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# Voronovskaja-Type Quantitative Results for Differences of Positive Linear Operators

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**Abstract:** We consider positive linear operators having the same fundamental functions and different functionals in front of them. For differences involving such operators, we obtain Voronovskaja-type quantitative results. Applications illustrating the theoretical aspects are presented.

**Keywords:** positive linear operators; Voronovskaja-type result; Bernstein operators; Kantorovich operators



**Citation:** Acu, A.M.; Başcanbaz-Tunca, G.; Raşa, I. Voronovskaja-Type Quantitative Results for Differences of Positive Linear Operators. *Symmetry* **2021**, *13*, 1392. https://doi.org/10.3390/sym13081392

Academic Editor: Vyacheslav Yukalov

Received: 15 June 2021
Accepted: 19 July 2021
Published: 31 July 2021



## 1. Introduction

The problem of studying the differences of positive linear operators was formulated firstly by Lupaş [1]. In particular, he was interested in the commutator $(A, B) := AB - BA$, due to its property of antisymmetry. Generally speaking, there are two approaches to estimate the difference of two positive linear operators. In this context, there are two approaches to estimate the difference of two operators. One of them deals with operators that have the same moments up to a certain order. For detailed historical background, we refer to the work of Acu et al. [2] and the references therein. The other approach considers those operators that have the same fundamental functions and different functionals in their construction (see [2,3]). In the second perspective, the *discrete operator associated with an integral operator* has important role in the study. Raşa [4] noticed the advantages of the discrete operators associated with certain integral operators in this area. In this sense, it is helpful to mention the work of Heilmann et al. [5] from which the notion of *discrete operator* is reproduced below [2]:

Let $I \subset \mathbb{R}$ denote an interval and $H$ be a subspace of $C(I)$ containing the monomials $e_i(x) = x^i,\ i = 0, 1, 2$. Consider a positive linear operator $L : H \to C(I)$ satisfying $Le_0 = e_0$ given by

$$Lf := \sum_{k=0}^{\infty} F_k(f) p_k,$$

where $F_k : H \to \mathbb{R}$ are positive linear functionals satisfying $F_k(e_0) = 1$, and $p_k \in C(I)$ are the fundamental functions such that $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = e_0$. The discrete operator associated with $L$ is denoted by $D$ and defined as

$$D : H \to C(I), \quad Df := \sum_{k=0}^{\infty} f\left(b^{F_k}\right) p_k,$$

where $b^{F_k} := F_k(e_1)$. Namely, the functional in the construction of the discrete operator is the point evaluation at $b_k$, which is obviously simpler than the functional $F_k$ of the corresponding operator $L$. Therefore, it is easier to work with the discrete operator associated with $L$. In [3], some useful estimates for the differences of certain positive linear operators with the same fundamental functions were studied.

In the present note, we study the difference of positive linear operators, with the same fundamental functions, by obtaining Voronovskaja-type quantitative estimates.

## 2. Preliminaries

Throughout the paper, we shall adopt the same notation of [3]. Thus, $E(I)$ will denote a space of real valued and continuous functions defined on $I$ containing the polynomials, and $E_b(I)$ will denote the space of all functions $f$ from $E(I)$ having

$$\|f\| := \sup_{x\in I}|f(x)| < \infty.$$

For a positive linear functional $F$ satisfying $F(e_0) = 1$, the following expressions will be used:

$$b^F := F(e_1) \text{ and } \mu_i^F := F\left(e_1 - b^F e_0\right)^i, \ i \in \mathbb{N}. \tag{1}$$

Obviously, we have $\mu_0^F = 1$, $\mu_1^F = 0$ and $\mu_2^F = F(e_2) - \left(b^F\right)^2 \geq 0$. Moreover, for convenience, we adopt the notation

$$\mu_i^F(x) := F(e_1 - xe_0)^i, \ x \in I. \tag{2}$$

Thus, since the functional $F$ is linear, one has

$$\begin{aligned} \mu_2^F(x) &= F(e_2) - 2xF(e_1) + x^2 \\ &= F(e_2) - (F(e_1))^2 + (F(e_1))^2 - 2xF(e_1) + x^2 \\ &= \mu_2^F + \left(b^F - x\right)^2. \end{aligned} \tag{3}$$

