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# Exact Solutions to the Navier–Stokes Equations with Couple Stresses

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**Abstract:** This article discusses the possibility of using the Lin–Sidorov–Aristov class of exact solutions and its modifications to describe the flows of a fluid with microstructure (with couple stresses). The presence of couple shear stresses is a consequence of taking into account the rotational degrees of freedom for an elementary volume of a micropolar liquid. Thus, the Cauchy stress tensor is not symmetric. The article presents exact solutions for describing unidirectional (layered), shear and three-dimensional flows of a micropolar viscous incompressible fluid. New statements of boundary value problems are formulated to describe generalized classical Couette, Stokes and Poiseuille flows. These flows are created by non-uniform shear stresses and velocities. A study of isobaric shear flows of a micropolar viscous incompressible fluid is presented. Isobaric shear flows are described by an overdetermined system of nonlinear partial differential equations (system of Navier–Stokes Equations and incompressibility equation). A condition for the solvability of the overdetermined system of equations is provided. A class of nontrivial solutions of an overdetermined system of partial differential equations for describing isobaric fluid flows is constructed. The exact solutions announced in this article are described by polynomials with respect to two coordinates. The coefficients of the polynomials depend on the third coordinate and time.

**Keywords:** exact solutions; Navier–Stokes Equations; couple stresses; micropolar fluid; non-symmetric stress tensor; isobaric flows; gradient flows; overdetermined system; solvability condition

## 1. Introduction

The overwhelming majority of studies dealing with fluid flows are based on the application of the conventional Navier–Stokes Equations supplemented by the incompressibility condition [1,2]. The Navier–Stokes Equations are derived from the postulates (hypotheses) of the Newtonian mechanics of continua, each particle of which is viewed as a material point. When the representative volume of a continuum is substituted by a material point, it is considered by default to have three degrees of freedom (translational degrees of freedom). The application of this approach imposes restrictions on studying changes in fluid viscosity, friction coefficients, and other surface effects [3–8]. The difference between the experimentally and theoretically obtained results reported in the pioneering study [8] stems from ignoring the rotational (orientational) degrees of freedom of the representative volume of a continuum. By taking into account additional degrees of freedom of the elementary volume of a deformable medium (continuum), one finds that the Cauchy stresses do not balance each other. The stress tensor becomes asymmetric in this case since additional stresses occur due to taking into account the deformation properties of the vortex velocities of elementary fluid volumes [9–11]. These media are currently termed micropolar [9–14]. As applied to

elastic bodies, media with additional tangential stresses were first described in [15]. It can be stated that micropolar fluids began to be studied as late as in the mid-1960s [3,4].

In [4], not only were the Navier–Stokes Equations derived to describe fluids with a representative volume having six degrees of freedom but also the first exact solutions were constructed and studied. Since [4] was published, steady-state and non-steady-state flows of micropolar viscous incompressible fluids have been studied in an exact formulation for unidirectional flows, e.g., in [4,8,16,17]. The exact solutions for Newtonian fluids were extended to micropolar media for Couette flows, the first and second Stokes problems, the Poiseuille flow, and their combinations and modifications.

There still appears to be no publications discussing classes of exact solutions to the Navier–Stokes Equations for shearing and three-dimensional micropolar viscous incompressible fluids. In this paper, the Lin–Sidorov–Aristov family found in [18–20] is chosen as a basis for constructing exact solutions. This class prescribes functional variable separation for the velocity field described by linear forms with respect to two (horizontal or longitudinal) coordinates $x$ and $y$. The coefficients of these forms depend on the third coordinate (vertical or transverse) $z$ and time $t$. In the Lin–Sidorov–Aristov class, pressure is a quadratic form similar to the velocity field. Various extensions and modifications of the Lin–Sidorov–Aristov family can be found in [21–25]. It was shown in [21,22] that the exact solution for the velocity field can nonlinearly depend on two coordinates.

In this paper, in view of the relevance of the research and on account of the insufficient completeness of the exact integration of micropolar fluid motion equations, we construct new exact solutions to the generalized Navier–Stokes Equations for unidirectional, shearing, and three-dimensional flows.

