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Refinements of Wilker–Huygens-Type Inequalities via Trigonometric Series

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Abstract: The study of even functions is important from the symmetry theory point of view because their graphs are symmetrical to the Oy axis; therefore, it is essential to analyse the properties of even functions for x greater than 0. Since the functions involved in Wilker–Huygens-type inequalities are even, in our approach, we use cosine polynomials expansion method in order to provide new refinements of the above-mentioned inequalities.

Keywords: trigonometric functions; hyperbolic functions; trigonometric series; analytic inequalities; even functions

MSC: 41A21; 42B05; 26D05; 26D15

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1. Introduction

The famous Huygens inequality for trigonometric functions states that for any $0 < x < \dfrac{\pi}{2}$ one has

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \tag{1}$$

while the Wilker inequality asserts that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{2}$$

In [1], S.-H. Wu and H. M. Srivastava established the following inequality, which is sometime known as the second Wilker inequality:

$$2\frac{x}{\sin x} + \frac{x}{\tan x} > 3,\ 0 < |x| < \frac{\pi}{2} \tag{3}$$

and the following inequality, which is also sometime known as the second Wilker inequality:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2,\ 0 < x < \frac{\pi}{2}. \tag{4}$$

In [2], the inequality (3) is established with another bound

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 > \frac{1}{60}x^3 \sin x, \tag{5}$$

for $|x| \in \left(0, \frac{\pi}{2}\right)$.

In [3], the inequality (4) is written with another bound

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 > \frac{2}{45}x^3 \sin x, \tag{6}$$

for $|x| \in \left(0, \frac{\pi}{2}\right)$.

In [4], E. Neumann and J. Sándor proved the following inequality

$$3\frac{x}{\sin x} + \cos x > 4 \text{ for } 0 < x < \frac{\pi}{2}. \tag{7}$$

In the paper [2], the inequality (7) is established with another bound

$$3\frac{x}{\sin x} + \cos x - 4 > \frac{1}{10}x^3 \sin x, \tag{8}$$

for $|x| \in \left(0, \frac{\pi}{2}\right)$.

In the same work, [4], E. Neumann and J. Sándor also showed the hyperbolic variants of the inequalities (3) and (4)

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 > 0, \text{ for all } x \neq 0 \tag{9}$$

and

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 > 0, \text{ for all } x \neq 0. \tag{10}$$

In the paper [3], the inequality (10) is written with another bound

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \frac{2}{45}x^3 \sinh x, \tag{11}$$

for $x > 0$.

The hyperbolic counterpart of the inequality (7) is:

$$3\frac{x}{\sinh x} + \cosh x - 4 > 0, \text{ for all } x \neq 0. \tag{12}$$

These inequalities were extended in different forms in the recent past. We refer to [1–17] and closely related references therein. Some of the recent improvements were obtained using Taylor's expansion or Padé approximation of the trigonometric functions involved.

In [6], we improved the Huygens and Wilker inequalities using the cosine polynomial method.

The aim of this work is to reformulate the inequalities (3)–(12) using again the cosine polynomial method. The main idea is that the functions involved in the above inequalities are even, so can be expanded in trigonometric series:

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 = a_1 + b_1 \cos x + c_1 \cos 2x + \ldots,$$

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 = a_2 + b_2 \cos x + c_2 \cos 2x + \ldots,$$

$$3\frac{x}{\sin x} + \cos x - 4 = a_3 + b_3 \cos x + c_3 \cos 2x + \ldots,$$

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 = a_4 + b_4 \cos x + c_4 \cos 2x + \ldots\ ,$$

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 = a_5 + b_5 \cos x + c_5 \cos 2x + \ldots,$$

$$3\frac{x}{\sinh x} + \cosh x - 4 = a_6 + b_6 \cos x + c_6 \cos 2x + \ldots\ .$$

The above functions can be also expanded as hyperbolic cosine polynomials:

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 = a_7 + b_7 \cosh x + c_7 \cosh 2x + \ldots,$$

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 = a_8 + b_8 \cosh x + c_8 \cosh 2x + \ldots,$$

$$3\frac{x}{\sin x} + \cos x - 4 = a_9 + b_9 \cosh x + c_9 \cosh 2x + \dots .$$

In the following we will present our method for the first function.