Recall that the remainder $R_2\left(f; b^F, .\right)$ of Taylor's formula is given by

$$\begin{aligned} R_2\left(f; b^F, x\right) &= f(x) - f\left(b^F\right) - f'\left(b^F\right)\left(x - b^F\right) - \frac{f''\left(b^F\right)}{2}\left(x - b^F\right)^2 \\ &= \frac{\left(x - b^F\right)^2}{2}\left(f''(\xi) - f''\left(b^F\right)\right), \end{aligned} \tag{4}$$

where $\xi$ is between $x$ and $b^F$. Therefore, since $\mu_1^F = 0$, one has

$$F(f) - f\left(b^F\right) - \frac{f''\left(b^F\right)}{2}\mu_2^F = F\left(R_2\left(f; b^F, \cdot\right)\right). \tag{5}$$

Using the fact that $|\xi - b^F| \leq |x - b^F|$, we have

$$|f''(\xi) - f''(b^F)| \leq \omega(f'', |x - b^F|). \tag{6}$$

Here $\omega(f,t) := \sup\{|f(x+h) - f(x)| : x, x+h \in [a,b], 0 \leq h \leq t\}$, $f \in C[a,b]$, $t \geq 0$ is the modulus of continuity of $f$.

Thus, for $\delta > 0$, it follows that

$$\left|R_2\left(f;b^F,x\right)\right| \leq \frac{\left(x-b^F\right)^2}{2}\omega\left(f'',\left|e_1-b^F\right|\right) \\ \leq \left((x-b^F)^2+\frac{(x-b^F)^4}{\delta^2}\right)\frac{\omega(f'',\delta)}{2}.$$

Therefore,

$$\left|F\left(R_2\left(f;b^F,\cdot\right)\right)\right| \leq F\left(\left(e_1-b^F e_0\right)^2+\frac{\left(e_1-b^F e_0\right)^4}{\delta^2}\right)\frac{\omega(f'',\delta)}{2} \\ = \left(\mu_2^F+\frac{\mu_4^F}{\delta^2}\right)\frac{\omega(f'',\delta)}{2}. \tag{7}$$

## 3. Main Result

As in [3], let $K$ denote a set of non-negative integers and

$$p_k \in C(I),\ p_k \geq 0,\ k \in K,$$

denote fundamental functions satisfying $\sum_{k\in K} p_k = e_0$. Let $F_k$ and $G_k$ be two positive linear functionals acting from $E(I)$ into $\mathbb{R}$ such that $F_k(e_0) = G_k(e_0) = 1$ for each $k \in K$. Moreover, let $D(I)$ denote the set of all $f \in E(I)$ such that $\sum_{k\in K} F_k(f)p_k$ and $\sum_{k\in K} G_k(f)p_k$ belong to the space $C(I)$. Now, we deal with the positive linear operators $U$ and $V$, acting from $D(I)$ to $C(I)$, given by

$$U(f;x) = \sum_{k\in K} F_k(f)p_k(x) \text{ and } V(f;x) = \sum_{k\in K} G_k(f)p_k(x).$$

Let $D_U$ and $D_V$ denote the discrete operators associated with $U$ and $V$, which are given by

$$D_U(f;x) = \sum_{k\in K} f\left(b^{F_k}\right)p_k(x) \text{ and } D_V(f;x) = \sum_{k\in K} f\left(b^{G_k}\right)p_k(x),$$

respectively. For future correspondence, we denote

$$\sigma(x) := \sum_{k\in K}\left(\mu_2^{F_k}+\mu_2^{G_k}\right)p_k(x), \quad \gamma(x) := \sum_{k\in K}\left(\mu_4^{F_k}+\mu_4^{G_k}\right)p_k(x) \tag{8}$$

and

$$\delta(x) := \sum_{k\in K}\left[\left(b^{F_k}-x\right)^2\mu_2^{F_k}(x)+\left(b^{G_k}-x\right)^2\mu_2^{G_k}(x)\right]p_k(x),\ \ x\in I. \tag{9}$$

