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Resummed Quantum Gravity: A Review with Applications

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Abstract: We summarize the status of the theory of resummed quantum gravity. In the context of the Planck scale cosmology formulation of Bonanno and Reuter, we review the use of our resummed quantum gravity approach to Einstein's general theory of relativity to estimate the value of the cosmological constant as $\rho_\Lambda = (0.0024\ \text{eV})^4$. Constraints on susy GUT models that follow from the closeness of the estimate to experiment are noted. Various consistency checks on the calculation are discussed. In particular, we use the Heisenberg uncertainty principle to remove a large part of the remaining uncertainty in our estimate of ρ_Λ.

Keywords: resummed; quantum; gravity

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1. Introduction

We use the well-known elementary example of "summation":

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \tag{1}$$

to illustrate why resummation can be worth its pursuit. Even though the mathematical tests for convergence of the series would only guarantee convergence for $|x| < 1$, this geometric series is summed to infinity to yield the analytic result that is well-defined except for a pole at $x = 1$. The result of the summation yields a function that is well-defined in the entire complex plane except for the simple pole at $x = 1$—infinite order summation has yielded behavior very much improved from what one sees order-by-order in the respective series.

We are thus motivated to 'resum' series that are already being summed to seek improvement in our knowledge of the represented function. This we illustrate as follows:

$$\sum_{n=0}^{\infty} C_n \alpha_s^n \begin{cases} = F_{\text{RES}}(\alpha_s) \sum_{n=0}^{\infty} B_n \alpha_s^n, \ \text{EXACT} \\ \cong G_{\text{RES}}(\alpha_s) \sum_{n=0}^{N} \tilde{B}_n \alpha_s^n, \ \text{APPROX.} \end{cases} \tag{2}$$

On the LHS (left-hand side) we have the original Feynman series for a process under study. On the RHS (right-hand side) are two versions of resumming this original series. One, labeled exact, is an exact re-arrangement of the original series. The other, labeled approx., only agrees with the LHS to some fixed order N in the expansion parameter α_s. For some time now, discussion has occurred as to which version is to be preferred [1]. Recently, a related more general version of this discussion occurs for quantum gravity.

Whether quantum gravity is even calculable in relativistic quantum field theory is a fair but difficult question. Answers vary. According to string theory [2,3], the answer is no, the true fundamental theory entails a one-dimensional Planck scale superstring. If we accept loop quantum gravity [4–7] we also find that the answer is no, the fundamental theory entails a space-time foam with a Planck scale loop structure. The answer is also no in the Horava–Lifshitz theory [8] because the fundamental theory requires Planck scale anisoptropic scaling for space and time. Kreimer [9,10] suggests that quantum gravity is leg-renormalizable, such that the answer is yes. Weinberg [11] suggests that quantum gravity may be asymptotically safe, with an S-matrix that depends only on a finite number of observable parameters, due to the presence of a non-trivial UV fixed point, with a finite dimensional critical surface; this is equivalent to an answer of yes. We would note that the authors in Refs. [12–23], using Wilsonian [24–28] field-space exact renormalization group methods, obtain results which support Weinberg's UV fixed-point. The results in Ref. [29] also give support to Weinberg's asymptotic safety suggestion.

In what follows, the YFS [30–45] version (YFS-type soft resummation and its extension to quantum gravity were also worked-out by Weinberg in Ref. [46]) of the exact example is extended to resum the Feynman series for the Einstein–Hilbert Lagrangian for quantum gravity. In conformity with the example in Equation (1), the resultant resummed theory, resummed quantum gravity (RQG), is very much better behaved in the UV compared to what one would estimate from that Feynman series. What we present here is a short review of the resummed theory and its predictions.

Specifically, as we show in Refs. [47–50] the RQG realization of quantum gravity leads to Weinberg's UV fixed-point behavior for the dimensionless gravitational and cosmological constants. The resummed theory is actually UV finite—the non-perturbative resummation in RQG changes the naive disperison relation for particle propagation in the deep UV so that the theory becomes UV finite. RQG and the latter UV fixed-point results are reviewed in Section 2.

