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Temporal Moduli of Non-Differentiability for Linearized Kuramoto–Sivashinsky SPDEs and Their Gradient

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Abstract: Let $U = U(t,x)$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\partial_x U = \partial_x U(t,x)$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ be the solution and gradient solution of the fourth order linearized Kuramoto–Sivashinsky (L-KS) SPDE driven by the space-time white noise in one-to-three dimensional spaces, respectively. We use the underlying explicit kernels and symmetry analysis, yielding exact, dimension-dependent, and temporal moduli of non-differentiability for $U(\cdot,x)$ and $\partial_x U(\cdot,x)$. It has been confirmed that almost all sample paths of $U(\cdot,x)$ and $\partial_x U(\cdot,x)$, in time, are nowhere differentiable.

Keywords: L-KS SPDEs; space-time white noise; temporal modulus of non-differentiability; Hölder regularity

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1. Introduction

We are concerned with delicate regularity properties of paths of the fourth order linearized Kuramoto–Sivashinsky (L-KS) SPDE driven by the space-time white noise in one-to-three dimensional spaces. The fundamental kernels related to the deterministic versions of this class are built on the Brownian-time process (BTP) in [1–3] and extensions thereof. In this article, we provide exact, dimension-dependent, and temporal moduli of non-differentiability for the important class of stochastic equations:

$$\begin{cases} \dfrac{\partial U}{\partial t} = -\dfrac{\varepsilon}{8}(\Delta + 2\vartheta)^2 U + \dfrac{\partial^{d+1} W}{\partial t \partial x}, & (t,x) \in \mathring{\mathbb{R}}_+ \times \mathbb{R}^d; \\ U(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where Δ is the d-dimensional Laplacian operator, $\mathring{\mathbb{R}}_+ = (0,\infty)$, $(\varepsilon,\vartheta) \in \mathring{\mathbb{R}}_+ \times \mathbb{R}$ is a pair of parameters, the noise term $\partial^{d+1}W/\partial t\partial x$ is the space-time white noise corresponding to the real-valued Brownian sheet W on $\mathring{\mathbb{R}}_+ \times \mathbb{R}^d$, $d = 1,2,3$, and the initial data u_0 is Borel measurable, deterministic, and twice continuously differentiable on $\mathbb{R}^d$.

This class of SPDEs is connected to the model of pattern formation phenomena accompanying the appearance of turbulence (see [1,4–8]) and was introduced by Allouba in a series of articles [1–6,9,10]. It includes stochastic versions of prominent nonlinear equations such as the Swift–Hohenberg PDE and variants of the L-KS PDE, as well as many new ones (see [4]). Among other things, [4,5] investigated classical examples of deterministic and stochastic pattern formation PDEs and [1,4–6] investigated the L-KS class and its connection to many classical and new examples of deterministic and stochastic pattern formation PDEs. The authors of [4,5,9,10] investigated the existence/uniqueness of sharp dimension-dependent Hölder regularity of the linear and nonlinear noise version of Equation (1). The authors of [6] investigated the exact, dimension-dependent, spatio-temporal, and uniform and local moduli of continuity for the L-KS SPDE in the time variable t and space variable x. The authors of [11] investigated the solutions to Equation (1) in time and possessing an infinite quadratic variation. Temporal asymptotic distributions for the realized power variations of the L-KS SPDEs Equation (1) were investigated in [11]. These results naturally cause the following motivating questions:

- Are the solutions to L-KS SPDE Equation (1) temporal continuously differentiable?
- What are the temporal moduli of continuity for L-KS SPDEs?
- What are the temporal moduli of non-differentiability for L-KS SPDEs?

The authors of [6] investigated the exact moduli of continuity for the fourth order L-KS SPDEs and their gradient. These results provided the answers to temporal continuity and exact moduli of continuity of the solutions to Equation (1) and provided partial answers to above questions. In this article, we investigate temporal differentiability of the solutions to Equation (1). We are concerned with the exact moduli of non-differentiability of the process U and its gradient in time. It builds on and complements works in [6] and together answers all of the above questions.

Here, we would like to mention Chung's law of the iterated logarithm (LIL) for $U(\cdot, x)$ provided by Allouba and Xiao [6]. Fix $x \in \mathbb{R}^d$. For $t, h \in \mathring{\mathbb{R}}_+$ and compact rectangle $I_{\text{time}} \subset \mathring{\mathbb{R}}_+$, we consider $V(t,h) = \sup_{s \in I_{\text{time}}:|s| \le 1} |U(t+hs, x) - U(t,x)|$ and $\mathcal{V}(t,h) = \sup_{s \in I_{\text{time}}:|s| \le 1} |\partial_x U(t+hs, x) - \partial_x U(t,x)|$. In [6], the following exact temporal Chung's LIL for L-KS SPDE $U(t,x)$ and the gradient process $\partial_x U(t,x)$ are obtained. (See Theorem 1 and 2).

Theorem 1 (Reference [7])**.** *Let $x \in \mathbb{R}^d$ be fixed and assume that $\vartheta = 0$ and $u_0 = 0$ in Equation (1).*

(a) Suppose $d \in \{1,2,3\}$. For any $t_0 \in \mathring{\mathbb{R}}_+$, we have the following:

$$\liminf_{h \to 0+} \rho(h) V(t_0, h) = c_{1,1} \quad a.s., \tag{2}$$

where $c_{1,1}$ is a positive finite d-dependent constant.

$$\rho(h) = (h|\log(|\log(h)|)|^{-1})^{-\frac{4-d}{8}}. \tag{3}$$

(b) Suppose $d = 1$. For any $t_0 \in \mathring{\mathbb{R}}_+$, we have the following:

$$\liminf_{h \to 0+} \nu(h) \mathcal{V}(t_0, h) = c_{1,2} \quad a.s., \tag{4}$$

where $c_{1,2}$ is a positive finite constant.

$$\nu(h) = (h|\log(|\log(h)|)|^{-1})^{-\frac{1}{8}}. \tag{5}$$

On the other hand, elementary calculations show that the sample paths of $U(\cdot, x)$ are both almost surely continuous and almost surely nowhere differentiable (see [6]). It is therefore natural to investigate, respectively, the modulus of continuity and the modulus of non-differentiability (in the sense of Csörgő-Révész, see [12]). This article is devoted to establishing the following exact temporal moduli of non-differentiability for L-KS SPDE $U(t,x)$ and the gradient process $\partial_x U(t,x)$.