## 2. The Navier–Stokes Equations with Couple Stresses

The incompressible Navier–Stokes Equations with tangential couple stresses are written as follows [3,4]:

$$\frac{d\boldsymbol{V}}{dt} = -\nabla P + \nu \Delta \boldsymbol{V} - \mu \Delta^2 \boldsymbol{V},$$

$$\nabla \cdot \boldsymbol{V} = 0,$$

(1)

where $\boldsymbol{V}(t,x,y,z) = (V_x, V_y, V_z)$ is the velocity vector; $P = p/\rho_0$ is the pressure normalized to the fluid density $\rho_0$; $\nu$ is kinematic viscosity ($\nu > 0$); $\mu$ is the couple stress viscosity parameter ($\mu > 0$); the symbols $\nabla$, $\Delta$, and $\Delta^2$ denote, respectively, the Hamilton operator, the Laplace operator, and the biharmonic operator:

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

$$\Delta^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + 2\frac{\partial^4}{\partial x^2 \partial z^2} + 2\frac{\partial^4}{\partial y^2 \partial z^2};$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \nabla)$ is the Lagrangian derivative; $\frac{\partial}{\partial t}$ is the local derivative, $(\boldsymbol{V} \cdot \nabla)$ is the convective derivative [1]. The contribution of the couple stresses $-\mu \Delta^2 \boldsymbol{V}$ is due to the possibility of fluid particles to have rotational interaction (the representative fluid volume has rotational degrees of freedom) [3,4,13,16,17,26–29]. In the limit case $\mu = 0$, system (1) passes to the standard Navier–Stokes Equations, describing flows of a Newtonian fluid.

Note that model (1) can be considered a suitable $\mu$-approximation of the Navier–Stokes Equations [30]. Ladyzhenskaya [31] established the global unique solvability of an initial boundary value problem for (1) with the following boundary conditions:

$$\boldsymbol{V}|_\Gamma = \boldsymbol{0}, \qquad \Delta \boldsymbol{V} \times \mathbf{n}|_\Gamma = \boldsymbol{0},$$

where $\Gamma$ is the boundary of the flow domain and $\mathbf{n}$ denotes the exterior unit normal to the surface $\Gamma$.

We write system (1) in coordinate form as follows:

$$\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} = -\frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) -$$
$$- \mu \left( \frac{\partial^4 V_x}{\partial x^4} + \frac{\partial^4 V_x}{\partial y^4} + \frac{\partial^4 V_x}{\partial z^4} + 2 \frac{\partial^4 V_x}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 V_x}{\partial x^2 \partial z^2} + 2 \frac{\partial^4 V_x}{\partial y^2 \partial z^2} \right), \quad (2)$$

$$\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} = -\frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right) -$$
$$- \mu \left( \frac{\partial^4 V_y}{\partial x^4} + \frac{\partial^4 V_y}{\partial y^4} + \frac{\partial^4 V_y}{\partial z^4} + 2 \frac{\partial^4 V_y}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 V_y}{\partial x^2 \partial z^2} + 2 \frac{\partial^4 V_y}{\partial y^2 \partial z^2} \right), \quad (3)$$

$$\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} = -\frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right) -$$
$$- \mu \left( \frac{\partial^4 V_z}{\partial x^4} + \frac{\partial^4 V_z}{\partial y^4} + \frac{\partial^4 V_z}{\partial z^4} + 2 \frac{\partial^4 V_z}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 V_z}{\partial x^2 \partial z^2} + 2 \frac{\partial^4 V_z}{\partial y^2 \partial z^2} \right), \quad (4)$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0. \quad (5)$$

System (2)–(5) consists of four scalar equations for the determination of pressure $P$ and three velocity vector projections $V_x$, $V_y$, and $V_z$. The integration of this system is complicated due to quadratic nonlinearity and the presence of the fourth derivatives of the velocity vector projections $V_x$, $V_y$, $V_z$ involved in the generalized Navier–Stokes Equations (2)–(4).