We introduce the function $F_1(x)$ by

$$F_1(x) = a_1 + b_1 \cos x + c_1 \cos 2x.$$

The power series expansion of $2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 - F_1(x)$ near 0 is

$$(-a_1 - b_1 - c_1) + \frac{1}{2}x^2(b_1 + 4c_1) + \frac{1}{120}x^4(-5b_1 - 80c_1 + 2) + O\left(x^6\right).$$

In order to increase the speed of the function $F_1(x)$ approximating $2\frac{x}{\sin x} + \frac{x}{\tan x} - 3$, we vanish the first coefficients as follows:

$$\begin{cases} -a_1 - b_1 - c_1 = 0 \\ b_1 + 4c_1 = 0 \\ -5b_1 - 80c_1 + 2 = 0 \end{cases}$$

and we obtain $a_1 = \frac{1}{10}$, $b_1 = -\frac{4}{30}$ and $c_1 = \frac{1}{30}$.

Then, we obtain

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 - \frac{1}{10} + \frac{4}{30}\cos x - \frac{1}{30}\cos 2x = \frac{1}{210}x^6 + \\ + \frac{17}{554400}x^{10} + \frac{31}{16511040}x^{12} + O(x^{14}),$$

or, equivalently,

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 - \frac{1}{15}(1 - \cos x)^2 = \frac{1}{210}x^6 + \frac{17}{554400}x^{10} + \frac{31}{16511040}x^{12} + O\left(x^{14}\right).$$

Using the same algorithm, we find

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \frac{8}{45}(1 - \cos x)^2 = \frac{1}{63}x^6 + \frac{1}{1400}x^8 + \frac{163}{831600}x^{10} + O\left(x^{12}\right),$$

$$3\frac{x}{\sin x} + \cos x - 4 - \frac{2}{5}(1 - \cos x)^2 = \frac{3}{140}x^6 - \frac{1}{1680}x^8 + \frac{19}{158400}x^{10} + O\left(x^{12}\right),$$

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 - \frac{1}{15}(1 - \cos x)^2 = \frac{1}{1260}x^6 - \frac{17}{1425600}x^{10} + \frac{31}{16511040}x^{12} + O\left(x^{14}\right),$$

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 - \frac{8}{45}(1 - \cos x)^2 = -\frac{1}{945}x^6 + \frac{1}{1400}x^8 - \frac{1093}{7484400}x^{10} + O\left(x^{12}\right),$$

$$3\frac{x}{\sinh x} + \cosh x - 4 - \frac{2}{5}(1 - \cos x)^2 = \frac{1}{84}x^6 - \frac{1}{1680}x^8 - \frac{1}{133056}x^{10} + O\left(x^{12}\right),$$

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 - \frac{1}{15}(1 - \cosh x)^2 = -\frac{1}{210}x^6 - \frac{17}{554400}x^{10} + \frac{31}{16511040}x^{12} + O\left(x^{14}\right),$$

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 - \frac{8}{45}(1 - \cosh x)^2 = -\frac{1}{63}x^6 + \frac{1}{1400}x^8 - \frac{163}{831600}x^{10} + O\left(x^{12}\right)$$

and

$$3\frac{x}{\sinh x} + \cosh x - 4 - \frac{2}{5}(1 - \cosh x)^2 = -\frac{3}{140}x^6 - \frac{1}{1680}x^8 - \frac{19}{158400}x^{10} + O\left(x^{12}\right).$$

2. Main Results

Using the Fourier trigonometric series method we can establish our main theorems, which are refined and simple forms of the inequalities (3)–(12).

Theorem 1. *(Wilker–Huygens-type inequalities)*

(i) *The following inequality*

$$2\frac{x}{\sin x} + \frac{x}{\tan x} - 3 > \frac{1}{15}(1 - \cos x)^2 \tag{13}$$

holds for all $0 < |x| < \frac{\pi}{2}$.

(ii) *The following inequality*

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 > \frac{8}{45}(1 - \cos x)^2 \tag{14}$$

holds for all $0 < |x| < \frac{\pi}{2}$.

(iii) *The following inequality*

$$3\frac{x}{\sin x} + \cos x - 4 > \frac{2}{5}(1 - \cos x)^2 \tag{15}$$

holds for all $0 < |x| < \frac{\pi}{2}$.

Theorem 2. *(Wilker–Huygens-type inequalities for hyperbolic functions)*

(i) *For all* $x \neq 0$, *one has*

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 < \frac{1}{15}(1 - \cosh x)^2. \tag{16}$$

(ii) *For all* $x \neq 0$, *one has*

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \frac{8}{45}(1 - \cosh x)^2. \tag{17}$$

(iii) *For all* $x \neq 0$, *one has*

$$3\frac{x}{\sinh x} + \cosh x - 4 < \frac{2}{5}(1 - \cosh x)^2. \tag{18}$$

Theorem 3. *(Mixed type of Wilker–Huygens inequalities)*

(i) *For all* $x \neq 0$, *one has*

$$2\frac{x}{\sinh x} + \frac{x}{\tanh x} - 3 > \frac{1}{15}(1 - \cos x)^2. \tag{19}$$

(ii) *For all* x, $0 < |x| < 1.50618$, *one has*

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \frac{8}{45}(1 - \cos x)^2. \tag{20}$$

(iii) *For all* $x \neq 0$, *one has*

$$3\frac{x}{\sinh x} + \cosh x - 4 > \frac{2}{5}(1 - \cos x)^2. \tag{21}$$

3. The Proofs of the Theorems

We first prove two lemmas.