Theorem 2. *(Temporal moduli of non-differentiability) Let $x \in \mathbb{R}^d$ be fixed and assume that $\vartheta = 0$ and $u_0 = 0$ in Equation (1).*

(a) Suppose $d \in \{1,2,3\}$. For any compact interval $I_{\text{time}} \subset \mathring{\mathbb{R}}_+$, we have the following:

$$\liminf_{h \to 0+} \beta(h) \inf_{t \in I_{\text{time}}} V(t,h) = c_{3,1}^{\frac{4-d}{8}} \quad a.s., \tag{6}$$

where $c_{3,1}$ is given in (19).

$$\beta(h) = (h|\log(h)|^{-1})^{-\frac{4-d}{8}}. \tag{7}$$

Consequently, the sample paths of $U(t,x)$ are almost surely nowhere differentiable in t.

(b) Suppose $d = 1$. For any compact interval $I_{\text{time}} \subset \mathring{\mathbb{R}}_+$, we have the following:

$$\liminf_{h\to 0+} \varrho(h) \inf_{t\in I_{\text{time}}} \mathcal{V}(t,h) = c_{3,2}^{\frac{1}{8}} \quad a.s., \tag{8}$$

where $c_{3,2}$ is given in (20).

$$\varrho(h) = (h|\log(h)|^{-1})^{-\frac{1}{8}}. \tag{9}$$

Consequently, the sample paths of $\partial_x U(t,x)$ are almost surely nowhere differentiable in t.

Remark 1. *For the above theorem, we have the following remarks:*

- *It is interesting to compare (2) and (6). The latter one states that the non-differentiability modulus of $U(\cdot,x)$ for any fixed x is not more than $\beta(h)$. On the other hand, the former tells us that at some given point the non-differentiability modulus of $U(\cdot,x)$ can be much smaller, namely $\rho(h)$. Similarly, by (4) and (8), the non-differentiability modulus of $\partial_x U(\cdot,x)$ for any fixed x is not more than $\varrho(h)$. On the other hand, at some given point, the non-differentiability modulus of $\partial_x U(\cdot,x)$ can be much smaller, namely $\nu(h)$.*
- *Equation (6) describe the size of the minimum oscillation of the L-KS SPDE solution $U(\cdot,x)$ over the compact rectangle I_{time} is $\beta(h)$ (up to a constant factor). Equation (8) describes the size of the minimum oscillation of the gradient of L-KS SPDE solution $\partial_x U(\cdot,x)$ over the compact rectangle I_{time} is $\varrho(h)$ (up to a constant factor).*
- *The constants $c_{1,1}$ in (2) and $c_{1,2}$ in (4) are equal to $c_{3,1}^{\frac{4-d}{8}}$ and $c_{3,2}^{\frac{1}{8}}$, respectively, by virtue of the existence of the small ball constants, one can calculate Chung's limit inferior LIL explicitly by making use of the standard Borel–Cantelli argument.*
- *Equation (6) implies that almost all sample paths $U(\cdot,x)$ are nowhere differentiable. Moreover, it quantifies precisely the roughness of the sample paths of $U(\cdot,x)$ by $\beta(h)$. For this reason, the function $\beta(h)$ is referred to as a modulus of non-differentiability of the L-KS SPDE solution. Similarly, (8) implies that almost all sample paths $\partial_x U(\cdot,x)$ are nowhere differentiable. The modulus of non-differentiability of the gradient of the L-KS SPDE solution is $\varrho(h)$.*

Throughout this article, an unspecified positive and finite constant will be denoted by c, which may not be the same in each occurrence. More specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$. Since we shall deal with index n which ultimately tends to infinity, our statements, sometimes without further mention, are valid only when n is sufficiently large.

The rest of this article is organized as follows. In Section 2, the rigorous L-KS SPDE kernel SIE (mild) formulation, temporal spectral density and bifractional Brownian motion (BFBM) link for L-KS SPDEs, and their gradient are discussed by using the L-KS SPDE kernel SIE formulation and symmetry analysis. In Section 3, we investigate the exact temporal small ball probability estimates and the exact temporal moduli of non-differentiability for L-KS SPDEs and their gradient by making use of the Gaussian correlation inequality [13] and the theory on limsup random fractals [14]. In Section 4, the results are summarized and discussed.

2. Methodology

2.1. Rigorous Kernel Stochastic Integral Equations Formulations

We use the L-KS kernel introduced in [1,4,5] to define their rigorous mild SIE formulation. The nonlinear drift diffusion L-KS SPDE is as follows.

$$\begin{cases} \dfrac{\partial U}{\partial t} = -\dfrac{\varepsilon}{8}(\Delta + 2\vartheta)^2 U + a(U) + \sigma(U)\dfrac{\partial^{d+1} W}{\partial t \partial x}, & (t,x) \in \mathring{\mathbb{R}}_+ \times \mathbb{R}^d; \\ U(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{10}$$

Then, the rigorous L-KS kernel SIE (mild) formulation is the following SIE:

$$\begin{aligned} U(t,x) &= \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}} u_0(y)dy \\ &+ \int_{\mathbb{R}^d}\int_0^t \mathbb{K}_{t-s;x,y}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}} [a(U(s,y))dsdy + \sigma(U(s,y))W(ds\times dy)] \end{aligned} \tag{11}$$

(see p. 530 in [7] and Definition 1.1 and Equation (1.11) in [4]). Here, $\mathbb{K}_{t;x,y}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}$ is the L-KS kernel given by the following:

$$\begin{aligned} \mathbb{K}_{t;x,y}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}} &= \int_{-\infty}^0 \frac{e^{\mathbf{i}\vartheta s}e^{-|x-y|^2/(2\mathbf{i}s)}}{(2\pi\mathbf{i}s)^{d/2}} K^{\mathrm{BM}}_{\varepsilon t;s}ds + \int_0^\infty \frac{e^{\mathbf{i}\vartheta s}e^{-|x-y|^2/(2\mathbf{i}s)}}{(2\pi\mathbf{i}s)^{d/2}} K^{\mathrm{BM}}_{\varepsilon t;s}ds \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2} e^{\mathbf{i}\langle\xi,x-y\rangle}d\xi \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2}\cos(\langle\xi,x-y\rangle)d\xi, \quad (\varepsilon,\vartheta)\in\mathring{\mathbb{R}}_+\times\mathbb{R}, \end{aligned} \tag{12}$$

where $\mathbf{i}=\sqrt{-1}$ and $K^{\mathrm{BM}}_{t;s} = \frac{e^{-s^2/(2t)}}{\sqrt{2\pi t}}$. Obviously, the mild formulation of (1) is then obtained by setting $\sigma\equiv 1$ and $a\equiv 0$ in (11).