## 3. Unidirectional Flows

In order to begin constructing exact solutions of system (2)–(5), we model unidirectional flows, i.e., we consider fluid flows with the velocity defined as follows:

$$\boldsymbol{V} = (V_x(x, y, z, t), 0, 0). \quad (6)$$

The velocity field (6) satisfies the incompressibility condition (5) if the following condition

$$\frac{\partial V_x}{\partial x} = 0$$

holds. Therefore $V_x$ depends only on two spatial coordinates and time, i.e.,

$$V_x = V_x(y, z, t). \quad (7)$$

By substituting Equation (6) into (3) and (4), we arrive at the conclusion that the pressure field is determined by only one spatial coordinate and time:

$$P = P(x, t).$$

In view of (7), Equation (2) also becomes simplified:

$$\frac{\partial V_x}{\partial t} = -\frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) - \mu \left( \frac{\partial^4 V_x}{\partial y^4} + \frac{\partial^4 V_x}{\partial z^4} + 2 \frac{\partial^4 V_x}{\partial y^2 \partial z^2} \right). \quad (8)$$

If the function $P$ is independent of the coordinate $x$ but determined only by the time $t$ (i.e., $P = P(t)$), then the first term in the right-hand side of (8) vanishes:

$$\frac{\partial V_x}{\partial t} = \nu \left( \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right) - \mu \left( \frac{\partial^4 V_x}{\partial y^4} + \frac{\partial^4 V_x}{\partial z^4} + 2 \frac{\partial^4 V_x}{\partial y^2 \partial z^2} \right). \tag{9}$$

Herewith, formally, the flow under study is not isobaric since the pressure is time dependent: $P = P(t)$.

Note that, due to the structure of Equation (9), steady-state flows $\boldsymbol{V} = (V_x(y, z), 0, 0)$ (as distinct from the classical Couette flow) depend on viscosity and velocity $V_x$, in general terms, is not a harmonic function,

$$\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} = \frac{\mu}{\nu} \left( \frac{\partial^4 V_x}{\partial y^4} + \frac{\partial^4 V_x}{\partial z^4} + 2 \frac{\partial^4 V_x}{\partial y^2 \partial z^2} \right) \neq 0.$$

In what follows, we shall present new exact solutions of Equation (9), which also satisfy the generalized Navier–Stokes Equations (1).

Let us now construct an extension of the classical Couette solution found in [32] to the case of couple stresses taken into account, i.e., consider a solution of the following form

$$V_x = U(z, t). \tag{10}$$

Note that the models for describing Couette flow (and some other flows) with couple stresses taken into account were discussed, e.g., in [4].

The substitution of Equation (10) into (9) produces the following heat-conduction-type equation

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial z^2} - \mu \frac{\partial^4 U}{\partial z^4}. \tag{11}$$

If herewith the flow represented by Equation (10) is steady-state ($V_x = U(z)$), the solution for velocity $U$ is a combination of linear and exponential functions:

$$U = C_1 z + C_2 + C_3 e^{z\sqrt{\nu/\mu}} + C_4 e^{-z\sqrt{\nu/\mu}}. \tag{12}$$

Moreover, the exponential terms in Equation (12) appear only when couple stresses are taken into account. If we solve Equation (11) for $\mu = 0$, the solution acquires the form of the classical Couette solution, where the velocity profile is defined by the linear function

$$U = C_1 z + C_2.$$

Here, $C_i$ denote integration constants.

Consider another particular exact solution of Equation (9)

$$V_x = y u_1(z, t). \tag{13}$$

By substituting (13) into Equation (9), we obtain

$$y \frac{\partial u_1}{\partial t} = y \left( \nu \frac{\partial^2 u_1}{\partial z^2} - \mu \frac{\partial^4 u_1}{\partial z^4} \right).$$

That is, the function $u_1$ satisfies Equation (11) and, hence, its solution can be represented as Equation (12). It can be easily shown that in view of the linearity of Equation (9), the linear combination of the solutions represented by Equations (10) and (13)

$$V_x = U(z, t) + y u_1(z, t) \tag{14}$$

also satisfies Equation (9).