The RQG theory, taken together with the Planck scale inflationary [51–53] cosmology formulation in Refs. [54,55] (the authors in Ref. [56] also proposed the attendant choice of the scale $k \sim 1/t$ used in Refs. [54,55]) from the asymptotic safety approach to quantum gravity in Refs. [12–23], allows us to predict [57] the cosmological constant Λ. See Refs. [58,59] for recent reviews of the status of the prediction. The prediction's closeness to the observed value [60,61] motivates us to discuss its reliability and we argue [62] that its uncertainty is at the level of a factor of $\mathcal{O}(10)$. Constraints on susy GUT's follow [58,59]. We present the Planck scale cosmology that we use and the latter results in Section 3.

Section 4 gives an outlook.

2. Overview of Resummed Quantum Gravity

As the Standard Theory (we follow D.J. Gross [63] and call the Standard Model the Standard Theory henceforth) of elementary particles contains many point particles, to investigate their graviton interactions, we consider (we treat spin as an inessential complication [64]) the Higgs-gravition extension of the Einstein–Hilbert theory, already studied in Refs. [65,66]:

$$\begin{aligned}\mathcal{L}(x) &= -\frac{1}{2\kappa^2} R\sqrt{-g} + \frac{1}{2}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m_o^2\varphi^2\right)\sqrt{-g} \\ &= \quad \frac{1}{2}\left\{h^{\mu\nu,\lambda}\bar{h}_{\mu\nu,\lambda} - 2\eta^{\mu\mu'}\eta^{\lambda\lambda'}\bar{h}_{\mu_\lambda,\lambda'}\eta^{\sigma\sigma'}\bar{h}_{\mu'\sigma,\sigma'}\right\} \\ &\quad + \frac{1}{2}\left\{\varphi_{,\mu}\varphi^{,\mu} - m_o^2\varphi^2\right\} - \kappa h^{\mu\nu}\left[\overline{\varphi_{,\mu}\varphi_{,\nu}} + \frac{1}{2}m_o^2\varphi^2\eta_{\mu\nu}\right] \\ &\quad - \kappa^2\left[\frac{1}{2}h_{\lambda\rho}\bar{h}^{\rho\lambda}\left(\varphi_{,\mu}\varphi^{,\mu} - m_o^2\varphi^2\right) - 2\eta_{\rho\rho'}h^{\mu\rho}\bar{h}^{\rho'\nu}\varphi_{,\mu}\varphi_{,\nu}\right] + \cdots\,.\end{aligned} \tag{3}$$

R is the curvature scalar, g is the determinant of the metric of space-time $g_{\mu\nu} \equiv \eta_{\mu\nu} + 2\kappa h_{\mu\nu}(x)$, and $\kappa = \sqrt{8\pi G_N}$. We expand [65,66] about Minkowski space with $\eta_{\mu\nu} = \mathrm{diag}\{1,-1,-1,-1\}$. $\varphi(x)$, our representative scalar field for matter, is the physical Higgs field and $\varphi(x)_{,\mu} \equiv \partial_\mu\varphi(x)$. We have introduced Feynman's notation $\bar{y}_{\mu\nu} \equiv \frac{1}{2}\left(y_{\mu\nu} + y_{\nu\mu} - \eta_{\mu\nu}y_\rho{}^\rho\right)$ for any tensor $y_{\mu\nu}$ (our conventions for raising and lowering indices in the second line of (3) are the same as those in Ref. [66]). In (3) and in what follows, $m_o(m)$ is the bare (renormalized) scalar boson mass. We set presently the small observed [60,61] value of the cosmological constant to zero so that our quantum graviton, $h_{\mu\nu}$, has zero rest mass in (3). The Feynman rules for (3) were essentially worked out by Feynman [65,66], including the rule for the famous Feynman–Faddeev–Popov [65,67,68] ghost contribution required for unitarity with the fixing of the gauge (we use the gauge in Ref. [65], $\partial^\mu\bar{h}_{\nu\mu} = 0$).