2.2. Temporal Spectral Density for L-KS SPDEs and Their Gradient

Our results are crucially dependent on the following temporal spectral density for L-KS SPDEs, which is Lemma 2.1 in [6].

Lemma 1. *Let $\mathbb{K}_{t;x}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}}$ be the (ε,ϑ) L-KS kernel. The spatial Fourier transform of the (ε,ϑ) L-KS kernel in (12) is provided by the following.*

$$\hat{\mathbb{K}}_{t;\xi}^{\mathrm{LKS}^d_{\varepsilon,\vartheta}} = (2\pi)^{-d/2}e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2}; \quad (\varepsilon,\vartheta)\in\mathring{\mathbb{R}}_+\times\mathbb{R}. \tag{13}$$

Here, the following symmetric form of the spatial Fourier transform has been used.

$$\hat{f}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(u)e^{-\mathbf{i}\xi\cdot u}du.$$

2.3. Bifractional Brownian Motion Link for L-KS SPDEs and Their Gradient

We consider the temporal probability law for L-KS SPDEs and their gradient in one-to-three dimensions. Recall that the BFBM $\{B^{H,K}(t), t\in[0,T]\}$ with indices $H\in(0,1)$ and $K\in(0,1]$ and introduced by Houdré and Villa in [15] is a centered Gaussian process with covariance.

$$R^{H,K}(t,s) := \frac{1}{2^K}\left(\left(t^{2H}+s^{2H}\right)^K - |t-s|^{2HK}\right), s,t\in[0,T]. \tag{14}$$

Lemma 2. *Let $x\in\mathbb{R}^d$ be fixed and assume that $\vartheta=0$ and $u_0=0$ in Equation (1).*

(a) Suppose $d\in\{1,2,3\}$. Then $U(\cdot,x)\overset{\mathcal{L}}{=} c_0B^{\frac{1}{2},\frac{4-d}{4}}$, where we have the following.

$$c_0 = (2\pi)^{-d}\frac{2\pi^{(d+1)/2}}{\varepsilon\Gamma(d/2)}8^{\frac{d-4}{4}}.$$

(b) Suppose $d=1$. Then $\partial_x U(\cdot,x)\overset{\mathcal{L}}{=} \mathcal{C}_0B^{\frac{1}{2},\frac{1}{4}}$, where we have the following.

$$\mathcal{C}_0 = \frac{1}{\varepsilon\Gamma(1/2)}8^{-\frac{1}{4}}.$$

Proof. In order to show (a), we use Parseval's identity to obtain the covariance function of U.

$$\begin{aligned}\mathbb{E}[U(t,x)U(s,x)] &= \int_{\mathbb{R}^d}\int_0^s \mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,y}\,\overline{\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,y}}dr\,dy \\ &= \int_0^s\int_{\mathbb{R}^d} \hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi}\,\overline{\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,\xi}}d\xi\,dr \\ &= (2\pi)^{-d}\int_0^s\int_{\mathbb{R}^d} e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4-\frac{\varepsilon(s-r)}{8}|\xi|^4}d\xi\,dr \\ &= (2\pi)^{-d}\int_0^s\int_{\mathbb{R}^d} e^{-\frac{\varepsilon(t+s-2r)}{8}|\xi|^4}d\xi dr.\end{aligned} \tag{15}$$

Thus, by using the following integral formula (see Corollary on page 23 in [16]), we have the following:

$$\int_{\mathbb{R}^d} f\Big(\sum_{i=1}^d u_i^2\Big)du_1\cdots du_d = \frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty y^{d/2-1}f(y)dy, \tag{16}$$

and (15) becomes the following.

$$\begin{aligned}\mathbb{E}[U(t,x)U(s,x)] &= (2\pi)^{-d}\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty y^{d/2-1}\int_0^s e^{-\frac{\varepsilon(t+s-2r)}{8}y^2}dr\,dy \\ &= \Big[(2\pi)^{-d}\frac{4\pi^{d/2}}{\varepsilon\Gamma(d/2)}8^{\frac{d-4}{4}}\int_0^\infty e^{-y^2}dy\Big][(t+s)^{1-\frac{d}{4}}-(t-s)^{1-\frac{d}{4}}].\end{aligned} \tag{17}$$

This yields (a). Similarly to (15), one has the following.

$$\begin{aligned}\mathbb{E}[\partial_x U(t,x)\partial_x U(s,x)] &= \int_{\mathbb{R}}\int_0^s \partial_x\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,y}\,\overline{\partial_x\mathbb{K}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,y}}dr\,dy \\ &= \int_0^s\int_{\mathbb{R}} \xi^2\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{t-r;x,\xi}\,\overline{\hat{\mathbb{K}}^{\mathrm{LKS}^d_{\varepsilon,0}}_{s-r;x,\xi}}d\xi\,dr \\ &= (2\pi)^{-1}\int_0^s\int_{\mathbb{R}} \xi^2 e^{-\frac{\varepsilon(t-r)}{8}|\xi|^4-\frac{\varepsilon(s-r)}{8}|\xi|^4}d\xi\,dr \\ &= (2\pi)^{-1}\int_0^s\int_{\mathbb{R}} \xi^2 e^{-\frac{\varepsilon(t+s-2r)}{8}|\xi|^4}d\xi dr.\end{aligned} \tag{18}$$

Thus, by using (16), (18) becomes the following.

$$\mathbb{E}[U(t,x)U(s,x)] = \Big[(2\pi)^{-1}\frac{4\pi^{1/2}}{\varepsilon\Gamma(1/2)}8^{\frac{-1}{4}}\int_0^\infty e^{-y^2}dy\Big][(t+s)^{\frac{1}{4}}-(t-s)^{\frac{1}{4}}].$$

This yields (b). □

3. Results

3.1. Extremes for L-KS SPDEs and Their Gradient

Our results are dependent on the following exact temporal small ball probability estimates for L-KS SPDEs and their gradient.