We now add a nonlinear (in the $y$ coordinate) term to (14), i.e., consider a solution of the following form

$$V_x = U(z,t) + yu_1(z,t) + \frac{y^2}{2}u_2(z,t). \tag{15}$$

Substitute the last sum into (9). Some simple transformations yield

$$\frac{\partial U}{\partial t} + y\frac{\partial u_1}{\partial t} + \frac{y^2}{2}\frac{\partial u_2}{\partial t} = \nu\left(u_2 + \frac{\partial^2 U}{\partial z^2} + y\frac{\partial^2 u_1}{\partial z^2} + \frac{y^2}{2}\frac{\partial^2 u_2}{\partial z^2}\right) -$$
$$- \mu\left(\frac{\partial^4 U}{\partial z^4} + y\frac{\partial^4 u_1}{\partial z^4} + \frac{y^2}{2}\frac{\partial^4 u_2}{\partial z^4}\right).$$

Applying the method of undetermined coefficients to this equation, we obtain the following system of equations:

$$\frac{\partial U}{\partial t} = \nu\left(u_2 + \frac{\partial^2 U}{\partial z^2}\right) - \mu\frac{\partial^4 U}{\partial z^4},$$
$$\frac{\partial u_1}{\partial t} = \nu\frac{\partial^2 u_1}{\partial z^2} - \mu\frac{\partial^4 u_1}{\partial z^4},$$
$$\frac{\partial u_2}{\partial t} = \nu\frac{\partial^2 u_2}{\partial z^2} - \mu\frac{\partial^4 u_2}{\partial z^4}.$$

The latter two equations in this system are equations of the form of (11) and the equation for determining the homogeneous term $U$ in (15) is not isolated any longer. Thus, the consideration of nonlinearity in the variable results in the fact that the solution represented by (15) stops being a superposition of the previously presented solutions. As the power at the $y$ coordinate increases, this tendency holds true. In other words, a unidirectional flow is described by the following exact solution

$$V_x = U(z,t) + \sum_{k=1}^{n} \frac{y^k}{k!}u_k(z,t).$$

This exact solution, as all those presented in what follows, need to be studied for stability. However, studying for stability questions is not the aim of this paper.

Note that unidirectional flows of non-Newtonian fluids have been studied in many works (see, e.g., [33–36] and the references therein).

## 4. Exact Solutions for Three-Dimensional Flows

In order to study the properties of three-dimensional flows of viscous fluids, one needs to have a reserve of exact solutions to the Navier–Stokes Equations. The exact solutions inheriting nonlinear properties of the motion equations help to understand better the equations structure in further analytical and numerical integrations [23–25,37–44].

As a preliminary, the exact solution of the system of nonlinear partial Equations (2)–(5) is sought within the Lin–Sidorov–Aristov family [18–20]:

$$\begin{aligned}
V_x(x,y,z,t) &= U(z,t) + xu_1(z,t) + yu_2(z,t),\\
V_y(x,y,z,t) &= V(z,t) + xv_1(z,t) + yv_2(z,t),\\
V_z(z,t) &= w(z,t).
\end{aligned} \tag{16}$$

Note that the expressions in Equation (16), they can be treated as the Taylor series expansion of the velocity vector components that is confined to linear terms [37].

We substitute the class (16) into (2) and immediately ignore the terms containing the second and fourth derivatives with respect to the $x$ and $y$ variables since the expressions in Equation (16) linearly depend on these coordinates; this gives

$$\frac{\partial(U + xu_1 + yu_2)}{\partial t} + (U + xu_1 + yu_2)\frac{\partial(U + xu_1 + yu_2)}{\partial x} +$$
$$+ (V + xv_1 + yv_2)\frac{\partial(U + xu_1 + yu_2)}{\partial y} + w\frac{\partial(U + xu_1 + yu_2)}{\partial z} =$$
$$= -\frac{\partial P}{\partial x} + \nu\frac{\partial^2(U + xu_1 + yu_2)}{\partial z^2} - \mu\frac{\partial^4(U + xu_1 + yu_2)}{\partial z^4}.$$