As we have shown in Refs. [47–49], the large virtual IR effects in the respective loop integrals for the scalar propagator in quantum general relativity can be resummed to the exact result:

$$\begin{aligned} i\Delta'_F(k) &= \frac{i}{k^2 - m^2 - \Sigma_s(k) + i\epsilon} \\ &= \frac{ie^{B''_g(k)}}{k^2 - m^2 - \Sigma'_s + i\epsilon} \end{aligned} \tag{4}$$

for $(\Delta = k^2 - m^2)$ where:

$$B''_g(k) = \frac{\kappa^2|k^2|}{8\pi^2} \ln\left(\frac{m^2}{m^2 + |k^2|}\right).$$

The form for $B''_g(k)$ holds for the UV (deep Euclidean) regime (by Wick rotation, the identification $-|k^2| \equiv k^2$ in the deep Euclidean regime gives immediate analytic continuation to the result for $B''_g(k)$ when the usual $-i\epsilon$, $\epsilon \downarrow 0$, is appended to m^2), so that $\Delta'_F(k)|_{\text{resummed}}$ falls faster than any power of $|k^2|$. See Ref. [47] for the analogous result for $m = 0$. Here, $-i\Sigma_s(k)$ is the 1PI scalar self-energy function so that $i\Delta'_F(k)$ is the exact scalar propagator. The residual Σ'_s starts in $\mathcal{O}(\kappa^2)$. We may drop it in calculating one-loop effects. When the respective analogs of $i\Delta'_F(k)|_{\text{resummed}}$ (these follow from the spin independence [46,47,69] of a particle's coupling to the graviton in the infrared regime) are used for the elementary particles, all quantum gravity loops are UV finite [47–49].

Specifically, we extend our resummed propagator results to all the particles in the ST Lagrangian and to the graviton itself and show in the Refs. [47–49] that (we use G_N for $G_N(0)$):

$$G_N(k) = G_N/(1 + \frac{c_{2,eff}k^2}{360\pi M_{Pl}^2}), \;\; g_* = \lim_{k^2\to\infty} k^2 G_N(k^2) = \frac{360\pi}{c_{2,eff}} \cong 0.0442. \tag{5}$$

In arriving at Equation (5), we used the result from Refs. [47–49] that the denominator for the propagation of transverse-traceless modes of the graviton becomes (M_{Pl} is the Planck mass):

$$q^2 + \Sigma^T(q^2) + i\epsilon \cong q^2 - q^4 \frac{c_{2,eff}}{360\pi M_{Pl}^2}, \tag{6}$$

where $c_{2,eff} \cong 2.56 \times 10^4$ is defined in Refs. [47–49].

For the dimensionless cosmological constant λ_* we use the VEV of Einstein's equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa^2 T_{\mu\nu}$, in a standard notation, to isolate [57] Λ. In this way, we find the deep UV limit of Λ then becomes, allowing $G_N(k)$ to run,

$$\Lambda(k) \underset{k^2\to\infty}{\longrightarrow} k^2\lambda_*, \;\; \lambda_* = -\frac{c_{2,eff}}{2880} \sum_j (-1)^{F_j} n_j/\rho_j^2 \cong 0.0817 \tag{7}$$

where F_j is the fermion number of particle j, n_j is the effective number of degrees of freedom of j, and $\rho_j = \rho(\lambda_c(m_j))$. λ_* vanishes in an exactly supersymmetric theory. Here, we have used the results that a scalar makes the contribution to Λ given by (we note the use here in the integrand of $2k_0^2$ rather than the $2(\vec{k}^2 + m^2)$ in Ref. [50], to be consistent with $\omega = -1$ [70] for the vacuum stress-energy tensor) $\Lambda_s \cong -8\pi G_N[\frac{1}{G_N^2 64\rho^2}]$ and that a Dirac fermion contributes -4 times Λ_s to Λ, where $\rho = \ln \frac{2}{\lambda_c}$ with $\lambda_c(j) = \frac{2m_j^2}{\pi M_{Pl}^2}$ for particle j with mass m_j.

We note that the UV fixed-point calculated here, $(g_*, \lambda_*) \cong (0.0442, 0.0817)$, and the estimate $(g_*, \lambda_*) \approx (0.27, 0.36)$ in Refs. [54,55] are similar in that in both of them g_* and λ_* are positive and are less than 1 in size. Further discussion of the relationship between the two fixed-point predictions can be found in Ref. [47].