Lemma 3. *Let $x \in \mathbb{R}^d$ be fixed and assume that $\vartheta = 0$ and $u_0 = 0$ in Equation (1).*

(a) Suppose $d \in \{1,2,3\}$. Then there exists a positive and finite constant $c_{3,1}$ such that for all $t_0 \in [0,1]$ and $r > 0$, whenever $u \to 0$, we have the following.

$$\mathbb{P}(V(t_0,r) \le u) \sim \exp\Big(-\frac{c_{3,1}r}{u^{\frac{8}{4-d}}}\Big). \tag{19}$$

(b) Suppose $d = 1$. Then there exists a positive and finite constant $c_{3,2}$ such that for all $t_0 \in [0,1]$ and $r > 0$, whenever $u \to 0$, we have the following.

$$\mathbb{P}(\mathcal{V}(t_0,r) \le u) \sim \exp\Big(-\frac{c_{3,2}r}{u^8}\Big). \tag{20}$$

Proof. It follows from Lemma 2 (a) that, up to a constant, the L-KS SPDE solution $\{U(t,x), t \in \mathbb{R}_+\}$ ($x \in \mathbb{R}^d$ fixed) is a BFBM with indices $H = \frac{1}{2}$ and $K = 1 - \frac{d}{4}$. It follows from Lemma 2 (b) that, up to a constant, the gradient of L-KS SPDE solution $\{\partial_x U(t,x), t \in \mathbb{R}_+\}$ ($x \in \mathbb{R}$ fixed) is a BFBM with indices $H = \frac{1}{2}$ and $K = \frac{1}{4}$. Then, by Proposition 2.1 in [17], one has (19) and (20) hold. This completes the proof. □

3.2. Temporal Moduli of Non-Differentiability for L-KS SPDEs and Their Gradient

We require the following lemma, which is Theorem 1.1 in [13].

Lemma 4. *Let $\mathbf{f}' = (\mathbf{f}_1', \mathbf{f}_2')$ be an $\mathbb{R}^n$-valued Gaussian random vector with mean $\mathbf{0}$, where $\mathbf{f}_1 = (f_1, ..., f_k)'$, $\mathbf{f}_2 = (f_{k+1}, ..., f_n)'$ and $1 \le k < n$. Then $\forall u > 0$ and we have the following:*

$$\mathbb{P}(\|\mathbf{f}\|_\infty \le u) \le \phi \mathbb{P}(\|\mathbf{f}_1\|_\infty \le u)\mathbb{P}(\|\mathbf{f}_2\|_\infty \le u), \tag{21}$$

where $\|\mathbf{f}\|_\infty$ denotes the maximum norm of a vector $\mathbf{f}$. the following is the case.

$$\phi = \left(\frac{\det(\mathbb{E}[\mathbf{f}_1\mathbf{f}_1'])\det(\mathbb{E}[\mathbf{f}_2\mathbf{f}_2'])}{\det(\mathbb{E}[\mathbf{f}\mathbf{f}'])} \right)^{1/2}. \tag{22}$$

We also need the following lemma, which is Lemma 2.4 in [18].

Lemma 5. *Let $D = (d_{ij}, 1 \le i,j \le 2m)$ be a positive semidefinite symmetric matrix provided by $D = (D_{ij}, 1 \le i,j \le 2)$, where D_{ij} and $1 \le i,j \le 2$ are $m \times m$ matrices. Substitute$B_i = \sum_{j=m+1}^{2m} |d_{ij}|$ for $1 \le i \le m$ and $= \sum_{j=1}^{m} |d_{ij}|$ for $m+1 \le i \le 2m$. Assume the following conditions are satisfied.*

(i) There is a constant b such that for all $1 \le i \le 2m$, $B_i < b$.

(ii) There exists a finite constant $\lambda > 0$ such that for all $1 \le i \le 2m$, the following is the case:

$$\frac{\det(D^{(i)})}{\det(D)} \le \lambda,$$

where $D^{(i)}$ is the submatrix of D obtained by deleting the ith *row and* ith *column.*

Then, the following obtains.

$$\det(D) \ge e^{-2b\lambda m}\det(D_{11})\det(D_{22}). \tag{23}$$

Proof of Theorem 2. Since the proof of (8) is similar to (6), we prove (6) only. To show (6), it suffices to show the following two inequalities $\forall 0 < \epsilon < 1$ is the case:

$$\liminf_{h \to 0+} \beta(h) \inf_{t \in I_{\text{time}}} V(t,h) \ge ((1-\epsilon)c_{3,1})^{\frac{4-d}{8}} \quad \text{a.s.} \tag{24}$$

and subsequently, we have the following.

$$\liminf_{h \to 0+} \beta(h) \inf_{t \in I_{\text{time}}} V(t,h) \le ((1+\epsilon)c_{3,1})^{\frac{4-d}{8}} \quad \text{a.s.} \tag{25}$$

In order to show the above two inequalities, without loss of generality, we assume $I_{\text{time}} = [0,1]$. We show (24) first. For $n \in \mathbb{Z}_+$, we define $h_n = \theta^{-n}$ and $b_n = \theta^n$, where $\theta > 1$ is an arbitrary constant and will be specified latter on. For $i \in \mathbb{Z}_+$ and $n \ge 1$, we substitute the following.

$$\begin{aligned} T_n &= \{h \in (0,1) : h_{n+1} < h \le h_n\}, \\ T_{i,n} &= \{t \in I_{\text{time}} : ib_n^{-1} < t \le (i+1)b_n^{-1}\}. \end{aligned}$$

Let us have $t_{i,n} := ib_n^{-1}$ be a point in $T_{i,n}$, $i \in [0, b_n] \cap \mathbb{Z}_+$. It follows from (19) that the following is the case.