Moreover, from (2), the $i$th central moment of each operator can be written as

$$U\left((e_1-xe_0)^i;x\right) = \sum_{k\in K}\mu_i^{F_k}(x)p_k(x), \\ V\left((e_1-xe_0)^i;x\right) = \sum_{k\in K}\mu_i^{G_k}(x)p_k(x),\ i\in\mathbb{N}.$$

In [3], the authors measured the distance $|U(f;x)-V(f;x)|$ using properties of the associated discrete operators. Specifically, they obtained the following result:

**Theorem 1.** *Let $f \in D(I)$ with $f'' \in E_b(I)$. Then,*

$$|(U-V)(f;x)| \leq \|f''\|\sigma(x)+\omega(f,t),$$

*where $\sigma(x)$ is given by (8) and $t := \sup_{k\in K}\left|b^{F_k} - b^{G_k}\right|$ (see [3] [Theorem 3]).*

A natural question arising here is to estimate the difference of positive linear operators in the sense of Videnskiĭ who stated the well-known result of Voronovskaja [6] for the Bernstein operators in the following quantitative form.

**Theorem 2** ([7])**.** *If $f \in C^2[0,1]$, then one has*

$$\left|n[B_n(f;x) - f(x)] - \frac{x(1-x)}{2}f''(x)\right| \le x(1-x)\omega\left(f'', \frac{2}{\sqrt{n}}\right),$$

*where $\omega(f'', .)$ is the modulus of continuity of $f''$.*

In this context, we give an expression for the difference of a positive linear operator and its discrete operator.

**Lemma 1.** *Let $x$ be an arbitrary point in $I$ and $f'' \in E_b(I)$. Then, we have*

$$\begin{aligned}
&(U - D_U)(f;x) - \sum_{p=1}^{2} \frac{f^{(p)}(x)}{p!}(U - D_U)\left((e_1 - xe_0)^p; x\right) \\
=\ & \sum_{k\in K}\left\{F\left(R_2\left(f; b^{F_k}, \cdot\right)\right) + \frac{1}{2}\left[f''\left(b^{F_k}\right) - f''(x)\right]\mu_2^{F_k}\right\}p_k(x),
\end{aligned}$$

*where $R_2\left(f; b^F, \cdot\right)$ is the remainder of Taylor's formula given by (4).*

**Proof.** Let $x \in I$ be a given point. Then, from (5), it readily follows that

$$\begin{aligned}
&(U - D_U)(f;x) - \sum_{p=1}^{2} \frac{f^{(p)}(x)}{p!}(U - D_U)\left((e_1 - xe_0)^p; x\right) \\
=\ & \sum_{k\in K}\Big\{F_k(f) - f'(x)F_k(e_1 - xe_0) - \frac{f''(x)}{2}F_k(e_1 - xe_0)^2 \\
& -\left[f\left(b^{F_k}\right) - f'(x)\left(b^{F_k} - x\right) - \frac{f''(x)}{2}\left(b^{F_k} - x\right)^2\right]\Big\}p_k(x) \\
=\ & \sum_{k\in K} F_k(f) - f\left(b^{F_k}\right) - \frac{f''(x)}{2}\left(F_k(e_2) - \left(b^{F_k}\right)^2\right)p_k(x) \\
=\ & \sum_{k\in K}\left\{F_k(f) - f\left(b^{F_k}\right) - \frac{f''(x)}{2}\mu_2^{F_k}\right\}p_k(x) \\
=\ & \sum_{k\in K}\left\{F_k\left(R_2\left(f; b^{F_k}, \cdot\right)\right) + \frac{1}{2}\left[f''\left(b^{F_k}\right) - f''(x)\right]\mu_2^{F_k}\right\}p_k(x).
\end{aligned} \tag{10}$$

□

Now we present a quantitative Voronovskaja-type theorem for the difference $U - V$.