Lemma 1. *(i)* *For every $x \geq 0$, one has*

$$2\sinh x \geq \sin 2x.$$

(ii) *For every $x \in \left(0, \frac{\pi}{2}\right)$, one has*

$$\frac{11}{2}\sin x + 45x + \frac{123}{2}x\cos x - 112\sin x\cos^2 x > 0.$$

(iii) *For every $|x| \in \left(0, \frac{\pi}{2}\right)$, one has*

$$4(1-\cos x)^2 > x^3 \sin x.$$

Proof. (i) We consider the function

$$g : [0,\infty) \to \mathbb{R}, g(x) = 2\sinh x - \sin 2x.$$

The derivative of the function g is

$$\begin{aligned} g'(x) &= 2\cosh x - 2\cos\ 2x \\ &= 2(\cosh x - 1) + 2(1-\cos 2x) \geq 0, \text{ for all } x \geq 0. \end{aligned}$$

Then g is increasing on $[0,\infty)$. As $g(0) = 0$, we find that $g \geq 0$ on $[0,\infty)$.

(ii) We define the function

$$p : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, p(x) = \frac{11}{2}\sin x + 45x + \frac{123}{2}x\cos x - 112\sin x\cos^2 x.$$

We can rearrange p as follows

$$p(x) = \frac{11}{2}\sin x\left(1-\cos^2 x\right) + \frac{123}{4}\cos x(2x - \sin 2x) + 45\left(x - \sin x\cos^2 x\right).$$

For $x \in \left(0, \frac{\pi}{2}\right)$, we have

$$\sin x\cos^2 x = \left|\sin x\cos^2 x\right| \leq |\sin x| = \sin x \leq x$$

It follows that $p > 0$ on $\left(0, \frac{\pi}{2}\right)$.

(iii) We introduce the function

$$h : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, h(x) = 4(1-\cos x)^2 - x^3 \sin x.$$

An alternate form of h is

$$h(x) = -2\sin\frac{x}{2}\left(x^3\cos\frac{x}{2} - 6\sin\frac{x}{2} + 2\sin\frac{3x}{2}\right).$$

Using the formula

$$\sin 3x = 3\sin x - 4\sin^3 x,$$

we have

$$h(x) = -2\sin\frac{x}{2}\left(x^3\cos\frac{x}{2} - 8\sin^3\frac{x}{2}\right).$$

The Adamović and Mitrinović inequality (see, e.g., ([8] p. 238)) asserts that

$$(\cos x)^{\frac{1}{3}} < \frac{\sin x}{x}$$

holds for every $\left(0, \dfrac{\pi}{2}\right)$.

Therefore, we obtain that $h(x) > 0$ for every $\left(0, \dfrac{\pi}{2}\right)$. $\square$

Lemma 2. *For every $x \neq 0$, one has*

$$4(1-\cos x)^2 < x^3 \sinh x.$$

Proof. We define the even function

$$r(x) = x^3 \sinh x - 2\cos 2x + 8\cos x - 6, x > 0.$$

We have

$$r'(x) = x^3 \cosh x - 8\sin x + 4\sin 2x + 3x^2 \sinh x,$$
$$r^{(2)}(x) = -8\cos x + 8\cos 2x + 6x^2 \cosh x + (6x + x^3)\sinh x,$$
$$r^{(3)}(x) = x(18 + x^2)\cosh x + 8(\sin x - 2\sin 2x) + (6 + 9x^2)\sinh x,$$
$$r^{(4)}(x) = 8(\cos x - 4\cos 2x) + 12(2 + x^2)\cosh x + x(36 + x^2)\sinh x,$$
$$r^{(5)}(x) = x(60 + x^2)\cosh x - 8(\sin x - 8\sin 2x) + 15(4 + x^2)\sinh x$$

and

$$r^{(6)}(x) = -8(\cos x - 16\cos 2x) + 6\left(20 + 3x^2\right)\cosh x + x\left(90 + x^2\right)\sinh x.$$

