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Regularity of Weak Solutions to the Inhomogeneous Stationary Navier–Stokes Equations

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Abstract: One of the most intriguing issues in the mathematical theory of the stationary Navier–Stokes equations is the regularity of weak solutions. This problem has been deeply investigated for homogeneous fluids. In this paper, the regularity of the solutions in the case of not constant viscosity is analyzed. Precisely, it is proved that for a bounded domain $\Omega \subset \mathbb{R}^2$, a weak solution $u \in W^{1,q}(\Omega)$ is locally Hölder continuous if $q = 2$, and Hölder continuous around x , if $q \in (1, 2)$ and $|\mu(x) - \mu_0|$ is suitably small, with μ_0 positive constant; an analogous result holds true for a bounded domain $\Omega \subset \mathbb{R}^n$ ($n > 2$) and weak solutions in $W^{1,n/2}(\Omega)$.

Keywords: stationary Navier–Stokes equations; weak solutions; regularity

MSC: Primary 76D03; 76D05; Secondary 35Q30; 35Q35



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1. Introduction and Statement of the Results

The stationary motions of an inhomogeneous, incompressible viscous fluid in a solid (bounded domain with connected boundary) Ω of \mathbb{R}^n ($n \geq 2$) are governed by the Navier–Stokes equations

$$\begin{aligned} 2\operatorname{div}(\mu \hat{\nabla} u) - \operatorname{div}(u \otimes u) - \nabla p &= 0, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1}$$

where $u : \Omega \rightarrow \mathbb{R}^n$, $p : \Omega \rightarrow \mathbb{R}$, are the velocity and pressure field, respectively, $\hat{\nabla} u = \frac{1}{2}(\nabla u + \nabla u^\top)$ the symmetric part of ∇u , and $\mu : \Omega \rightarrow \mathbb{R}^+$ is the viscosity coefficient, satisfying the natural assumption

$$\mu_i \leq \mu(x) \leq \mu_e, \tag{2}$$

almost everywhere in Ω . Unless otherwise specified, we will essentially use a standard notation as in [1]. In particular, $B_R(x) = \{y \in \mathbb{R}^n : |y - x| < R\}$; if \mathcal{F} is a function space the subscript σ in \mathcal{F}_σ stands for the (weakly) divergence-free condition; the integral mean of the field u over A will be denoted by $u_A = \frac{1}{|A|} \int_A u$. Throughout the paper, the symbol c will denote a positive constant whose numerical value is unessential for our purposes and may change from line to line and in the same line, too.

Observe that we are considering the case of viscosity exclusively depending on the position x (see [2,3], for unsteady flows). This is the case, for example, of immiscible fluids or fluid mixtures where, in the first instance, we can neglect the natural dependence of viscosity on pressure and temperature (see, e.g., [4] and the references therein). Limiting ourselves to isothermal flows, the dependence of viscosity on pressure was already highlighted by G.G. Stokes in the seminal work [5] of 1845. In the last decades, many efforts have been done to take into account the effects of a pressure-dependent viscosity. In particular, we quote [6,7], where a global in time existence of solutions for the evolution problem was proved (a previous result [8] regarded a local in time existence and uniqueness result).

In [6,7], besides on pressure, a further dependence of the viscosity on the shear rate was required. More recently, a pure dependence of viscosity on pressure was instead supposed in [9], where an existence and uniqueness result for the stationary Stokes system, obtained from (1) by neglecting the inertial term, was presented by requiring general assumptions on μ that are satisfied by the *Barus formula* and other empiric laws; in the case of a thin straight pipe with variable cross-section the effective behavior of the flow was found via a rigorous asymptotic analysis with respect to the pipe's thickness [10].

In order to deal with the problem of regularity of weak solutions to system (1) with variable viscosity μ , in this paper, we consider the simplified situation of viscosity depending on the point $\mu = \mu(x)$. Besides including meaningful cases as immiscible fluids in the model, it is worth pointing out that the problem here analyzed is also interesting from a purely mathematical point of view. Indeed, as far as we are aware, up to now the regularization problem has been considered only for variational solutions to (1) (see, e.g., [11], where it is considered a more general non linear system of the type of the Navier–Stokes one; however, it is not difficult to see that our technique can be used to deal with the equations here considered).