$$\begin{aligned}&\mathbb{P}\Big(\beta(h_n)\min_{i\in[0,b_n]\cap\mathbb{Z}_+}V(t_{i,n},h_n)\le((1-\epsilon)c_{3,1})^{\frac{4-d}{8}}\Big)\\&\le\sum_{i\in[0,b_n]\cap\mathbb{Z}_+}\mathbb{P}(V(t_{i,n},h_n)\le((1-\epsilon)c_{3,1})^{\frac{4-d}{8}}\beta(h_n)^{-1})\\&\le\theta^{(1-\frac{1}{1-\epsilon})n}.\end{aligned}$$

Hence, by Borel–Cantelli lemma, one has the following.

$$\liminf_{n\to\infty}\beta(h_n)\min_{i\in[0,b_n]\cap\mathbb{Z}_+}V(t_{i,n},h_n)\ge((1-\epsilon)c_{3,1})^{\frac{4-d}{8}}\quad\text{a.s.}\tag{26}$$

It follows from Theorem 4.1 in Meerschaert et al. [19] that the following is the case.

$$\limsup_{n\to\infty}\beta(h_n)\sup_{t\in[0,2]}V(t,h_n)=0\quad\text{a.s.}\tag{27}$$

Observe that for all $h\in(0,1)$, there exists a set T_n such that $h\in T_n$ and for all $x\in I$, there exists a set $T_{i,n}$ such that $x\in T_{i,n}$. One has the following.

$$\begin{aligned}&\liminf_{h\to0+}\beta(h)\inf_{t\in[0,1]}V(t,h)\\&\ge\liminf_{n\to\infty}\inf_{h\in T_n}\min_{i\in[0,b_n]\cap Z_+}\inf_{t\in T_{i,n}}\beta(h)V(t,h)\\&\ge\liminf_{n\to\infty}\min_{i\in[0,b_n]\cap Z_+}\inf_{t\in T_{i,n}}(\beta(h_{n+1})/\beta(h_n))\beta(h_n)V(t,h_n)\\&\ge\liminf_{n\to\infty}\min_{i\in[0,b_n]\cap Z_+}(\beta(h_{n+1})/\beta(h_n))\beta(h_n)V(t_{i,n},h_n)\\&\quad-\limsup_{n\to\infty}\max_{i\in[0,b_n]\cap Z_+}\sup_{t\in T_{i,n}}(\beta(h_{n+1})/\beta(h_n))\beta(h_n)|V(t,h_n)-V(t_{i,n},h_n)|\\&\ge\liminf_{n\to\infty}\min_{i\in[0,b_n]\cap Z_+}\beta(h_n)V(t_{i,n},h_n)\\&\quad-2\limsup_{n\to\infty}\sup_{t\in[0,2]}\beta(h_n)V(t,h_n).\end{aligned}\tag{28}$$

It follows from (26)–(28) that (24) holds.

Next we show (25). For every $n\ge1$, we define $h_n=2^{-n}$, $\Theta=\Theta_n=|\log(h_n)|^{-1}$, $A_n=\{1,...,\nu_n\}$, and $B_n=\{1,...,\lfloor\Theta^{-1}\rfloor\}$, where $\lfloor a\rfloor$ denotes the integer part of $a\in\mathbb{R}_+$ satisfying $\lfloor a\rfloor\le a<\lfloor a\rfloor+1$. For every $i\in A_n$, we define the following:

$$t_{i,n}=ih_n\Theta,$$

and define a Bernoulli random variable $Y_{i,n}$ which takes the value 1 or 0 according to the following:

$$\max_{k\in B_n}\beta(h_n)|U(t_{i,n}+kh_n\Theta,x)-U(t_{i,n},x)|\le((1+\epsilon)c_{3,1})^{\frac{4-d}{8}}$$

and whether it is the case or not. For every $n\ge1$, we define $S_n:=\sum_{i\in A_n}Y_{i,n}$ and $p_n:=\mathbb{E}[Y_{i,n}]$. Then, by (19), one has uniformly over $i\in A_n$.

$$\begin{aligned}p_n&\ge\mathbb{P}\Big(\beta(h_n)V(t_{i,n},\nu)\le((1+\epsilon)c_{3,1})^{\frac{4-d}{8}}\Big)\\&\ge\exp(\tfrac{1}{1+\epsilon}\log(h_n)).\end{aligned}\tag{29}$$

We want to show that almost surely $S_n>0$ for infinitely many n's. To this end, we first estimate the following.

$$\mathrm{Var}(S_n)=\sum_{i,j\in A_n}\mathrm{Cov}(Y_{i,n},Y_{j,n})=\sum_{i=1}^{\nu_n}\mathrm{Cov}(Y_{i,n},Y_{i,n})+2\sum_{i=1}^{\nu_n}\sum_{j=i+1}^{\nu_n}\mathrm{Cov}(Y_{i,n},Y_{j,n}).\tag{30}$$

Let $\mu > 21/(4+d)$ be a constant and $\nu_n = |\log(h_n)|^{\mu+1}$, $n \geq 1$. We make the following claim: $\forall\delta > 0$, whenever $j > i \geq \nu_n$ and $j - i \geq \nu_n$, one obtains the following.

$$\mathbb{P}(Y_{i,n} = 1, Y_{j,n} = 1) \leq (1+\delta)\mathbb{P}(Y_{i,n} = 1)\mathbb{P}(Y_{j,n} = 1) \tag{31}$$

We postpone the verification of (31) and prove (25) first.

It follows from (31) that $\mathrm{Cov}(Y_{i,n}, Y_{j,n}) \leq \delta\mathbb{E}[Y_{i,n}]\mathbb{E}[Y_{j,n}]$ for all $i, j \in A_n$ that satisfy $j > i \geq \nu_n$ and $j - i \geq \nu_n$. Thus, by (30), one has the following.

$$\mathrm{Var}(S_n) \leq 2\delta\Big(\sum_{i=1}^{\nu_n} p_{i,n}\Big)^2 + \sum_{i=1}^{\nu_n}\mathrm{Cov}(Y_{i,n}, Y_{i,n}) + 2\sum_{i=1}^{\nu_n}\sum_{j=i+1}^{\nu_n}\mathrm{Cov}(Y_{i,n}, Y_{j,n}) + 2\sum_{i=\nu_n+1}^{\nu_n}\sum_{j=i+1}^{i+\nu_n}\mathrm{Cov}(Y_{i,n}, Y_{j,n}).$$