From Lemma 1, (i), we deduce that $\cosh x \geq \frac{5-\cos 2x}{4}$ for all $x \in \mathbb{R}$.
Then,

$$r^{(6)}(x) \geq -8(\cos x - 16\cos 2x) + 120 \cdot \frac{5-\cos 2x}{4} + 18x^2 \cosh x + x(90 + x^2)\sinh x,$$

or, equivalently,

$$r^{(6)}(x) > \left(14\cos x - \frac{2}{7}\right)^2 + \frac{2544}{49} + 18x^2 \cosh x + x(90 + x^2)\sinh x > 0 \text{ for all } x > 0.$$

Therefore, $r^{(5)}$ is strictly increasing on $(0, \infty)$. Since $r^{(5)}(0) = 0$, it follows that $r^{(5)}(x) > 0$ for all $x > 0$. Continuing the algorithm, we finally find that $r > 0$ on $(0, \infty)$. $\square$

Proof of Theorem 1. (i) Due to the form of the inequality (13), if the inequality (13) holds for $0 < x < \dfrac{\pi}{2}$, then it holds for $-\dfrac{\pi}{2} < x < 0$.

Therefore, we can consider $x > 0$.

The inequality (13) takes the following equivalent form:

$$30x + 15x\cos x - 45\sin x - \sin x(1-\cos x)^2 > 0,\ 0 < x < \frac{\pi}{2}.$$

We introduce the function

$$f_1 : \left(0, \frac{\pi}{2}\right) \to \mathbb{R},\ f_1(x) = 30x + 15x\cos x - 45\sin x - \sin x(1-\cos x)^2.$$

The derivative of the function f_1 is

$$\begin{aligned} f_1'(x) &= 30+15(\cos x - x\sin x) - 45\cos x - \\ &\quad -\Big(\cos x(1-\cos x)^2 + 2\sin x(1-\cos x)\sin x\Big) \\ &= 2\sin\frac{x}{2}\Big(30\sin\frac{x}{2} - 15x\cos\frac{x}{2} - \sin\frac{x}{2}\Big(\cos x - \cos^2 x + 2\sin^2 x\Big)\Big) \\ &= 2\sin\frac{x}{2}\Big(\sin\frac{x}{2}\Big(3\cos^2 x - \cos x + 28\Big) - 15x\cos\frac{x}{2}\Big). \end{aligned}$$

The function

$$f_2 : \Big(0, \frac{\pi}{2}\Big) \to \mathbb{R},\ f_2(x) = \sin\frac{x}{2}\Big(3\cos^2 x - \cos x + 28\Big) - 15x\cos\frac{x}{2}$$

has the derivative

$$\begin{aligned} f_2'(x) &= \cos\frac{x}{2}\left(\frac{3}{2}\cos^2 x - \frac{1}{2}\cos x - 1\right) + \sin\frac{x}{2}\left(-6\sin x\cos x + \sin x + \frac{15}{2}x\right) \\ &= -\cos\frac{x}{2}\sin^2\frac{x}{2}(3\cos x + 2) + \sin\frac{x}{2}\left(-6\sin x\cos x + \sin x + \frac{15}{2}x\right) \\ &= \frac{15}{2}\sin\frac{x}{2}(x - \sin x\cos x) \\ &= \frac{15}{2}\sin\frac{x}{2}(x - \sin x + \sin x(1-\cos x)). \end{aligned}$$

Since $f_2' > 0$ on $\Big(0, \frac{\pi}{2}\Big)$, it follows that f_2 is strictly increasing on $\Big(0, \frac{\pi}{2}\Big)$. As $f_2(0) = 0$ we obtain that $f_2 > 0$ on $\Big(0, \frac{\pi}{2}\Big)$.

Then, $f_1' > 0$ on $\Big(0, \frac{\pi}{2}\Big)$. Using the same arguments, we finally find that $f_1 > 0$ on $\Big(0, \frac{\pi}{2}\Big)$.

(ii) The functions involved in the inequality (14) are even functions, so it is sufficient to prove for $x \in \Big(0, \frac{\pi}{2}\Big)$.

