w(X) = \sup\left\{\lambda > 0 : \lambda \cdot \liminf_{n \rightarrow \infty} \|z_n + z\| \leq \liminf_{n \rightarrow \infty} \|z_n - z\|\right\}$$

where the supremum takes over all the weakly null sequences z_n in X and all the elements z of X . It was proven that $\frac{1}{3} \leq w(X) \leq 1$ for all Banach space X .

We begin by starting with two lemmas, which will be our main tools.

Lemma 3 ([40]). Let X be a Banach space without weak normal structure; then, for any $0 < \varepsilon < 1$, there exists a sequence $\{z_n\} \subseteq S_X$ with $z_n \xrightarrow{w} 0$, and

$$1 - \varepsilon < \|z_{n+1} - z\| < 1 + \varepsilon$$

for sufficiently large n , and any $z \in \text{co}\{z_k\}_{k=1}^n$.

From Lemma 3, we obtain the following:

Lemma 4. Let X be a Banach space without weak normal structure; then, for any $0 < \varepsilon < w(X)$, there exists an z_n in S_X , satisfying

$$(i) \quad 1 - \varepsilon \leq \|z_n - z\| \leq 1 + \varepsilon;$$

$$(ii) \quad \|z_n - z_1\| \leq 1 + \varepsilon;$$

$$(iii) \quad \|z_n + x_1\| \leq \frac{1+\varepsilon}{w(X)-\varepsilon},$$

$$\text{where } z \in \text{co}\{z_k\}_{k=1}^n.$$

Proof. By defining $w(X)$ and Lemma 3, we can easily obtain the result, so we omit the proof.

By characterizing the relation between weak orthogonal coefficient $w(X)$ and constant $Jin(X)$, a sufficient condition for Banach space to have normal structure is given. \square

Theorem 2. If

$$Jin(X) < 10w(X)^2 + 4w(X) + 10,$$

then the Banach space X has normal structure.

Proof. Let z_1 and z_n be as in Lemma 4, and let

$$x_1 = \frac{z_n - z_1}{\|z_n - z_1\|}, x_2 = \frac{-(w(X) - \varepsilon)(z_n + z_1)}{\|(w(X) - \varepsilon)(z_n + z_1)\|}$$

and

$$x_3 = \frac{z_1 - z_n}{\|z_n - z_1\|}, x_4 = \frac{(w(X) - \varepsilon)(z_n + z_1)}{\|(w(X) - \varepsilon)(z_n + z_1)\|}.$$

Then, x_1, x_2, x_3, x_4 belong to S_X and $x_1 + x_2 + x_3 + x_4 = 0$. We set

$$A = \frac{1}{\|z_n - z_1\|} + \frac{w(X) - \varepsilon}{\|(w(X) - \varepsilon)(z_n + z_1)\|}$$

and

$$B = \frac{1}{\|z_n - z_1\|} - \frac{w(X) - \varepsilon}{\|(w(X) - \varepsilon)(z_n + z_1)\|}.$$

Then, we obtain

$$\begin{aligned} \|x_1 - x_2\|^2 &= (A\|z_n - \frac{B}{A}z_1\|)^2 \\ &= \left(A \left\| z_n - \left(\frac{B}{A}z_1 + \frac{A-B}{A} \cdot 0 \right) \right\| \right)^2 \\ &\geq (A(1 - \varepsilon))^2 \end{aligned} \quad \text{(by Lemma 4(i))}$$

and

$$\begin{aligned} \|x_1 - x_4\|^2 &= \|Az_1 - Bz_n\|^2 \\ &\geq (\|Az_1\| - \|Bz_n\|)^2 \\ &= (A - B)^2 \\ &= \left(\frac{2w(X) - 2\varepsilon}{\|(w(X) - \varepsilon)(z_n + z_1)\|} \right)^2. \end{aligned}$$

Similarly, we can deduce

$$\|x_2 - x_3\|^2 \geq \left(\frac{2w(X) - 2\varepsilon}{\|(w(X) - \varepsilon)(z_n + z_1)\|} \right)^2, \quad \|x_3 - x_4\|^2 \geq (A(1 - \varepsilon))^2.$$

Furthermore,

$$\|x_1 - x_3\|^2 = 4, \quad \|x_2 - x_4\|^2 = 4.$$

Note that if $\varepsilon \rightarrow 0$, then $A \geq 1 + w(X)$. Therefore, from the definition of $Jin(X)$ and letting $\varepsilon \rightarrow 0$, we have

$$Jin(X) \geq 10w(X)^2 + 4w(X) + 10.$$

In the case of $Jin(X) < 10w(X)^2 + 4w(X) + 10$, we have $Jin(X) < 24$, and so X is uniformly nonsquare. Therefore, X is reflexive [26], which implies that normal structure equals weak normal structure, as required. \square

5. Conclusions

Based on the characterization of inner product space and the relative background of classical Euclidean geometry, we define a new geometric constant $Jin(X)$ for a Banach space X . It is remarkable that the constant $Jin(X)$ is symmetric and related to the length of the inscribed quadrilateral side of the unit ball. The introduction of the constant $Jin(X)$ takes the bridge, attempting to establish a new relationship with the classical constant. In some ways, it provides some ideas for studying the geometric constants of the four variables.

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