For the covariances on the right-hand side, we use the fact that $\mathrm{Cov}(Y_{i,n}, Y_{j,n}) \leq \mathbb{E}[Y_{i,n}] = p_{i,n}$ to derive the following.

$$\mathrm{Var}(S_n) \leq 2\delta\Big(\sum_{i=1}^{\nu_n} p_{i,n}\Big)^2 + 5\nu_n\sum_{i=1}^{\nu_n} p_{i,n}. \tag{32}$$

It follows from the Pale–Zygmund inequality (see p. 8 in [20] or [14]) that the following will obtain.

$$\mathbb{P}(S_n > 0) \geq \frac{(\mathbb{E}[S_n])^2}{\mathbb{E}[S_n^2]}.$$

Combining this with (32) we obtain the following:

$$\mathbb{P}(S_n = 0) \leq \frac{\mathrm{Var}(S_n)}{(\mathbb{E}[S_n])^2} \leq 2\delta + \frac{5\nu_n}{\mathbb{E}[S_n]}$$

since $\mathbb{E}[S_n] = \sum_{i=1}^{\nu_n} p_{i,n}$. Thus, by making use of (29) and the arbitrariness of δ, we see that $\mathbb{P}(S_n = 0) \to 0$ as $n \to \infty$. By Fatou's lemma, one has the following.

$$\mathbb{P}(S_n > 0 \ \text{ i.o.}) \geq \limsup_{n\to\infty}\mathbb{P}(S_n > 0) = 1.$$

This implies that the following obtains.

$$\liminf_{n\to\infty}\min_{i\in A_n}\max_{k\in B_n}\beta(h_n)|U(t_{i,n} + kh_n\Theta, x) - U(t_{i,n}, x)| \leq ((1+\epsilon)c_{3,1})^{\frac{4-d}{8}} \quad \text{a.s.} \tag{33}$$

Thus, the following is the case.

$$\liminf_{n\to\infty}\inf_{t\in[0,1]}\max_{k\in B_n}\beta(h_n)|U(t + kh_n\Theta, x) - U(t, x)| \leq ((1+\epsilon)c_{3,1})^{\frac{4-d}{8}} \quad \text{a.s.} \tag{34}$$

Note that the following obtains.

$$\begin{aligned}
&\liminf_{n\to\infty}\inf_{t\in[0,1]}\beta(h_n)V(t, h_n) \\
&\leq \liminf_{n\to\infty}\inf_{t\in[0,1]}\max_{k\in B_n}\sup_{k-1\leq u\leq k}\beta(h_n)|U(t + uh_n\Theta, x) - U(t, x)| \\
&\leq \liminf_{n\to\infty}\inf_{t\in[0,1]}\max_{k\in B_n}\beta(h_n)|U(t + kh_n\Theta, x) - U(t, x)| \\
&\quad + \limsup_{n\to\infty}\sup_{t\in[0,1]}\max_{k\in B_n}\sup_{k-1\leq u\leq k}\beta(h_n)|U(t + uh_n\Theta, x) - U(t + kh_n\Theta, x)| \\
&\leq \liminf_{n\to\infty}\inf_{t\in[0,1]}\max_{k\in B_n}\beta(h_n)|U(t + kh_n\Theta, x) - U(t, x)| \\
&\quad + \limsup_{n\to\infty}\sup_{t\in[0,2]}\sup_{|y|\leq h_n\Theta}\beta(h_n)|U(t + y, x) - U(t, x)|.
\end{aligned} \tag{35}$$

It follows from Theorem 4.1 in Meerschaert et al. [19] that the following will obtain.

$$\limsup_{n\to\infty} \sup_{t\in[0,2]} \sup_{|y|\le h_n\Theta} \beta(h_n)|U(t+y,x)-U(t,x)| = 0 \quad \text{a.s.} \tag{36}$$

Hence, by (34)–(36), (25) holds.

Therefore, it remains to show (31). We will make use of Lemma 5 (with $m = \Theta^{-1}$) to consider the determinant of $(2\Theta^{-1}) \times (2\Theta^{-1})$ matrix Σ. We first verify that the positive semidefinite matrix Σ satisfies Conditions (i)–(ii) of Lemma 5.

Consider the following points

$$u_{i,n} = i\Theta, \quad i \in B_n,$$

and the Gaussian processes $\mathcal{F}_{i,n}$ defined by the following.

$$\mathcal{F}_{i,n}(t) = U(u_{i,n}+t,x) - U(u_{i,n},x), \quad \forall 0 \le t \le 1.$$

By (17), for all $i,j \in \mathcal{F}_n$ and $s,t \in [0,1]$, one has the following:

$$\mathbb{E}[\mathcal{F}_{i,n}(s)\mathcal{F}_{j,n}(t)] = c_0(C_{i,j,n}(s,t) + D_{i,j,n}(s,t)), \tag{37}$$

where is the case.

$$\begin{aligned} C_{i,j,n}(s,t) := &-|(u_{j,n}-u_{i,n})+(t-s)|^{\frac{4-d}{4}} + |(u_{j,n}-u_{i,n})-s|^{\frac{4-d}{4}} \\ &+ |(u_{j,n}-u_{i,n})+t|^{\frac{4-d}{4}} - |u_{j,n}-u_{i,n}|^{\frac{4-d}{4}}, \end{aligned}$$

Similarly, the following also is the case.

$$\begin{aligned} D_{i,j,n}(s,t) := &[(u_{i,n}+s)+(u_{j,n}+t)]^{\frac{4-d}{4}} - [(u_{i,n}+s)+u_{j,n}]^{\frac{4-d}{4}} \\ &- [u_{i,n}+(u_{j,n}+t)]^{\frac{4-d}{4}} + (u_{i,n}+u_{j,n})^{\frac{4-d}{4}}. \end{aligned}$$

Thus, by Taylor's expansion, we derive that if $i<j$ and $s\le t$, we will have the following.

$$\begin{aligned} C_{i,j,n}(s,t) &= \frac{4-d}{4}s\{|(u_{j,n}-u_{i,n})+(t-\eta_1 s)|^{\frac{4-d}{4}-1} - |(u_{j,n}-u_{i,n})-\eta_2 s|^{\frac{4-d}{4}-1}\} \\ &= \frac{4-d}{4}(\frac{4-d}{4}-1)s(t-\eta_1 s+\eta_2 s)|(u_{j,n}-u_{i,n})+\eta_3(t-\eta_1 s+\eta_2 s)|^{\frac{4-d}{4}-2}, \end{aligned} \tag{38}$$