We write the inequality (14) as follows:

$$45x^2 + 45x\sin x\cos x - 90\sin^2 x - 8\sin^2 x(1-\cos x)^2 > 0,\ 0 < x < \frac{\pi}{2}.$$

We define the function

$$f_3 : \Big(0, \frac{\pi}{2}\Big) \to \mathbb{R},\ f_3(x) = 45x^2 + 45x\sin x\cos x - 90\sin^2 x - 8\sin^2 x(1-\cos x)^2.$$

Elementary calculations reveal that

$$f_3'(x) = 45x(2+\cos 2x) - \frac{135}{2}\sin 2x - 8(2\sin x - \sin 2x)(\cos x - \cos 2x),$$

$$f_3^{(2)}(x) = 8\sin\frac{x}{2}\Big(45\cos\frac{x}{2}(\sin x - x\cos x) - 8\sin^3\frac{x}{2}\cos x(8\cos x + 7)\Big).$$

The function

$$f_4 : \Big(0, \frac{\pi}{2}\Big) \to \mathbb{R},\ f_4(x) = 45\cos\frac{x}{2}(\sin x - x\cos x) - 8\sin^3\frac{x}{2}\cos x(8\cos x + 7)$$

has the derivative

$$\begin{aligned} f_4'(x) &= \sin\frac{x}{2}\left[\frac{45}{2}\left(-\sin x + x\cos x + 4x\cos^2\frac{x}{2}\right) - \right. \\ &\left. -8\sin\frac{x}{2}\left(\frac{3}{2}\cos\frac{x}{2}\left(8\cos^2 x + 7\cos x\right) - \sin\frac{x}{2}(16\sin x\cos x + 7\sin x)\right)\right] \\ &= \sin\frac{x}{2}\left(\frac{11}{2}\sin x + 45x + \frac{123}{2}x\cos x - 112\sin x\cos^2 x + 6\cos x(x - \sin x)\right). \end{aligned}$$

According to the second part of the Lemma 1, we have

$$\frac{11}{2}\sin x + 45x + \frac{123}{2}x\cos x - 112\sin x\cos^2 x > 0$$

on $\left(0, \frac{\pi}{2}\right)$. We also have

$$6\cos x(x - \sin x) > 0$$

on $\left(0, \frac{\pi}{2}\right)$.

We obtain that $f_4' > 0$ on $\left(0, \frac{\pi}{2}\right)$, then f_4 is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

As $f_4(0) = 0$, we prove that $f_4 > 0$ on $\left(0, \frac{\pi}{2}\right)$.

Therefore, $f_3^{(2)} > 0$ on $\left(0, \frac{\pi}{2}\right)$. Using the same arguments, we finally find that $f_3 > 0$ on $\left(0, \frac{\pi}{2}\right)$.

(iii) We can assume that $x \in \left(0, \frac{\pi}{2}\right)$.

We rewrite the inequality (15) as follows:

$$15(x - \sin x) - \sin x(1 - \cos x)(7 - 2\cos x) > 0,\ 0 < x < \frac{\pi}{2}.$$

The function

$$f_5 : \left(0, \frac{\pi}{2}\right) \to \mathbb{R},\ f_5(x) = 15(x - \sin x) - \sin x(1 - \cos x)(7 - 2\cos x)$$

has the derivative

$$\begin{aligned} f_5'(x) &= 15(1 - \cos x) - \\ &-\left(\cos x(1 - \cos x)(7 - 2\cos x) + \sin^2 x(7 - 2\cos x) + 2\sin^2 x(1 - \cos x)\right) \\ &= 15(1 - \cos x) - \\ &-(\cos x(1 - \cos x)(7 - 2\cos x) + (1 - \cos x)(1 + \cos x)(9 - 4\cos x)) \\ &= (1 - \cos x)\left(6 - 12\cos x + 6\cos^2 x\right) \\ &= 6(1 - \cos x)^3. \end{aligned}$$

The function f_5' is > 0 on $\left(0, \frac{\pi}{2}\right)$, hence f_5 is strictly increasing on $\left(0, \frac{\pi}{2}\right)$. Since $f_5(0) = 0$, we find that $f_5 > 0$ on $\left(0, \frac{\pi}{2}\right)$.

This completes the proof of the Theorem 1. □

Proof of Theorem 2. (i) We assume that $x > 0$. We have to prove the following inequality:

$$30(x - \sinh x) + 15(x\cosh x - \sinh x) - \sinh x(1 - \cosh x)^2 < 0$$

for all $x > 0$.

We introduce the function $f_6 : (0, \infty) \to \mathbb{R}$,

$$f_6(x) = 30(x - \sinh x) + 15(x \cosh x - \sinh x) - \sinh x(1 - \cosh x)^2.$$

The derivative of the function f_6 is

$$f_6'(x) = 2 \sinh \frac{x}{2} \left(15x \cosh \frac{x}{2} + \sinh \frac{x}{2} \left(-28 - 3 \cosh^2 x + \cosh x\right)\right).$$

The function $f_7 : (0, \infty) \to \mathbb{R}$,

$$f_7(x) = 15x \cosh \frac{x}{2} + \sinh \frac{x}{2} \left(-28 - 3 \cosh^2 x + \cosh x\right)$$

has the derivative

$$f_7'(x) = -\frac{15}{4} \sinh \frac{x}{2} (\sinh 2x - 2x).$$

Since $f_7'(x) < 0$ on $(0, \infty)$, it follows that f_7 is strictly decreasing on $(0, \infty)$. Since $f_7(0) = 0$, we have $f_7 < 0$ on $(0, \infty)$, then $f_6' < 0$ on $(0, \infty)$.