A weak solution (variational if $q = 2$) to (1) is a field $u \in W_{\sigma}^{1,q}(\Omega)$, $q \geq 2n/(n+2)$, which satisfies the relation

$$2 \int_{\Omega} \mu \hat{\nabla} u \cdot \nabla \varphi + \int_{\Omega} u \cdot \nabla u \cdot \varphi = 0, \quad \forall \varphi \in C_{\sigma,0}^{\infty}(\Omega). \quad (3)$$

To a solution to (3) is associated a scalar field $p \in L^q(\Omega)$ such that [1]

$$2 \int_{\Omega} \mu \hat{\nabla} u \cdot \nabla \varphi + \int_{\Omega} u \cdot \nabla u \cdot \varphi - \int_{\Omega} p \operatorname{div} \varphi = 0, \quad \forall \varphi \in C_0^{\infty}(\Omega). \quad (4)$$

It is natural to ask whether and under what condition a weak solution to (1) is more regular, for instance Hölder continuous. For homogeneous fluids, this issue has been the object of several papers, starting from the classical one of D. Serre (1983) [12], who proved regularity of a weak solution $u \in W^{1,q}(\Omega)$, $q > 3/2$, in the physical case $n = 3$. The regularity of variational solutions for $n = 2, 3$ was proved by O.A. Ladyzhenskaia [13] and for $n = 4$ by C. Gerhardt [14] (see also [11]). Regularity of variational solutions for $n > 4$ is an open problem [1]. More recently, Serre's results were generalized to distributional solutions in $L^n(\Omega)$ by G. P. Galdi [1] and also in $L_w^n(\Omega)$ (weak $L^n(\Omega)$ space) by H. Kim and H. Kozono [15], under the assumption of smallness of the norm $\|u\|_{L_w^n(\Omega)}$.

The main purpose of this paper is to extend the results of [1] to system (1). To be precise, starting from the technique introduced in [16], we shall prove

Theorem 1. *Let $n = 2$. A variational solution u to Equation (1) is locally Hölder continuous. A weak solution $u \in W^{1,q}(\Omega)$, $q \in (1, 2)$, is Hölder continuous around x if there is a positive constant μ_0 such that $|\mu - \mu_0|$ is small in a neighborhood of x .*

Theorem 2. *Let $n > 2$. A weak solution $u \in W^{1,n/2}(\Omega)$ to Equation (1) is Hölder continuous around x if there is a positive constant μ_0 such that $|\mu - \mu_0|$ is small in a neighborhood of x .*

From Theorem 2 it easily follows

Corollary 1. *Let $u \in W^{1,n/2}(\Omega)$, $n > 2$, be a weak solution to Equation (1). If $\mu \in C(\Omega)$, then $u \in C_{\text{loc}}^{0,\alpha}(\Omega)$ for every $\alpha \in (0, 1)$.*

2. Preliminary Results

We collect in this section the main tools we need to prove Theorems 1 and 2 in the next sections.

Consider the linear system [17] (see also [18,19])

$$\begin{aligned} 2\operatorname{div}(\mu \hat{\nabla} u) - \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{5}$$

A weak solution (variational if $q = 2$) to (5) is a field $u \in W_{\sigma}^{1,q}(\Omega)$ such that

$$\int_{\Omega} \mu \hat{\nabla} u \cdot \nabla \varphi = 0, \quad \forall \varphi \in C_{\sigma,0}^{\infty}(\Omega). \tag{6}$$

It is well-known that for constant μ a weak solution to (5) is real analytic [1].

Lemma 1. *If u is a variational solution to (5) in $B_{R'}(x)$, then there is a positive constant γ depending only on n, μ_i and μ_e such that*

$$\int_{B_{\rho}(x)} |\nabla u|^2 \leq c \left(\frac{\rho}{R}\right)^{\gamma} \int_{B_R(x)} |\nabla u|^2, \tag{7}$$

for all $0 < \rho \leq R \ll R'$.