We now make the following simple transformations in this equation by computing the partial derivatives and combining similar terms at the identical powers of the independent variables $x$ and $y$:

$$\left(\frac{\partial U}{\partial t} + Uu_1 + Vu_2 + w\frac{\partial U}{\partial z}\right) + x\left(\frac{\partial u_1}{\partial t} + u_1^2 + v_1 u_2 + w\frac{\partial u_1}{\partial z}\right) +$$
$$+ y\left(\frac{\partial u_2}{\partial t} + u_1 u_2 + v_2 u_2 + w\frac{\partial u_2}{\partial z}\right) = -\frac{\partial P}{\partial x} + \left(\nu\frac{\partial^2 U}{\partial z^2} - \mu\frac{\partial^4 U}{\partial z^4}\right) + \tag{17}$$
$$+ x\left(\nu\frac{\partial^2 u_1}{\partial z^2} - \mu\frac{\partial^4 u_1}{\partial z^4}\right) + y\left(\nu\frac{\partial^2 u_2}{\partial z^2} - \mu\frac{\partial^4 u_2}{\partial z^4}\right).$$

The structure of the obtained expression suggests that the pressure should be viewed as a quadratic form of the $x$ and $y$ variables. In other words, we have

$$P(x, y, z, t) = P_0(z, t) + xP_1(z, t) + yP_2(z, t) +$$
$$+ \frac{x^2}{2}P_{11}(z, t) + xyP_{12}(z, t) + \frac{y^2}{2}P_{22}(z, t). \tag{18}$$

Taking into account (18), when the method of undetermined coefficients is applied, Equation (17) splits into the following three equations:

$$\frac{\partial U}{\partial t} + Uu_1 + Vu_2 + w\frac{\partial U}{\partial z} = -P_1 + \nu\frac{\partial^2 U}{\partial z^2} - \mu\frac{\partial^4 U}{\partial z^4},$$
$$\frac{\partial u_1}{\partial t} + u_1^2 + v_1 u_2 + w\frac{\partial u_1}{\partial z} = -P_{11} + \nu\frac{\partial^2 u_1}{\partial z^2} - \mu\frac{\partial^4 u_1}{\partial z^4}, \tag{19}$$
$$\frac{\partial u_2}{\partial t} + u_1 u_2 + v_2 u_2 + w\frac{\partial u_2}{\partial z} = -P_{12} + \nu\frac{\partial^2 u_2}{\partial z^2} - \mu\frac{\partial^4 u_2}{\partial z^4}.$$

Similar operations with the substitution of Equations (16) and (18) into the other two equations of system (2)–(4) yield, respectively, the following two systems:

$$\frac{\partial V}{\partial t} + Uv_1 + Vv_2 + w\frac{\partial V}{\partial z} = -P_2 + \nu\frac{\partial^2 V}{\partial z^2} - \mu\frac{\partial^4 V}{\partial z^4},$$
$$\frac{\partial v_1}{\partial t} + u_1 v_1 + v_1 v_2 + w\frac{\partial v_1}{\partial z} = -P_{12} + \nu\frac{\partial^2 v_1}{\partial z^2} - \mu\frac{\partial^4 v_1}{\partial z^4}, \tag{20}$$
$$\frac{\partial v_2}{\partial t} + u_2 v_1 + v_2^2 + w\frac{\partial v_2}{\partial z} = -P_{22} + \nu\frac{\partial^2 v_2}{\partial z^2} - \mu\frac{\partial^4 v_2}{\partial z^4};$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{\partial P_0}{\partial z} + \nu\frac{\partial^2 w}{\partial z^2} - \mu\frac{\partial^4 w}{\partial z^4},$$
$$\frac{\partial P_1}{\partial z} = 0, \quad \frac{\partial P_2}{\partial z} = 0, \quad \frac{\partial P_{11}}{\partial z} = 0, \quad \frac{\partial P_{12}}{\partial z} = 0, \quad \frac{\partial P_{22}}{\partial z} = 0. \tag{21}$$

Note that the last equations in system (21) indicate that the overwhelming majority of the coefficients in Equation (18) are independent of the vertical coordinate; consequently, we gain a possibility of correcting the structure of the pressure field:

$$P(x,y,z,t) = P_0(z,t) + xP_1(t) + yP_2(t) + \frac{x^2}{2}P_{11}(t) + xyP_{12}(t) + \frac{y^2}{2}P_{22}(t). \quad (22)$$

Herewith, the homogeneous component of this field (background pressure $P_0$) is determined by integrating the first equation of system (21) up to the additive function of time,

$$P_0 = -\int \frac{\partial w}{\partial t}\,dz - \frac{w^2}{2} + v\frac{\partial w}{\partial z} - \mu\frac{\partial^3 w}{\partial z^3} + c_0(t).$$

In addition to the equations contained in systems (19)–(21), the coefficients in Equation (16) must satisfy the following equation

$$u_1 + v_2 + \frac{\partial w}{\partial z} = 0. \quad (23)$$

Equation (23) is obtained from the substitution of class (16) into the incompressibility condition (5).