**Theorem 3.** *Let $f'' \in E_b(I)$. Then, we have*

$$\begin{aligned}
&\left|(U - V)(f;x) - \sum_{p=0}^{2} \frac{f^{(p)}(x)}{p!}(U - V)\left((e_1 - xe_0)^p; x\right)\right| \\
\le\ & \frac{\omega\left(f'', \sqrt{\gamma(x)}\right)}{2}[1 + \sigma(x)] \\
& + \frac{\omega\left(f'', \sqrt{\delta(x)}\right)}{2}\left[1 + (U + V)\left((e_1 - xe_0)^2; x\right)\right],
\end{aligned}$$

*where $\sigma(x)$, $\gamma(x)$, and $\delta(x)$ are given in (8) and (9), respectively.*

**Proof.** Let $x$ be an arbitrary fixed point in $I$. Using (5), we obtain

$$\begin{aligned}
&(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\left((e_1-xe_0)^p;x\right)\\
=\;&\sum_{k\in K}\left\{F_k(f)-G_k(f)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(F_k-G_k)(e_1-xe_0)^p\right\}p_k(x)\\
=\;&\sum_{k\in K}\left\{\left[F_k(f)-f\left(b^{F_k}\right)\right]-\left[G_k(f)-f\left(b^{G_k}\right)\right]-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(F_k-G_k)(e_1-xe_0)^p\right\}p_k(x)\\
&+\sum_{k\in K}\left[f\left(b^{F_k}\right)-f\left(b^{G_k}\right)\right]p_k(x).
\end{aligned}$$

The above formula can be expressed as

$$\begin{aligned}
&(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\left((e_1-xe_0)^p;x\right)\\
=\;&\sum_{k\in K}\left\{\left[F_k(f)-f\left(b^{F_k}\right)\right]-\left[G_k(f)-f\left(b^{G_k}\right)\right]\right\}p_k(x)\\
&+\sum_{k\in K}\left\{\left[f\left(b^{F_k}\right)-f(x)-f'(x)F_k(e_1-xe_0)-\frac{f''(x)}{2}F_k(e_1-xe_0)^2\right]\right.\\
&\left.-\left[f\left(b^{G_k}\right)-f(x)-f'(x)G_k(e_1-xe_0)-\frac{f''(x)}{2}G_k(e_1-xe_0)^2\right]\right\}p_k(x)\\
=\;&\sum_{k\in K}\left\{\left[F_k(f)-f\left(b^{F_k}\right)\right]-\left[G_k(f)-f\left(b^{G_k}\right)\right]\right\}p_k(x)\\
&+\sum_{k\in K}\left\{\left[f\left(b^{F_k}\right)-f(x)-f'(x)\left(b^{F_k}-x\right)-\frac{f''(x)}{2}\mu_2^{F_k}(x)\right]\right.\\
&\left.-\left[f\left(b^{G_k}\right)-f(x)-f'(x)\left(b^{G_k}-x\right)-\frac{f''(x)}{2}\mu_2^{G_k}(x)\right]\right\}p_k(x).
\end{aligned}$$

By using (3), the last formula can be written as

$$\begin{aligned}
&(U-V)(f;x)-\sum_{p=1}^{2}\tfrac{f^{(p)}(x)}{p!}(U-V)\left((e_1-xe_0)^p;x\right)\\
=\;&\sum_{k\in K}\left\{\left[F_k(f)-f\left(b^{F_k}\right)-\tfrac{f''(x)}{2}\mu_2^{F_k}\right]-\left[G_k(f)-f\left(b^{G_k}\right)-\tfrac{f''(x)}{2}\mu_2^{G_k}\right]\right\}p_k(x)\\
&+\sum_{k\in K}\left\{\left[f\left(b^{F_k}\right)-f(x)-f'(x)\left(b^{F_k}-x\right)-\tfrac{f''(x)}{2}\left(b^{F_k}-x\right)^2\right]\right.\\
&\left.-\left[f\left(b^{G_k}\right)-f(x)-f'(x)\left(b^{G_k}-x\right)-\tfrac{f''(x)}{2}\left(b^{G_k}-x\right)^2\right]\right\}p_k(x).
\end{aligned} \tag{11}$$