3. Review of Planck Scale Cosmology and an Estimate of Λ

The authors in Refs. [54,55], using the exact renormalization group for the Wilsonian [24–28] coarse grained effective average action in field space in the Einstein–Hilbert theory, as discussed in Section 1, have argued that the dimensionless Newton and cosmological constants approach UV fixed points as the attendant scale k goes to infinity in the deep Euclidean regime. This is also in agreement with what we have found in RQG. The contact with cosmology one may facilitate via a connection between the momentum scale k, characterizing the coarseness of the Wilsonian graininess of the average effective action and the cosmological time t. The authors in Refs. [54,55], using this latter connection, arrive at the following extension of the standard cosmological equations:

$$
\begin{aligned}
(\frac{\dot{a}}{a})^2 + \frac{K}{a^2} &= \frac{1}{3}\Lambda + \frac{8\pi}{3}G_N\rho, \\
\dot{\rho} + 3(1+\omega)\frac{\dot{a}}{a}\rho = 0,\ & \dot{\Lambda} + 8\pi\rho\dot{G_N} = 0, \\
G_N(t) = G_N(k(t)),\ & \Lambda(t) = \Lambda(k(t)).
\end{aligned} \tag{8}
$$

Here, ρ is the density and $a(t)$ is the scale factor with the Robertson–Walker metric given as:

$$
ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{9}
$$

where $K = 0, 1, -1$ correspond respectively to flat, spherical, and pseudo-spherical 3-spaces for constant time t. The attendant equation of state is:

$$
p(t) = \omega\rho(t), \tag{10}
$$

where p is the pressure. The aforementioned relationship between k and the cosmological time t is:

$$
k(t) = \frac{\xi}{t} \tag{11}
$$

with the constant $\xi > 0$ determined from constraints on physical observables. Note that the physical meaning of this scale $k(t)$ is the causal limit for the respective Wilsonian field-space coarse graining as explained in Ref. [54].

Using the UV fixed points for $k^2G_N(k) \equiv g_*$ and $\Lambda(k)/k^2 \equiv \lambda_*$ obtained independently, the authors in Refs. [54,55] solve the cosmological system in Equation (8). They find, for $K = 0$, a solution in the Planck regime where $0 \le t \le t_{\rm class}$, with $t_{\rm class}$ a "few" times the Planck time t_{Pl}, which joins smoothly onto a solution in the classical regime, $t > t_{\rm class}$, which coincides with standard Friedmann–Robertson–Walker phenomenology but with the horizon, flatness, scale free Harrison–Zeldovich spectrum, and entropy problems all solved purely by Planck scale quantum physics. We now recapitulate how to use the Planck scale cosmology of Refs. [54,55] and the UV limits $\{g_*,\ \lambda_*\}$ in RQG [47–49] in Ref. [50] to predict [57] the current value of Λ.

Specifically, the transition time between the Planck regime and the classical Friedmann–Robertson–Walker (FRW) regime is determined as $t_{tr} \sim 25t_{Pl}$ in the Planck scale cosmology description of inflation in Ref. [55]. In Ref. [57], we show that, starting with the quantity $\rho_\Lambda(t_{tr}) \equiv \frac{\Lambda(t_{tr})}{8\pi G_N(t_{tr})}$, we get, following the arguments in Ref. [71] (t_{eq} is the time of matter-radiation equality),

$$
\begin{aligned}
\rho_\Lambda(t_0) &\cong \frac{-M_{Pl}^4(1 + c_{2,eff}k_{tr}^2/(360\pi M_{Pl}^2))^2}{64}\sum_j \frac{(-1)^F n_j}{\rho_j^2} \times \frac{t_{tr}^2}{t_{eq}^2} \times (\frac{t_{eq}^{2/3}}{t_0^{2/3}})^3 \\
&\cong \frac{-M_{Pl}^2(1.0362)^2(-9.194\times 10^{-3})}{64}\frac{(25)^2}{t_0^2} \cong (2.4\times 10^{-3}eV)^4.
\end{aligned} \tag{12}
$$

$t_0 \cong 13.7 \times 10^9$ yrs. is the age of the universe. The estimate in (12) is close to the experimental result [61] (the analysis in Ref. [72] also gives a value for $\rho_\Lambda(t_0)$ that is qualitatively similar to this experimental result) $\rho_\Lambda(t_0)|_{\text{expt}} \cong ((2.37 \pm 0.05) \times 10^{-3}\ \text{eV})^4$.