By analogy with the derivation of systems (19)–(21), other solutions with the velocity field arbitrarily dependent on the horizontal coordinates can be constructed [21,22]. For example, the following exact solution of system (2)–(4) is valid:

$$V_x = \sum_{k=0}^{n} U_k, \quad V_y = \sum_{k=0}^{n} V_k, \quad V_z = \sum_{k=0}^{n-1} W_k,$$

$$A = \sum_{k=0}^{n^2-n} A_k, \quad B = \sum_{k=0}^{n^2-n} B_k, \quad C = \sum_{k=0}^{n^2-n+1} C_k, \quad P = \sum_{k=0}^{n^2-n+1} P_k.$$

Here, the forms of $U_k$, $V_k$, $W_k$, $P_k$ are determined by the following expressions:

$$U_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i U_{i(k-i)}(z,t)x^i y^{k-i}, \quad V_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i V_{i(k-i)}(z,t)x^i y^{k-i},$$

$$W_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i W_{i(k-i)}(z,t)x^i y^{k-i}, \quad A_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i A_{i(k-i)}(z,t)x^i y^{k-i},$$

$$B_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i B_{i(k-i)}(z,t)x^i y^{k-i}, \quad C_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i C_{i(k-i)}(z,t)x^i y^{k-i}, \quad (24)$$

$$P_k = \frac{1}{k!}\sum_{i=0}^{k} C_k^i P_{i(k-i)}(z,t)x^i y^{k-i},$$

where $C_k^i = \frac{k!}{(k-i)!i!}$ is a binomial coefficient.

## 5. Exact Solutions for Shearing Flows

Let us now consider an important particular case of isothermal flows of viscous fluids defined by Equations (16)–(23), namely, shearing flows. To perform this, we assume that the vertical velocity $V_z$ is zero, i.e.,

$$V_z(z,t) = w(z,t) = 0.$$

Using this assumption, we derive from (19) and (20) the following two systems:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + u_1^2 + v_1 u_2 &= -P_{11} + v\frac{\partial^2 u_1}{\partial z^2} - \mu\frac{\partial^4 u_1}{\partial z^4}, \\
\frac{\partial u_2}{\partial t} + u_1 u_2 + v_2 u_2 &= -P_{12} + v\frac{\partial^2 u_2}{\partial z^2} - \mu\frac{\partial^4 u_2}{\partial z^4}, \\
\frac{\partial v_1}{\partial t} + u_1 v_1 + v_1 v_2 &= -P_{12} + v\frac{\partial^2 v_1}{\partial z^2} - \mu\frac{\partial^4 v_1}{\partial z^4}, \\
\frac{\partial v_2}{\partial t} + u_2 v_1 + v_2^2 &= -P_{22} + v\frac{\partial^2 v_2}{\partial z^2} - \mu\frac{\partial^4 v_2}{\partial z^4}
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
\frac{\partial U}{\partial t} + U u_1 + V u_2 &= -P_1 + v\frac{\partial^2 U}{\partial z^2} - \mu\frac{\partial^4 U}{\partial z^4}, \\
\frac{\partial V}{\partial t} + U v_1 + V v_2 &= -P_2 + v\frac{\partial^2 V}{\partial z^2} - \mu\frac{\partial^4 V}{\partial z^4}.
\end{aligned}
\tag{26}
$$

All the coefficients in Equation (22) for the pressure depend only on the time coordinate $t$. The scientific literature on hydrodynamics reports that shearing flows are used in oceanology, atmosphere physics, astrophysics, and other sciences studying large-scale flows [23–25,45,46]. Large-scale flows are not only fluid motions at the planetary or galactic scale. The class of large-scale fluxes encompasses all flows with geometric anisotropy (thin layer approximation). In this case, if one filters out surface waves and neglects free surface deformation, the hydrodynamics of large-scale flows (fluid motions in thin layers) are describable by shearing flows $\boldsymbol{V}(x, y, z, t) = (V_x, V_y, 0)$.