The term

$$f\left(b^F\right)-f(x)-f'(x)\left(b^F-x\right)-\frac{f''(x)}{2}\left(b^F-x\right)^2$$

in (11) is the remainder $R_2\left(f;x,b^F\right)$ of Taylor's formula for $x$ (fixed) and $b^F$, given by

$$R_2\left(f;x,b^F\right)=\frac{\left(b^F-x\right)^2}{2}\left(f''(\xi)-f''(x)\right),$$

where $\xi$ is a point between $x$ and $b^F$. Therefore, we have

$$\left|R_2\left(f;x,b^F\right)\right|\le\frac{\left(b^F-x\right)^2}{2}\omega\left(f'',\left|b^F-x\right|\right). \tag{12}$$

The formula (11) can be written as

$$
\begin{aligned}
&(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\big((e_1-xe_0)^p;x\big)\\
=\;&\sum_{k\in K}\Big[F_k\Big(R_2\Big(f;b^{F_k},\cdot\Big)\Big)-G_k\Big(R_2\Big(f;b^{G_k},\cdot\Big)\Big)\Big]p_k(x)\\
&+\frac{1}{2}\sum_{k\in K}\Big[\Big(f''\Big(b^{F_k}\Big)-f''(x)\Big)\mu_2^{F_k}-\Big(f''\Big(b^{G_k}\Big)-f''(x)\Big)\mu_2^{G_k}\Big]p_k(x)\\
&+\sum_{k\in K}\Big[R_2\Big(f;x,b^{F_k}\Big)-R_2\Big(f;x,b^{G_k}\Big)\Big]p_k(x).
\end{aligned}
$$

Taking into account (6), (7), and (12), we obtain

$$
\begin{aligned}
&\left|(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\big((e_1-xe_0)^p;x\big)\right|\\
\le\;&\frac{1}{2}\sum_{k\in K}\left[\left(\mu_2^{F_k}+\frac{\mu_4^{F_k}}{\gamma(x)}\right)+\left(\mu_2^{G_k}+\frac{\mu_4^{G_k}}{\gamma(x)}\right)\right]p_k(x)\omega(f'',\sqrt{\gamma(x)})\\
&+\frac{1}{2}\sum_{k\in K}\Big[\mu_2^{F_k}\omega\Big(f'',\Big|b^{F_k}-x\Big|\Big)+\mu_2^{G_k}\omega\Big(f'',\Big|b^{G_k}-x\Big|\Big)\Big]p_k(x)\\
&+\frac{1}{2}\sum_{k\in K}\Big[\Big(b^{F_k}-x\Big)^2\omega\Big(f'',\Big|b^{F_k}-x\Big|\Big)+\Big(b^{G_k}-x\Big)^2\omega\Big(f'',\Big|b^{G_k}-x\Big|\Big)\Big]p_k(x)\\
=\;&\frac{1}{2}\sum_{k\in K}\left[\left(\mu_2^{F_k}+\frac{\mu_4^{F_k}}{\gamma(x)}\right)+\left(\mu_2^{G_k}+\frac{\mu_4^{G_k}}{\gamma(x)}\right)\right]p_k(x)\omega(f'',\sqrt{\gamma(x)})\\
&+\frac{1}{2}\sum_{k\in K}\Big[F_k(e_1-xe_0)^2\omega\Big(f'',\Big|b^{F_k}-x\Big|\Big)+G_k(e_1-xe_0)^2\omega\Big(f'',\Big|b^{G_k}-x\Big|\Big)\Big]p_k(x).
\end{aligned}
$$

Since

$$
F(e_1-xe_0)^2\omega(f'',|b^F-x|)\le\mu_2^{F_k}(x)\left(1+\frac{(b^F-x)^2}{\delta(x)}\right)\omega\Big(f'',\sqrt{\delta(x)}\Big),
$$

we obtain

$$
\begin{aligned}
&\left|(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\big((e_1-xe_0)^p;x\big)\right|\\
\le\;&\frac{\omega\Big(f'',\sqrt{\gamma(x)}\Big)}{2}\sum_{k\in K}\left[\mu_2^{F_k}+\mu_2^{G_k}+\frac{1}{\gamma(x)}\Big(\mu_4^{F_k}+\mu_4^{G_k}\Big)\right]p_k(x)\\
&+\frac{\omega\Big(f'',\sqrt{\delta(x)}\Big)}{2}\sum_{k\in K}\left[\mu_2^{F_k}(x)+\mu_2^{G_k}(x)+\frac{\big(b^{F_k}-x\big)^2\mu_2^{F_k}(x)+\big(b^{G_k}-x\big)^2\mu_2^{G_k}(x)}{\delta(x)}\right]p_k(x).
\end{aligned}
$$