The following will also obtain:

$$\begin{aligned} D_{i,j,n}(s,t) = &\frac{4-d}{4}\{[(u_{i,n}+s)+u_{j,n}+\eta_4((u_{j,n}+t)-u_{j,n})]^{\frac{4-d}{4}-1} \\ &\times [(u_{j,n}+t)-u_{j,n}] \\ &- [u_{i,n}+u_{j,n}+\eta_5((u_{j,n}+t)-u_{j,n})]^{\frac{4-d}{4}-1}[(u_{j,n}+t)-u_{j,n}]\} \\ = &\frac{4-d}{4}(\frac{4-d}{4}-1)[u_{i,n}+u_{j,n}+(\eta_5+\eta_6(\eta_4-\eta_5))((u_{j,n}+t)-u_{j,n}) \\ &+\eta_6((u_{i,n}+s)-u_{i,n})]^{\frac{4-d}{4}-2} \\ &\times [((u_{i,n}+s)-u_{i,n})+(\eta_4-\eta_5))((u_{j,n}+t)-u_{j,n})][(u_{j,n}+t)-u_{j,n}] \\ = &\frac{4-d}{4}(\frac{4-d}{4}-1)[u_{i,n}+u_{j,n}+(\eta_5+\eta_6(\eta_4-\eta_5))((u_{j,n}+t)-u_{j,n}) \\ &+\eta_6((u_{i,n}+s)-u_{i,n})]^{\frac{4-d}{4}-2}[s+(\eta_4-\eta_5)t], \end{aligned} \tag{39}$$

where $\eta_\ell \in [0,1]$ for all $1 \le \ell \le 9$. Thus, by noting $\frac{4-d}{4} \in (0,1)$, one has the following.

$$|D_{i,j,n}(s,t)| \le \frac{4-d}{2}(1-\frac{4-d}{4})u_{j,n}^{\frac{4-d}{4}-2}. \tag{40}$$

Consider Gaussian random vectors $\mathbf{f}_1' := (\mathcal{F}_{i,n}(\Theta), \mathcal{F}_{i,n}(2\Theta), ..., \mathcal{F}_{i,n}(1))$ and $\mathbf{f}_2' := (\mathcal{F}_{j,n}(\Theta), \mathcal{F}_{j,n}(2\Theta), ..., \mathcal{F}_{j,n}(1))$. Let $\mathbf{f}' = (\mathbf{f}_1', \mathbf{f}_2')$ and $\mathbf{\Sigma}$ the covariance matrix of $\mathbf{f}$. Then, we have the following:

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{\Sigma}_2 \\ \mathbf{\Sigma}_2' & \mathbf{\Sigma}_1 \end{pmatrix},$$

where $\mathbf{\Sigma}_1 = \mathbb{E}[\mathbf{f}_1\mathbf{f}_1'] = \mathbb{E}[\mathbf{f}_2\mathbf{f}_2']$ and $\mathbf{\Sigma}_2 = \mathbb{E}[\mathbf{f}_1\mathbf{f}_2']$. For simplicity of notation, set $\psi(h_n) = ((1+\epsilon)c_{3,1}/|\log(h_n)|)^{\frac{4-d}{8}}$. By Lemma 4, one obtains the following:

$$\begin{aligned} &\mathbb{P}\Big(\max_{1\le k\le 1/\Theta} |\mathcal{F}_{i,n}(k\Theta)| \le \psi(h_n), \max_{1\le m\le 1/\Theta} |\mathcal{F}_{j,n}(m\Theta)| \le \psi(h_n)\Big) \\ &\le \phi\mathbb{P}\Big(\max_{1\le k\le 1/\Theta} |\mathcal{F}_{i,n}(k\Theta)| \le \psi(h_n)\Big)\mathbb{P}\Big(\max_{1\le m\le 1/\Theta} |\mathcal{F}_{j,n}(m\Theta)| \le \psi(h_n)\Big), \end{aligned} \tag{41}$$

where the following is the case.

$$\phi = \Big(\frac{\det(\Sigma_1)\det(\Sigma_1)}{\det(\Sigma)}\Big)^{1/2}.$$

It follows from (38) that for all $i,j \in B_n$ with $j-i \ge \nu_n$ and $1 \le k \le m \le 1/\Theta$, we have the following.

$$\begin{aligned} &|C_{i,j,n}(k\Theta, m\Theta)| \\ &= |\frac{4-d}{4}(\frac{4-d}{4}-1)k(m-\eta_1 k+\eta_2 k)\Theta^2|\,|(j-i)\Theta+\eta_3(m-\eta_1 k+\eta_2 k)\Theta|^{\frac{4-d}{4}-2} \\ &\le c_{3,3}|\log(h_n)|^{-(\mu-2)(2-\frac{4-d}{4})}. \end{aligned} \tag{42}$$

It follows from (40) that for all $i,j \in B_n$ with $j > i \ge \nu_n$ and $1 \le k \le m \le 1/\Theta$, we have the following.

$$\begin{aligned} &|D_{i,j,n}(k\Theta, m\Theta)| \\ &\le \frac{d(4-d)}{8} j^{-(1+\frac{d}{4})}\Theta^{-(1+\frac{d}{4})} \\ &\le c_{3,4}|\log(h_n)|^{-(\mu-2)(1+\frac{d}{4})}. \end{aligned} \tag{43}$$

Thus, combining (37) with (49) and (43), one obtains the following.

$$\begin{aligned} |b_{k,m}| &:= |\mathbb{E}[\mathcal{F}_{i,n}(k\Theta)\mathcal{F}_{j,n}(m\Theta)]| \\ &\le c_{3,5}\,|\log(h_n)|^{-(\mu-2)(1+\frac{d}{4})}. \end{aligned} \tag{44}$$

Thus, by (44), we observe the following.

$$\sum_{k\in B_n}\sum_{m\in B_n} b_{k,m} \le c_{3,5}\,|\log(h_n)|^{-(\mu-2)(1+\frac{d}{4})+2}. \tag{45}$$

This verifies Condition (i) in Lemma 5 with $b = c_{3,5}\,|\log(h_n)|^{-(\mu-2)(1+\frac{d}{4})+2}$.
In order to verify Condition (ii) in Lemma 5, we make use of the following fact on the conditional variance.

$$\mathrm{Var}\Big(\mathcal{F}_{i,n}(k\Theta)|\mathcal{F}_{u,n}(m\Theta), m\ne k, m\in B_n, u\in\{i,j\}\Big) = \frac{\det(\Sigma)}{\det(\Sigma^{(l)})}. \tag{46}$$

Thus, the following obtains.

$$\frac{\det(\Sigma)}{\det(\Sigma^{(l)})} \geq \mathrm{Var}\Big(U(k\Theta, x)|U(m\Theta, x), m \neq k, m \in B_n\Big) \geq c_{3,6}\, \Theta^{\frac{4-d}{4}}. \tag{47}$$

This verifies Condition (ii) with $\lambda = c_{3,6}\, \Theta^{\frac{4-d}{4}}$.