Hence, f_6 is strictly decreasing on $(0, \infty)$.

As $f_6(0) = 0$, we finally obtain that $f_6 < 0$ on $(0, \infty)$.

(ii) We have to prove that

$$45x^2 + 45x \sinh x \cosh x - 90 \sinh^2 x - 8 \sinh^2 x(1 - \cosh x)^2 < 0$$

for all $x > 0$.

The function $f_8 : (0, \infty) \to \mathbb{R}$,

$$f_8(x) = 45x^2 + 45x \sinh x \cosh x - 90 \sinh^2 x - 8 \sinh^2 x(1 - \cosh x)^2$$

has the derivatives:

$$f_8'(x) = \frac{135}{2}(2x - \sinh 2x) + 45x(\cosh 2x - 1) - 16 \sinh x(1 - \cosh x)^2(2 \cosh x + 1)$$

and

$$f_8^{(2)}(x) = 90(1 - \cosh 2x) + 90x \sinh 2x - 16(1 - \cosh x)^2 \left(7 \cosh x + 8 \cosh^2 x\right).$$

To find critical points of the function $f_8^{(3)}$, first, we calculate the derivative $f_8^{(3)}$:

$$f_8^{(3)}(x) = -2(2 \sinh x + 61 \sinh 2x - 54 \sinh 3x + 32 \sinh 4x - 90x \cosh 2x).$$

Solving the equation $f_8^{(3)}(x) = 0$ yields $x = 0$.

Therefore, the only critical point of the function $f_8^{(2)}$ is $x = 0$. Then, we evaluate $f_8^{(2)}$ at the critical point and at the endpoint of the domain:

$$f_8^{(2)}(0) = 0, \ \lim_{x \to \infty} f_8^{(2)}(x) = -\infty.$$

Hence, the function $f_8^{(2)}$ has a global maximum at $x = 0$: $f_8^{(2)}(0) = 0$.

Then, f_8' is strictly decreasing on $(0, \infty)$. As $f_8'(0) = 0$, we obtain $f_8' < 0$ on $(0, \infty)$.

It follows that f_8 is strictly decreasing on $(0, \infty)$. As $f_8(0) = 0$, we find $f_8 < 0$ on $(0, \infty)$.

(iii) We have to prove that

$$15(x - \sinh x) + 5 \sinh x(\cosh x - 1) - 2 \sinh x(1 - \cosh x)^2 < 0$$

for $x > 0$.

We consider the function

$$f_9 : (0, \infty) \to \mathbb{R},\ f_9(x) = 15(x - \sinh x) + 5 \sinh x(\cosh x - 1) - 2 \sinh x(1 - \cosh x)^2.$$

The derivative of the function f_9 is

$$f_9'(x) = -48 \sinh^6 \frac{x}{2}.$$

Then $f_9' < 0$ on $(0, \infty)$, hence f_9 is strictly decreasing on $(0, \infty)$.

As $f_9(0) = 0$, we find that $f_9 < 0$ on $(0, \infty)$.

The proof of the Theorem 2 is complete. □

Proof of Theorem 3. (i) Since the functions involved in the inequality (19) are even functions, we can assume that $x > 0$.

The inequality (19) takes the equivalent form:

$$30x + 15x \cosh x - 45 \sinh x - \sinh x(1 - \cos x)^2 > 0 \text{ for all } x > 0.$$

We consider the function

$$f_{10} : (0, \infty) \to \mathbb{R},\ f_{10}(x) = 30x + 15x \cosh x - 45 \sinh x - \sinh x(1 - \cos x)^2.$$