Proof. Let $g(|y - x|)$ be a regular cut-off function, vanishing for $|y - x| > 2R$, equal to 1 for $|y - x| < R$ and such that $|\nabla g| \leq cR^{-1}$. The system

$$\operatorname{div}(h + g^2(u - \kappa)) = 0 \quad \text{in } T_R(x),$$

where $T_R(x) = B_{2R}(x) \setminus \overline{B_R(x)}$ and κ is a constant, has a solution $W_0^{1,2}(T_R(x))$ such that [1] (Chapter III, Section 3, Theorem III.3.1)

$$\|\nabla h\|_{L^2(T_R(x))} \leq c \|(u - \kappa) \cdot \nabla g^2\|_{L^2(T_R(x))}, \tag{8}$$

where c is independent of R . In writing (8), we used the divergence free condition (1)₂. Since the field $h + g^2(u - \kappa)$ is permissible in (6), one has

$$\begin{aligned} \int_{B_{R'}(x)} \mu \hat{\nabla} u \cdot \nabla(h + g^2(u - \kappa)) &= \int_{T_R(x)} \mu \nabla h \cdot \hat{\nabla} u \\ &+ 2 \int_{T_R(x)} g \mu (u - \kappa) \cdot (\hat{\nabla} u) \nabla g + \int_{B_{R'}(x)} \mu g^2 |\hat{\nabla} u|^2 = 0. \end{aligned} \tag{9}$$

Therefore, taking into account (2),

$$\mu_i \int_{B_{R'}(x)} |g \hat{\nabla} u|^2 \leq -2 \int_{T_R(x)} g \mu (u - \kappa) \cdot (\hat{\nabla} u) \nabla g - \int_{T_R(x)} \mu \nabla h \cdot \hat{\nabla} u. \tag{10}$$

Since

$$2|g \hat{\nabla} u|^2 = |g \nabla u|^2 + \operatorname{div}[g^2(u - \kappa) \cdot \nabla u] - 2g(u - \kappa) \cdot (\nabla u) \nabla g,$$

from (10), it follows that

$$\begin{aligned} \int_{B_{R'}(x)} |g \nabla u|^2 &\leq 2 \int_{T_R(x)} g(u - \kappa) \cdot (\nabla u) \nabla g \\ &- \frac{4}{\mu_i} \int_{T_R(x)} g \mu (u - \kappa) \cdot (\hat{\nabla} u) \nabla g - \frac{2}{\mu_i} \int_{T_R(x)} \mu \nabla h \cdot \hat{\nabla} u, \end{aligned} \tag{11}$$

where we also took into account that, applying the divergence theorem and the condition $g = 0$ outside $B_{2R}(x)$,

$$\int_{B_{R'}(x)} \operatorname{div}[g^2(u - \kappa) \cdot \nabla u] = 0. \tag{12}$$

By the arithmetic–geometric mean inequality and (8), for every $\alpha_1, \alpha_2 > 0$

$$\begin{aligned} 2 \int_{T_R(x)} |g(u - \kappa) \cdot (\nabla u) \nabla g| &\leq \alpha_1 \int_{T_R(x)} |g \nabla u|^2 + \alpha_1^{-1} \int_{T_R(x)} |u - \kappa|^2 |\nabla g|^2, \\ \frac{4}{\mu_i} \int_{T_R(x)} |g \mu(u - \kappa) \cdot (\hat{\nabla} u) \nabla g| &\leq \frac{4\mu_e}{\mu_i} \int_{T_R(x)} |g(u - \kappa) \cdot (\hat{\nabla} u) \nabla g| \\ &\leq \alpha_2 \int_{T_R(x)} |g \nabla u|^2 + c \int_{T_R(x)} |u - \kappa|^2 |\nabla g|^2, \\ \frac{2}{\mu_i} \int_{T_R(x)} |\mu \nabla h \cdot \hat{\nabla} u| &\leq c \int_{T_R(x)} |u - \kappa|^2 |\nabla g|^2 + \int_{T_R(x)} |\nabla u|^2. \end{aligned}$$

Therefore, choosing $\alpha_1, \alpha_2 \ll 1$, (11) implies

$$\int_{B_{R'}(x)} |g \nabla u|^2 \leq c \int_{T_R(x)} |u - \kappa|^2 |\nabla g|^2 + \int_{T_R(x)} |\nabla u|^2. \tag{13}$$