Note that, when shearing flows are considered ($V_z(z, t) = 0$), the problem of integrating the generalized Navier–Stokes Equations (2)–(5) becomes noticeably more complicated; namely, we obtain an overdetermined system since it follows from Equation (4) that the pressure function must be specified by boundary conditions. This is what motivates us to start from constructing exact solutions for three-dimensional and unidirectional fluid flows. In other words, it has been demonstrated that the intermediate reduction of some components of the velocity field to zero does not always simplify the problems; it may even complicate them significantly. In what follows, we demonstrate that there exists a nontrivial exact solution in the Lin–Sidorov–Aristov class for shearing flows of the form $(V_x(x, y, z, t), V_y(x, y, z, t), 0)$ by the mathematics developed in [23–25,47–49].

In the case of shearing flows, Equation (23) allows the number of unknowns in system (25) to be reduced by using the relation between the spatial accelerations:

$$
v_2 = -u_1.
\tag{27}
$$

Equation (23) follows from the incompressibility condition (5).

System (25) is transformed in view of (27) as follows:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} - v\frac{\partial^2 u_1}{\partial z^2} + \mu\frac{\partial^4 u_1}{\partial z^4} + u_1^2 + v_1 u_2 &= -P_{11}, \\
\frac{\partial u_1}{\partial t} - v\frac{\partial^2 u_1}{\partial z^2} + \mu\frac{\partial^4 u_1}{\partial z^4} - (u_1^2 + v_1 u_2) &= P_{22}, \\
\frac{\partial u_2}{\partial t} - v\frac{\partial^2 u_2}{\partial z^2} + \mu\frac{\partial^4 u_2}{\partial z^4} &= -P_{12}, \\
\frac{\partial v_1}{\partial t} - v\frac{\partial^2 v_1}{\partial z^2} + \mu\frac{\partial^4 v_1}{\partial z^4} &= -P_{12}.
\end{aligned}
\tag{28}
$$

Comparing the first two equations of this system with one another, we can conclude that they will be simultaneous only if the following condition

$$
u_1^2 + v_1 u_2 = -\frac{P_{11} + P_{22}}{2}.
\tag{29}
$$

holds.

Let us define a linear differential operator $\mathscr{L}_{\nu,\mu}$ by the following formula

$$\mathscr{L}_{\nu,\mu} = \frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial z^2} + \mu\frac{\partial^4}{\partial z^4}.$$

In view of the consistency condition (29), system (28) can be represented as follows:

$$\mathscr{L}_{\nu,\mu}u_1 = -\frac{P_{11} - P_{22}}{2}, \qquad \mathscr{L}_{\nu,\mu}u_2 = -P_{12}, \qquad \mathscr{L}_{\nu,\mu}v_1 = -P_{12}. \tag{30}$$

Similar results for isobaric two-dimensional and quasi-two-dimensional flows were obtained earlier (see, e.g., [22,47,50]), but the possibility of the rotational interaction of fluid particles was not then taken into account. Herewith, system (30) generalizes the results reported in [47–49] and it can be reduced to them by passing to the limit as $\mu \to 0$ in the expression for the operator $\mathscr{L}_{\nu,\mu}$:

$$\mathscr{L}_{\nu,0} = \frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial z^2}.$$

Note that the consistency condition (29) for the overdetermined system of Equations (25)–(27) can be obtained on other grounds. Let us write the Navier–Stokes equations (2)–(4) for shearing flows as follows:

$$\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) - $$
$$- \mu\left(\frac{\partial^4 V_x}{\partial x^4} + \frac{\partial^4 V_x}{\partial y^4} + \frac{\partial^4 V_x}{\partial z^4} + 2\frac{\partial^4 V_x}{\partial x^2\partial y^2} + 2\frac{\partial^4 V_x}{\partial x^2\partial z^2} + 2\frac{\partial^4 V_x}{\partial y^2\partial z^2}\right), \tag{31}$$

$$\frac{\partial V_y}{\partial t} + V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} = -\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right) - $$
$$- \mu\left(\frac{\partial^4 V_y}{\partial x^4} + \frac{\partial^4 V_y}{\partial y^4} + \frac{\partial^4 V_y}{\partial z^4} + 2\frac{\partial^4 V_y}{\partial x^2\partial y^2} + 2\frac{\partial^4 V_y}{\partial x^2\partial z^2} + 2\frac{\partial^4 V_y}{\partial y^2\partial z^2}\right). \tag{32}$$