The derivatives of the function f_{10} is

$$f_{10}'(x) = 30 - 30 \cosh x + 15x \sinh x - (1 - \cos x)(\cosh x - \cosh x \cos x + 2 \sinh x \sin x),$$

$$\begin{aligned} f_{10}^{(2)}(x) \quad = \quad & -15 \sinh x + 15x \cosh x - \\ & -\Big(\sin x \cosh x - \sin x \cosh x \cos x + 2 \sinh x \sin^2 x + \\ & +(1 - \cos x)(\sinh x + \sinh x \cos x + 3 \cosh x \sin x)\Big), \end{aligned}$$

$$f_{10}^{(3)}(x) = \cosh x\Big(-6 \sin^2 x + 5 \cos^2 x - 4 \cos x - 1\Big) + \sinh x(15x - 2 \sin x \cos x - 4 \sin x),$$

$$f_{10}^{(4)}(x) = \sinh x\Big(-4 \sin^2 x + 3 \cos^2 x - 8 \cos x + 14\Big) + 3 \cosh x(5x - 8 \sin x \cos x),$$

$$f_{10}^{(5)}(x) = \cosh x(1 - \cos x)(49 + 41 \cos x) + (15x + 8 \sin x - 38 \sin x \cos x) \sinh x,$$

The function

$$s : (0, \infty) \to \mathbb{R}, s(x) = 15x + 8 \sin x - 38 \sin x \cos x$$

has the positive roots $x = 0$, $x \approx 0.85321$. Then, $s < 0$ on $(0, 0.85321)$ and $s > 0$ on $(0.85321, \infty)$. It follows that $f_{10}^{(5)}(x) > 0$ on $(0.85321, \infty)$.

If $x \in (0, 0.85321) \subset \left(0, \frac{\pi}{2}\right)$, then $f_{10}^{(5)}(x)$ can be rewritten as

$$\begin{aligned} f_{10}^{(5)}(x) \quad = \quad & \cosh x(1 - \cos x)(49 + 41 \cos x) - \\ & -15x \sinh x + (15(2x - \sin 2x) + 8 \sin x(1 - \cos x)) \sinh x. \end{aligned}$$

The function

$$t : (0, \infty) \to \mathbb{R},\ t(x) = \cosh x(1 - \cos x)(49 + 41 \cos x) - 15x \sinh x$$

has the positive roots $x = 0$, $x \approx 2.34534$ and $t > 0$ on $(0, 2.34534)$.

Hence, $f_{10}^{(5)}(x) > 0$ on $(0,\infty)$. It follows that $f_{10}^{(4)}$ is strictly increasing on $(0,\infty)$. As $f_{10}^{(4)}(0) = 0$, we find $f_{10}^{(4)} > 0$ on $(0,\infty)$. Continuing the algorithm we finally obtain that $f_{10} > 0$ on $(0,\infty)$.

(ii) We also can assume $x > 0$.

We write the inequality (20) as follows:

$$45x^2 + 45x\sinh x\cosh x - 90\sinh^2 x - 8\sinh^2 x(\cos x - 1)^2 < 0, \text{ for } 0 < x < 1.50618.$$

The function

$$f_{11} : (0,\infty) \to \mathbb{R},\ f_7(x) = 45x^2 + 45x\sinh x\cosh x - 90\sinh^2 x - 8\sinh^2 x(\cos x - 1)^2.$$

has the derivative

$$f_{11}'(x) = 90x - \frac{135}{2}\sinh 2x + 45x\cosh 2x - 8(\cos x - 1)^2\sinh 2x + 8\sin x(\cos x - 1)(\cosh 2x - 1).$$

The equation $f_{11}'(x) = 0$ has the positive roots $x = 0$, $x \approx 1.35234$. Moreover, $f_{11}'(x) < 0$ for $x \in (0, 1.35234)$. Then, f_{11} is strictly decreasing on $(0, 1.35234)$ and it is strictly increasing on $(1.35234, 1.50618)$. Since $f_{11}(0) = 0$ and $f_{11}(1.50618) = 0$, it follows that $f_{11} < 0$ on $(0, 1.50618)$.

(iii) As in the above theorems, we can assume $x > 0$.

We rearrange the inequality (21) as follows:

$$15x + 5\sinh x\cosh x - 20\sinh x - 2\sinh x(1-\cos x)^2 > 0, \text{ for all } x > 0.$$

We introduce the function

$$f_{12} : (0,\infty) \to \mathbb{R},\ f_8(x) = 15x + 5\sinh x\cosh x - 20\sinh x - 2\sinh x(1-\cos x)^2.$$