Using the properties of the function g and, in particular, that $g = 1$ in $B_R(x)$ and $|\nabla g| \leq cR^{-1}$, it follows

$$\int_{B_R(x)} |\nabla u|^2 = \int_{B_R(x)} |g \nabla u|^2 \leq \int_{B_{R'}(x)} |g \nabla u|^2 \leq \frac{c}{R^2} \int_{T_R(x)} |u - \kappa|^2 + \int_{T_R(x)} |\nabla u|^2.$$

Hence, choosing $k = u_{T_R(x)}$, and using Poincaré’s inequality $\|u - u_{T_R(x)}\|_{L^2(T_R(x))}^2 \leq cR^2 \|\nabla u\|_{L^2(T_R(x))}^2$ (see, e.g., [20]), it follows

$$\int_{B_R(x)} |\nabla u|^2 \leq c \int_{T_R(x)} |\nabla u|^2. \tag{14}$$

Adding $c \int_{B_R(x)} |\nabla u|^2$ to both sides of (14) yields

$$\int_{B_R(x)} |\nabla u|^2 \leq \tau \int_{B_{2R}(x)} |\nabla u|^2, \quad \tau = \frac{c}{c+1}. \tag{15}$$

Now, to (15) we can apply a classical result in the theory of regularity of weak solutions to elliptic systems (see, e.g., [21], Lemma 8.23), to conclude that (7) holds, with $\gamma = -\log \tau / \log 2$. \square

Lemma 2. (Campanato’s inequality)– If μ is constant and $u \in W^{1,q}(B_{R'}(x))$ is a weak solution to (5), then for every $p \in [1, +\infty)$

$$\int_{B_\rho(x)} |\nabla u|^p \leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x)} |\nabla u|^p, \tag{16}$$

for all $0 < \rho \leq R \ll R'$ and for some positive constant c depending only on n and p .

Lemma 3. If $F \in L^q(B_R)$, $q \in (1, +\infty)$, then the system

$$\begin{aligned} 2 \operatorname{div}(\mu \hat{\nabla} u) - \nabla p + \operatorname{div} F &= 0 \quad \text{in } B_R(x), \\ \operatorname{div} u &= 0 \quad \text{in } B_R(x), \\ u &= 0 \quad \text{on } \partial B_R(x), \end{aligned}$$

has a unique solution $u \in W_0^{1,q}(B_R(x))$ and

$$\|\nabla u\|_{L^q(B_R(x))} \leq c \|F\|_{L^q(B_R(x))}, \tag{17}$$

for some constant c independent of R .

A proof of Lemmas 2 and 3 can be found in [11,22] (see Remark 10.1 in [22]).

Lemma 4. Let $u \in W^{1,q}(B_{\bar{R}}(x))$, $q \in [1, n]$. If

$$\int_{B_{\rho}(y)} |\nabla u|^q \leq c \left(\frac{\rho}{R}\right)^{n-q+\alpha q} \int_{B_R(y)} |\nabla u|^q,$$

for every $0 \leq \rho \leq R = \bar{R} - |x - y|$ and for every $y \in B_{\bar{R}}(x)$, then u is Hölder continuous around x with exponent α .

Lemma 4 is due to C. Morrey (see [23], p. 79).

Lemma 5. Let Φ be a non negative and non decreasing function. Then there exists ϵ_0 such that if

$$\Phi(\rho) \leq c_0 \left[\left(\frac{\rho}{R}\right)^\alpha + \epsilon \right] \Phi(R) + c_1 R^\beta,$$

for all $\rho \leq R \leq \bar{R}$ with c_0, c_1, α, β positive constants, $\beta < \alpha$ and $\epsilon < \epsilon_0$, then for all $\rho \leq R \leq \bar{R}$,

$$\Phi(\rho) \leq c \left[\left(\frac{\rho}{R}\right)^\beta \Phi(R) + c_1 \rho^\beta \right],$$

where c is a constant depending on α, β, c_0 .

Lemma 5 is due to S. Campanato [24] (see also [11], p. 179).