These equations are closed by the incompressibility condition

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \tag{33}$$

We differentiate Equation (31) with respect to the variable $x$ and Equation (32) with respect to $y$ and add up the obtained expressions. Some simple transformations and the use of the incompressibility condition (33) result in the following relationship (the consistency condition):

$$\frac{\partial V_x}{\partial y}\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial x}\frac{\partial V_y}{\partial y} = -\frac{1}{2}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right). \tag{34}$$

The particular case of the condition represented by Equation (34) for isobaric flows was obtained in [22].

Substituting expressions (16) and (18) into Equation (34), we obtain

$$\frac{\partial(U + xu_1 + yu_2)}{\partial y} \cdot \frac{\partial(V + xv_1 + yv_2)}{\partial x} - \frac{\partial(U + xu_1 + yu_2)}{\partial x} \cdot \frac{\partial(V + xv_1 + yv_2)}{\partial y} =$$

$$= -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(P_0 + xP_1 + yP_2 + \frac{x^2}{2}P_{11} + xyP_{12} + \frac{y^2}{2}P_{22}\right).$$

Elementary transformations for calculating partial derivatives produce the following relation

$$u_2 v_1 - u_1 v_2 = -\frac{P_{11} + P_{22}}{2}.$$

Finally, by substituting Equation (27) into the latter equation, we arrive at the consistency condition (29).

By analogy with the results reported in [22], the class of exact solutions (16) for system (19)–(21) can be expanded. Note the following velocity field

$$V_x^{(i)} = \sum_{k=0}^{n} U_k^{(i)}(z, t)\frac{y^k}{k!}, \quad V_y^{(i)} = V^{(i)}(z, t)$$

and the pressure field

$$P(x, y, z, t) = P_0(z, t) + xP_1(t) + yP_2(t) + \frac{x^2}{2}P_{11}(t) + xyP_{12}(t) + \frac{y^2}{2}P_{22}(t)$$

satisfy the reduced Navier–Stokes Equations system and the incompressibility condition, i.e., system (25)–(27). If a rotational transformation is performed for the coordinates and velocities according to the following rule:

$$x \longrightarrow x\cos\varphi - y\sin\varphi, \quad y \longrightarrow x\sin\varphi + y\cos\varphi,$$
$$V_x \longrightarrow V_x\cos\varphi - V_y\sin\varphi, \quad V_y \longrightarrow V_x\sin\varphi + V_y\cos\varphi,$$

then we obtain a family of exact solutions of the form (24) with a nonlinear dependence on two horizontal coordinates (the $x$ and $y$ coordinates).

## 6. Conclusions

The paper has presented families of exact solutions to the Navier–Stokes Equations for describing viscous fluid flows with rotational interaction of fluid particles taken into account. Models describing unidirectional vertical vortex flows have been separately discussed. The class proposed for such flows is characterized by a polynomial dependence of velocity on one of the horizontal coordinates. The polynomial can be of any order; the coefficients in the polynomial representation are arbitrarily related to the vertical coordinate and time. It has been demonstrated that quadratic and higher-power polynomial solutions cannot be obtained by superposition of lower-power solutions.

The solution families for describing three-dimensional flows have been considered. The Lin–Sidorov–Aristov family of exact solutions is chosen as the basic class and extended to the case of arbitrary-power dependences of the velocity field on the horizontal coordinates. The case of shearing flows has been separately discussed. It has been shown that the system of constitutive relations for spatial accelerations is reducible to a system of operator equations for which the solutions must satisfy a consistency condition of a special form. The consistency condition itself is derived in several ways.

It has also been shown that the proposed classes are a generalization of our previous results. Moreover, in the passage to the limit, which enables one to ignore the possibility of rotational interaction among fluid particles, these classes coincide with previously published results.

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