Using (8) and (9), the theorem is proved. □

## 4. Examples

### *4.1. Quantitative Voronovskaja-Type Result for the Differences of Bernstein Operators and Kantorovich Operators*

The well-known Bernstein operators $B_n : C[0,1] \to C[0,1],\ n \in \mathbb{N}$ are given by

$$
B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x),
$$

where the fundamental functions $p_{n,k}(x)$ are

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

The Kantorovich operators $K_n : L_1[0,1] \to C[0,1],\ n \in \mathbb{N}$ are defined as ([8])

$$K_n(f;x) = \sum_{k=0}^{n} \left( (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right) p_{n,k}(x).$$

In [3] [Proposition 8], as an application of Theorem 1, the authors expressed the difference between Bernstein and its Kantorovich variant as

$$|(K_n - B_n)(f;x)| \leq \frac{1}{24(n+1)^2} \|f''\| + \omega\left(f, \frac{1}{2(n+1)}\right), \ \ f'' \in C[0,1].$$

Now, we give an estimate of this difference with the help of Theorem 3.

**Proposition 1.** *Let $f'' \in C[0,1]$. Then, for Bernstein operators and Kantorovich operators we have*

$$\begin{aligned}
&\left| (K_n - B_n)(f;x) - \sum_{p=1}^{2} \frac{f^{(p)}(x)}{p!} (K_n - B_n)\left((e_1 - xe_0)^p; x\right) \right| \\
\leq\ & \frac{\omega\left(f'', \sqrt{\gamma(x)}\right)}{2} \left[ 1 + \frac{1}{12(n+1)^2} \right] \\
&+ \frac{\omega\left(f'', \sqrt{\delta(x)}\right)}{2} \left[ 1 + \frac{x(1-x)(2n^2+n+1)}{n(n+1)^2} + \frac{1}{3(n+1)^2} \right],
\end{aligned}$$

*where*

$$\gamma(x) = \frac{1}{80(n+1)^4},$$

*and*

$$\begin{aligned}
\delta(x) = &\frac{1}{12(n+1)^4} \Big\{ 36x^2(1-x)^2 n^2 + 5x(1-x)(48x^2 - 48x + 11)n \\
&+ (1-2x)^2(3x^2 - 3x + 1) \Big\}.
\end{aligned}$$

**Proof.** Denoting the functional of Bernstein operators by $F_k(f) := f\left(\frac{k}{n}\right)$ and Kantorovich operators by $G_k(f) = (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt$, we can express these operators as

$$B_n(f;x) = \sum_{k=0}^{n} F_k(f) p_{n,k}(x)$$

and

$$K_n(f;x) = \sum_{k=0}^{n} G_k(f) p_{n,k}(x).$$

Thus, according to (1), (8), and (9), we obtain

$$b^{F_k} = \frac{k}{n},\ \mu_2^{F_k} = 0,\ \mu_4^{F_k} = 0,$$
$$b^{G_k} = \frac{2k+1}{2(n+1)},\ \mu_2^{G_k} = \frac{1}{12(n+1)^2},\ \mu_4^{G_k} = \frac{1}{80(n+1)^4}$$
$$\sigma(x) = \frac{1}{12(n+1)^2},\quad \gamma(x) = \frac{1}{80(n+1)^4},$$

and

$$\begin{aligned}\delta(x) &= \sum_{k=0}^{n}\left[\left(\frac{k}{n}-x\right)^2\mu_2^{F_k}(x) + \left(\frac{2k+1}{2(n+1)}-x\right)^2\mu_2^{G_k}(x)\right]p_{n,k}(x)\\ &= \frac{1}{12(n+1)^4}\Big\{36x^2(1-x)^2n^2 + 5x(1-x)(48x^2-48x+11)n\\ &+(1-2x)^2(3x^2-3x+1)\Big\}.\end{aligned}$$