3. Proof of Theorem 1

• Let $u \in W^{1,2}(\Omega)$ and $x \in \Omega$. By uniqueness $u = u_1 + u_2$, with u_1 and u_2 solutions to

$$\begin{aligned} 2\operatorname{div}(\mu \hat{\nabla} u_1) - \nabla p_1 &= 0 && \text{in } B_R(x), \\ \operatorname{div} u_1 &= 0 && \text{in } B_R(x), \\ u_1 &= u && \text{on } \partial B_R(x), \end{aligned} \tag{18}$$

and

$$\begin{aligned} 2\operatorname{div}(\mu \hat{\nabla} u_2) - \nabla p_2 + \operatorname{div}[u \otimes (u - \kappa)] &= 0 && \text{in } B_R(x), \\ \operatorname{div} u_2 &= 0 && \text{in } B_R(x), \\ u_2 &= 0 && \text{on } \partial B_R(x), \end{aligned} \tag{19}$$

respectively, for every constant κ (this follows from the divergence free condition (1)₂). By Lemma 1 there exist positive constants c and γ , such that

$$\int_{B_{\rho}(x)} |\nabla u_1|^2 \leq c \left(\frac{\rho}{R}\right)^\gamma \int_{B_R(x)} |\nabla u_1|^2, \tag{20}$$

and by Lemma 3 and Schwarz’s inequality,

$$\int_{B_R(x)} |\nabla u_2|^2 \leq c \int_{B_R(x)} |u|^2 |u - \kappa|^2 \leq c \|u\|_{L^4(B_R(x))}^2 \|u - \kappa\|_{L^4(B_R(x))}^2. \tag{21}$$

Choosing $\kappa = u_{B_R(x)}$, by Ladyzhenskaia’s inequality (see [1], p. 55) and Poincaré’s inequality

$$\begin{aligned} \|u - u_{B_R(x)}\|_{L^4(B_R(x))}^2 &\leq c \|u - u_{B_R(x)}\|_{L^2(B_R(x))} \|\nabla u\|_{L^2(B_R(x))} \\ &\leq cR \|\nabla u\|_{L^2(B_R(x))}^2. \end{aligned}$$

Therefore, (21) yields

$$\int_{B_R(x)} |\nabla u_2|^2 \leq cR \|u\|_{L^4(B_R(x))}^2 \int_{B_R(x)} |\nabla u|^2 = c\omega(R) \int_{B_R(x)} |\nabla u|^2, \tag{22}$$

where $\omega(R) = R \|u\|_{L^4(B_R(x))}^2$. Putting together (20) and (22) and using the inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, for all $a, b \in \mathbb{R}$, one has

$$\begin{aligned} \int_{B_\rho(x)} |\nabla u|^2 &\leq 2 \int_{B_\rho(x)} (|\nabla u_1|^2 + |\nabla u_2|^2) \\ &\leq 2c \left(\frac{\rho}{R}\right)^\gamma \int_{B_R(x)} |\nabla u_1|^2 + 2c\omega(R) \int_{B_R(x)} |\nabla u|^2 \\ &\leq c \left(\frac{\rho}{R}\right)^\gamma \int_{B_R(x)} |\nabla u|^2 + c\omega(R) \int_{B_R(x)} |\nabla u|^2. \end{aligned} \tag{23}$$

Hence, taking into account Lemma 5, it follows that

$$\int_{B_\rho(x)} |\nabla u|^2 \leq c \left(\frac{\rho}{R}\right)^{\gamma'} \int_{B_R(x)} |\nabla u|^2, \tag{24}$$

for some $\gamma' \in (0, \gamma)$.

By a well-known argument (see, e.g., [22], pp. 313–314) (24) implies that

$$\sup_{y \in \mathcal{J}, \rho \leq \tilde{R}} \frac{1}{\rho^{\gamma'}} \int_{B_\rho(y)} |\nabla u|^2 < +\infty,$$

where \mathcal{J} is a neighborhood of x and \tilde{R} such that $B_\rho(y) \subset \mathcal{J}$. Hence by Lemma 4 it follows that u is Hölder continuous around x with exponent $\frac{\gamma'}{2}$ depending only on μ_i, μ_e .