Easy computation yields

$$f_{12}'(x) = 5\sinh^2 x + 5\cosh^2 x - 2\left(\cos^2 x - 2\cos x + 11\right)\cosh x + 4\sin x(\cos x - 1)\sinh x + 15,$$

$$f_{12}^{(2)}(x) = 2\sinh x\left(-2\sin^2 x + \cos^2 x - 11\right) + 4\cosh x(5\sinh x + 2\sin x(\cos x - 1)),$$

$$\begin{aligned} f_{12}^{(3)}(x) \quad = \quad & 20\sinh^2 x + 20\cosh^2 x - 20\cosh x - 2(\cos x - 1)^2\cosh x + \\ & +12\sin x(\cos x - 1)\sinh x - 2\sinh x(2\sin x(\cos x - 1) + 6\sin x\cos x) - \\ & -6\cosh x\left(2\sin^2 x - 2(\cos x - 1)\cos x\right) \end{aligned}$$

and

$$f_{12}^{(4)}(x) = 2\sinh x\left(-4\sin^2 x + 3\cos^2 x - 8\cos x - 11\right) + \cosh x(80\sinh x - 48\sin x\cos x).$$

According to the first part of the Lemma 1 we have $2\sinh x \geq \sin 2x$ for all $x \geq 0$. Then,

$$\begin{aligned} \cosh x(80\sinh x - 48\sin x\cos x) \quad &= \quad \cosh x(32\sinh x + 48\sinh x - 24\sin 2x) \\ &\geq \quad 32\cosh x\sinh x. \end{aligned}$$

Therefore, we find that

$$f_{12}^{(4)}(x) \geq 2\sinh x\left(-4\sin^2 x + 3\cos^2 x - 8\cos x - 11 + 16\cosh x\right).$$

In the following, we will prove that the function

$$h:(0,\infty)\to\mathbb{R},\ h(x)=-4\sin^2 x+3\cos^2 x-8\cos x-11+16\cosh x$$

is positive on $(0,\infty)$.

The derivatives of the function h are

$$h'(x)=8\sin x-7\sin 2x+16\sinh x$$

and

$$\begin{aligned} h^{(2)}(x) &= 8\cos x-14\cos 2x+16\cosh\ x \\ &= 2\Big(4\cos x(1-\cos x)+7\sin^2 x+8(\cosh x-1)+8\Big(1-\sin^2 x\Big)+5\sin^2 x\Big). \end{aligned}$$

Since the function $h^{(2)}>0$ on $(0,\infty)$ it follows that h' is strictly increasing on $(0,\infty)$. As $h'(0)=0$, we get $h'>0$ on $(0,\infty)$. Continuing the algorithm, we finally obtain that $h>0$ on $(0,\infty)$.

Hence, we deduce that $f_{12}^{(4)}>0$ on $(0,\infty)$. Using the same arguments as above, we finally find that $f_{12}>0$ on $(0,\infty)$.

The proof of the Theorem 3 is complete. □

Remark 1. *(1) From the Lemma 1, (iii), it follows that*

$$\frac{1}{15}(1-\cos x)^2>\frac{1}{60}x^3\sin x,\ \textit{for}\ 0<|x|<\frac{\pi}{2},$$

$$\frac{8}{45}(1-\cos x)^2>\frac{2}{45}x^3\sin x,\ \textit{for}\ 0<|x|<\frac{\pi}{2},$$

$$\frac{2}{5}(1-\cos x)^2>\frac{1}{10}x^3\sin x,\ \textit{for}\ 0<|x|<\frac{\pi}{2}.$$

Therefore, we also improved the inequalities (5), (6) and (8).

(2) From the Lemma 2 and Theorem 3, we find that

$$\left(\frac{x}{\sinh x}\right)^2+\frac{x}{\tanh x}-2<\frac{8}{45}(1-\cos x)^2<\frac{2}{45}x^3\sinh x$$

for all x, $0<|x|<1.50618$, *hence we improved the inequality (11).*

4. Conclusions

The function

$$sinc(x)=\frac{\sin x}{x}$$

occurs in Fourier analysis and its applications in signal processing. The Fourier transform of the *sinc* function is a rectangle, and the Fourier transform of a rectangular pulse is a *sinc* function. The *sinc* function also appears in analysis of digital-to-analogue conversion.

In our work, Taylor expansion of the error function between the truncated sum of the first terms of the cosine series of the functions involved in Wilker–Huygens-type inequalities and the functions themselves is carried out. Then the best approximation of the functions which improve Wilker–Huygens-type inequalities is obtained.

These new approximations give sharp bounds to the functions $sinc(x)$ and

$$tanc(x)=\frac{\tan x}{x}.$$

For example, the inequalities

$$\frac{\sin x}{x} < \frac{3}{4 - \cos x + \frac{2}{5}(1 - \cos x)^2}, 0 < |x| < \frac{\pi}{2}$$

and

$$\frac{x}{\tan x} > 3 - 2\frac{x}{\sin x} + \frac{1}{15}(1 - \cos x)^2, 0 < |x| < \frac{\pi}{2}$$

are very sharp and interesting for further studies.

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