Applying Lemma 5 with $m = \Theta^{-1}$, $b = c_{3,5}\, |\log(h_n)|^{-(\mu-2)(1+\frac{d}{4})+2}$ and $\lambda = c_{3,6}\, \Theta^{\frac{4-d}{4}}$, we obtain the following.

$$\det(\Sigma) \geq e^{-2b\lambda m}(\det(\Sigma_1))^2. \tag{48}$$

This, together with (48), yields the following.

$$\phi \leq e^{b\lambda m}. \tag{49}$$

Notice that $b\lambda m \to 0$ as $n \to \infty$. This, together with (49) and (37), yields that (31) holds. The proof of Theorem 2 is completed. □

4. Conclusions

In this article, we have presented that the L-KS SPDE solutions and their gradients are almost surely nowhere differentiable in time variable t. We have established the exact temporal small ball probability estimates and the exact, dimension dependent, and temporal moduli of non-differentiability for L-KS SPDEs and their gradient. They complement Allouba's earlier works on the spatio-temporal Hölder regularity of L-KS SPDEs and their gradient. Together with the temporal Khinchin-type law of the iterated logarithm and the uniform temporal moduli of continuity, they provide complete information on the regularity properties of L-KS SPDEs and their gradient in time.

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Abbreviations

The following abbreviations are used in this manuscript:

L-KS	linearized Kuramoto–Sivashinsky;
SPDE	stochastic partial differential equation;
SIE	stochastic integral equation;
BFBM	bifractional Brownian motion;
LIL	law of the iterated logarithm.

References

1. Allouba, H. A linearized Kuramoto-Sivashinsky PDE via an imaginary-Brownian-time-Brownian-angle process. *Comptes Rendus Math. Acad. Sci.* **2003**, *336*, 309–314. [CrossRef]
2. Allouba, H. Brownian-time processes: The PDE connection II and the corresponding Feynman-Kac formula. *Trans. Am. Math. Soc.* **2002**, *354*, 4627–4637. [CrossRef]
3. Allouba, H.; Zheng, W. Brownian-time processes: The PDE connection and the half-derivative generator. *Ann. Probab.* **2001**, *29*, 1780–1795. [CrossRef]
4. Allouba, H. L-Kuramoto-Sivashinsky SPDEs in one-to-three dimensions: L-KS kernel, sharp Hölder regularity, and Swift-Hohenberg law equivalence. *J. Differ. Equ.* **2015**, *259*, 6851–6884. [CrossRef]
5. Allouba, H. A Brownian-time excursion into fourth-order PDEs, linearized Kuramoto-Sivashinsky, and BTPSPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$. *Stoch. Dyn.* **2006**, *6*, 521–534. [CrossRef]
6. Allouba, H.; Xiao, Y. L-Kuramoto-Sivashinsky SPDEs v.s. time-fractional SPIDEs: Exact continuity and gradient moduli, 1/2-derivative criticality, and laws. *J. Differ. Equ.* **2017**, *263*, 15521610. [CrossRef]
7. Duan, J.; Wei, W. *Effective Dynamics of Stochastic Partial Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2014.
8. Temam, R. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed.; Springer: New York, NY, USA, 1997.
9. Allouba, H. Time-fractional and memoryful Δ^{2^k} SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: How far can we push white noise? *Ill. J. Math.* **2013**, *57*, 919–963.
10. Allouba, H. Brownian-time Brownian motion SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: Ultra regular direct and lattice-limits solutions and fourth order SPDEs links. *Discret. Contin. Dyn. Syst.* **2013**, *33*, 413–463. [CrossRef]
11. Wang, W.; Wang, D. Asymptotic distributions for power variations of the solutions to linearized Kuramoto-Sivashinsky SPDEs in one-to-three dimensions. *Symmetry* **2021**, *13*, 73. [CrossRef]
12. Csörgő, M.; Révész, P. How small are the increments of a Wiener process? *Stoch. Process. Appl.* **1979**, *8*, 119–129. [CrossRef]
13. Shao, Q.-M. A Gaussian correlation inequality and its applications to the existence of small ball constant. *Stoch. Process. Appl.* **2003**, *107*, 269–287. [CrossRef]
14. Khoshnevisan, D.; Peres, Y.; Xiao, Y. Limsup random fractals. *Elect. J. Probab.* **2000**, *5*, 1–24. [CrossRef]
15. Houdré, C.; Villa, J. An example of infinite dimensional quasi-helix. *Contemp. Math.* **2003**, *336*, 195–202.
16. Fang, K.T.; Kotz, S.; NG, K.W. *Symmetric Multivariate and Related Distribution*; Chapman and Hall Ltd.: London, UK, 1990.
17. Tudor, C.A.; Xiao, Y. Sample path properties of bifractional Brownian motion. *Bernoulli* **2008**, *14*, 865–898. [CrossRef]
18. Wang, W.; Xiao, Y. The Csörgő-Révész moduli of non-differentiability of fractional Brownian motion. *Stat. Probab. Lett.* **2019**, *150*, 81–87. [CrossRef]
19. Meerschaert, M.M.; Wang, W.; Xiao, Y. Fernique type inequality and moduli of continuity for anisotropic Gaussian random fields. *Trans. Am. Math. Soc.* **2013**, *365*, 1081–1107. [CrossRef] [PubMed]
20. Kahane, J.P. *Some Random Series of Functions*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1985.