• Let $u \in W^{1,q}(\Omega)$, $q \in (1, 2)$, and let $|\mu - \mu_0|$ be suitably small in a neighborhood of x . Now $u = u_1 + u_2$, with u_1 and u_2 solutions to

$$\begin{aligned} \mu_0 \Delta u_1 - \nabla p_1 &= 0 && \text{in } B_R(x), \\ \operatorname{div} u_1 &= 0 && \text{in } B_R(x), \\ u_1 &= u && \text{on } \partial B_R(x), \end{aligned} \tag{25}$$

and

$$\begin{aligned} \mu_0 \Delta u_2 - \nabla p_2 + \operatorname{div} G &= 0 && \text{in } B_R(x), \\ \operatorname{div} u_2 &= 0 && \text{in } B_R(x), \\ u_2 &= 0 && \text{on } \partial B_R(x), \end{aligned} \tag{26}$$

respectively, with

$$G = 2(\mu - \mu_0) \hat{\nabla} u + u \otimes (u - \kappa), \tag{27}$$

for every constant κ (this follows from the divergence free condition (1)₂). By Lemma 2

$$\int_{B_\rho(x)} |\nabla u_1|^q \leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x)} |\nabla u_1|^q, \tag{28}$$

and by Lemma 3

$$\int_{B_R(x)} |\nabla u_2|^q \leq c \int_{B_R(x)} |G|^q. \tag{29}$$

By Hölder's inequality,

$$\int_{B_R(x)} |u \otimes (u - \kappa)|^q \leq \left\{ \int_{B_R(x)} |u|^2 \right\}^{q/2} \left\{ \int_{B_R(x)} |u - \kappa|^{2q/(2-q)} \right\}^{(2-q)/2}.$$

Therefore, choosing $\kappa = u_{B_R(x)}$ and using Sobolev's inequality,

$$\int_{B_R(x)} |u - u_{B_R(x)}|^{2q/(2-q)} \leq c \left\{ \int_{B_R(x)} |\nabla u|^q \right\}^{2/(2-q)},$$

one has

$$\int_{B_R(x)} |u \otimes (u - u_{B_R(x)})|^q \leq c \left\{ \int_{B_R(x)} |u|^2 \right\}^{q/2} \int_{B_R(x)} |\nabla u|^q. \quad (30)$$

Moreover, since

$$2 \int_{B_R(x)} |\mu - \mu_0|^q |\nabla u|^q \leq \epsilon^q \int_{B_R(x)} |\nabla u|^q,$$

taking into account that $\lim_{R \rightarrow 0} \|u\|_{L^2(B_R(x))}^q = 0$, (29) writes

$$\int_{B_R(x)} |\nabla u_2|^q \leq c \left[\epsilon^q + \|u\|_{L^2(B_R(x))}^q \right] \int_{B_R(x)} |\nabla u|^q \leq c\epsilon' \int_{B_R(x)} |\nabla u|^q, \quad (31)$$

with ϵ' suitably small. Putting together (28) and (31) and using the inequality $|a + b|^q \leq 2^{q-1}(|a|^q + |b|^q)$, for all $a, b \in \mathbb{R}$, one gets

$$\begin{aligned} \int_{B_\rho(x)} |\nabla u|^q &\leq c \int_{B_\rho(x)} (|\nabla u_1|^q + |\nabla u_2|^q) \\ &\leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x)} |\nabla u_1|^q + c\epsilon' \int_{B_R(x)} |\nabla u|^q \\ &\leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x)} |\nabla u|^q + c\epsilon' \int_{B_R(x)} |\nabla u|^q. \end{aligned} \quad (32)$$