In Ref. [57], detailed discussions are given of the three issues of the effect of various spontaneous symmetry breaking energies on Λ, the effect of our approach to Λ on big bang nucleosynthesis (BBN) [73], and the effect of the time dependence of Λ and G_N on the covariance [74–76] of the theory. We refer the reader to the respective discussions in Ref. [57].

In Ref. [62], we have argued, regarding the issue of the error on our estimate, that the structure of the solutions of Einstein's equation [77–80], taken together with the Heisenberg uncertainty principle,

$$\Delta p \Delta q \geq \frac{1}{2}, \tag{13}$$

implies the constraint:

$$k \geq \frac{\sqrt{5}}{2w_0} \equiv \frac{\sqrt{5}}{2} \frac{1}{\sqrt{3/\Lambda(k)}} \tag{14}$$

where $\Lambda(k)$ follows from (12) (see Equation (52) in Ref. [57]). For, in a de Sitter universe, which we describe here with the metric [77,81]:

$$g_{\mu\nu} dx^\mu dx^\nu = dt^2 - e^{2t/b}\left[dw^2 + w^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

in an obvious notation, with $b = \sqrt{3/\Lambda}$, a light ray starting at the orign $w = 0$ never gets past $w = w_0 \equiv b$ if travels uninterruptedly along its geodesic. Taking $q = w\cos\theta$ where θ is the polar angle when $\vec{k} \equiv k\hat{z}$, we may identify Δp as our effective k, as k represents the size of the mean squared momentum fluctuations in the universe that are effective for the running of the universe observables $G_N(k),\ \Lambda(k)$. For the universe in the Planck regime, from the explicit solutions of the field equations in Refs. [78–80], we arrive at the estimate [62], at any given time, using an obvious notation,

$$(\Delta q)^2 \cong \frac{\int_0^{w_0} dw w^2 w^2 < \cos^2\theta >}{\int_0^{w_0} dw w^2} = \frac{1}{5} w_0^2. \tag{15}$$

From this estimate and Equation (13), we get the Einstein–Heisenberg consistency condition in Equation (14). This constraint's equality gives the estimate [57,62] of the transition time, $t_{tr} = \alpha/M_{Pl} = 1/k_{\text{tr}}$, from the Planck scale inflationary regime [54,55] to the Friedmann–Robertson–Walker regime via the implied value of α. On solving this equality for α we get $\alpha \cong 25.3$, in agreement with the value $\alpha \cong 25$ implied by the numerical studies in Refs. [54,55]. This agreement suggests an error on t_{tr} at the level of a factor $\mathcal{O}(3)$ or less and an uncertainty on Λ reduced from a factor of $\mathcal{O}(100)$ [57] to a factor of $\mathcal{O}(10)$.

One may ask what would happen to our estimate if there were a susy GUT theory at high scales. Even though the LHC has yet to see [82,83] any trace of susy, it may still appear. In Ref. [57], for definiteness and purposes of illustration, we use the susy SO(10) GUT model in Ref. [84] to illustrate how such a theory might affect our estimate of Λ. We show that either one needs a very high mass for the gravitino or one needs twice the usual particle content with the susy partners of the new quarks and leptons at masses much lower than their partners' masses. This allows for the cancellation of the respective contributions to the cosmological constant—see Ref. [57].

4. Outlook

We have presented a review and update of the current status of the resummed quantum gravity approach to the quantum theory of general relativity. It can be seen as what Prof. John Wheeler has called a radically conservative approach [85]: It is conservative because it is based on well-established exact resummation methods in relativistic quantum field theory. It is radical because it applies these methods in a completely new and different way: The respective resummation in the infrared (IR) is based on the part of the quantum amplitudes that would be IR divergent if it were on-shell but it is not actually IR divergent because it is still off-shell. This tames the UV regime in quantum general relativity. It also allows us to predict the value of the cosmological constant with a good accuracy.

There may be connections with the quantum cosmology approach in Ref. [86] where a similar dependence of ρ_Λ on t_0 and the transition time t_{tr} is found. In addition, the relation of the running scale k to the cosmological time via $k = \xi/t$ as given in Equation (11) may well be related to the phenomenologically successful scale invariant vacuum paradigm exhibited and used in Refs. [87,88], where we also note that the latter paradigm may very well be connected with the behavior of the quantum gravity in the conformal sector as presented in Refs. [89–91]. These connections and relations are ripe for further theoretical and phenomenological investigation.

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