Hence, Hölder continuity of u around x follows from Lemmas 4 and 5. \square

4. Proof of Theorem 2

Let u_1 and u_2 be the solutions to (25) and (26), respectively. By Lemma 2

$$\int_{B_\rho(x)} |\nabla u_1|^{n/2} \leq c \left(\frac{\rho}{R}\right)^n \int_{B_\rho(x)} |\nabla u_1|^{n/2}, \quad (33)$$

and by Lemma 3

$$\int_{B_R(x)} |\nabla u_2|^{n/2} \leq c \int_{B_R(x)} |G|^{n/2}. \quad (34)$$

By Hölder's inequality,

$$\int_{B_R(x)} |u \otimes (u - u_{B_R(x)})|^{n/2} \leq c \left\{ \int_{B_R(x)} |u|^n \right\}^{1/2} \left\{ \int_{B_R(x)} |u - u_{B_R(x)}|^n \right\}^{1/2}.$$

Therefore, by Sobolev's inequality,

$$\left\{ \int_{B_R(x)} |u - u_{B_R(x)}|^n \right\}^{1/2} \leq c \int_{B_R(x)} |\nabla u|^{n/2},$$

one has

$$\int_{B_R(x)} |u \otimes (u - u_{B_R(x)})|^{n/2} \leq c \left\{ \int_{B_R(x)} |u|^n \right\}^{1/2} \int_{B_R(x)} |\nabla u|^{n/2}. \tag{35}$$

Moreover, since

$$2 \int_{B_R(x)} |\mu - \mu_0|^{n/2} |\hat{\nabla} u|^{n/2} \leq \epsilon^{n/2} \int_{B_R(x)} |\nabla u|^{n/2},$$

(34) writes

$$\int_{B_R} |\nabla u_2|^{n/2} \leq c \left[\epsilon^{n/2} + \|u\|_{L^n(B_R(x))}^{n/2} \right] \int_{B_R} |\nabla u|^{n/2} \leq c\epsilon' \int_{B_R} |\nabla u|^{n/2}, \tag{36}$$

with ϵ' suitably small. Putting together (33) and (36) and repeating the steps yielding (32), one has

$$\begin{aligned} \int_{B_\rho(x)} |\nabla u|^{n/2} &\leq c \int_{B_\rho(x)} (|\nabla u_1|^{n/2} + |\nabla u_2|^{n/2}) \\ &\leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x)} |\nabla u_1|^{n/2} + c\epsilon' \int_{B_R(x)} |\nabla u|^{n/2} \\ &\leq c \left(\frac{\rho}{R}\right)^n \int_{B_R(x)} |\nabla u|^{n/2} + c\epsilon' \int_{B_R(x)} |\nabla u|^{n/2}, \end{aligned}$$

for small positive ϵ' . Hence the desired result follows from Lemmas 4 and 5. □

It should be of some interest to detect whether for $n > 2$ the above regularity results can be extended to function spaces larger than $W^{1,n/2}(\Omega)$, like the grand Sobolev spaces introduced by T. Iwaniec and C. Sbordone [25]. In this connection see [26,27].

5. Conclusions

In this paper, the regularity properties of the solutions to the steady Navier–Stokes equations with a variable viscosity coefficient $\mu = \mu(x)$ have been examined (see [2,3]). By only requiring $\mu \in L^\infty(\Omega)$, it has been proved that if $\Omega \subset \mathbb{R}^2$ the weak solutions $u \in W^{1,2}(\Omega)$ are locally Hölder continuous and the weak solutions $u \in W^{1,q}(\Omega)$, $q \in (1,2)$, are Hölder continuous around x if there exists a constant μ_0 such that $|\mu(x) - \mu_0|$ is suitably small; analogously for weak solutions $u \in W^{1,n/2}(\Omega)$ if $\Omega \subset \mathbb{R}^n$, $n > 2$.

The interest in considering models with variable viscosity goes back to the researches by G.G. Stokes [5] and all the derived studies where, for instance, a pressure-dependent viscosity is considered (see [6–10] and the references therein for a qualitative analysis of the solutions). The idea is to use the results obtained in this paper, regarding the simplified case of viscosity depending on the point—that is, however, interesting itself, as pointed out in the introduction—to tackle the problem of regularity of solutions in the case of more general inhomogeneities, as the one due to the variation of viscosity with pressure. Obviously, a pressure-dependent viscosity brings an additional nonlinearity to system (1) and makes the pressure no more, merely a Lagrange multiplier, so that the mathematical analysis of the system is more complicated and this work can be considered a starting point in this perspective.

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