Moreover, making use of the well-known second central moments of Bernstein and Kantorovich operators given by

$$B_n\Big((e_1-xe_0)^2;x\Big) = \frac{x(1-x)}{n},\quad K_n\Big((e_1-xe_0)^2;x\Big) = \frac{1}{(n+1)^2}\Big(x(1-x)(n-1)+\frac{1}{3}\Big),$$

the proof follows from Theorem 3. □

*4.2. Quantitative Voronovskaja-Type Result for the Differences of Bernstein Operators and Genuine Bernstein–Durrmeyer Operators*

For $f \in C[0,1]$, the genuine Bernstein–Durrmeyer operators are defined as

$$U_n(f;x) = f(0)(1-x)^n + f(1)x^n + \sum_{k=1}^{n-1}\left((n-1)\int_0^1 f(t)p_{n-2,k-1}(t)dt\right)p_{n,k}(x).$$

In [3] [Proposition 4], as an application of Theorem 1, the authors expressed the difference between Bernstein and Bernstein–Durrmeyer operators as

$$|(U_n-B_n)(f;x)| \le \frac{x(1-x)(n-1)}{2n(n+1)}\|f''\|,\quad f'' \in C[0,1].$$

Below, we estimate this difference via the above quantitative Voronovskaja-type result.

**Proposition 2.** *Let $f'' \in C[0,1]$. Then, for Bernstein operators and genuine Bernstein–Durrmeyer operators we have*

$$\begin{aligned}&\left|(U_n-B_n)(f;x) - \sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U_n-B_n)\big((e_1-xe_0)^p;x\big)\right|\\ \le\ &\frac{\omega\Big(f'',\sqrt{\gamma'(x)}\Big)}{2}\left[1+\frac{x(1-x)(n-1)}{n(n+1)}\right]\\ &+\frac{\omega\Big(f'',\sqrt{\delta(x)}\Big)}{2}\left[1+\frac{x(1-x)(3n+1)}{n(n+1)}\right],\end{aligned}$$

*where*

$$\gamma(x) \le \gamma'(x) := \frac{x(1-x)3(n-1)}{4n(n+1)^2} \le \frac{1}{16}\frac{3}{n+1},$$

*and*

$$\delta(x) = \frac{2x(1-x)(2nx - 2nx^2 + 5x^2 - 5x + 1)}{n^2(n+1)},$$

*in which $\gamma(x)$ and $\delta(x)$ are given in (8) and (9), respectively.*

**Proof.** As in the previous proposition, denoting the functional of Bernstein operators by $F_k(f) := f\left(\frac{k}{n}\right)$ and genuine Bernstein–Durrmeyer operators by

$$G_k(f) = \begin{cases} (n-1)\int\limits_0^1 f(t)p_{n-2,k-1}(t)dt, & 1 \le k \le n-1 \\ f(0), & k = 0 \\ f(1), & k = 1 \end{cases},$$

we have $B_n(f;x) = \sum\limits_{k=0}^{n} F_k(f)p_{n,k}(x)$ and $U_n(f;x) = \sum\limits_{k=0}^{n} G_k(f)p_{n,k}(x)$. Hence, we obtain

$$\begin{aligned} b^{F_k} &= \frac{k}{n},\ \mu_2^{F_k} = \mu_4^{F_k} = 0, \\ b^{G_k} &= \frac{k}{n},\ \mu_2^{G_k} = \frac{k(n-k)}{n^2(n+1)},\ \mu_4^{G_k} = \frac{3k(k-n)\left[k(n-6)(k-n) - 2n^2\right]}{n^4(n+1)(n+2)(n+3)}, \\ \sigma(x) &= \frac{x(1-x)(n-1)}{n(n+1)}. \end{aligned}$$

In order to obtain simpler upper bounds, we can majorize $\gamma(x)$. For this, as in the proof of [3] [Proposition 5], we obtain

$$\begin{aligned} \mu_4^{G_k} &= \frac{\frac{k}{n}\left(1-\frac{k}{n}\right)\left[3(n-6)\frac{k}{n}\left(1-\frac{k}{n}\right)+6\right]}{(n+1)(n+2)(n+3)} \\ &\le \frac{1}{(n+1)(n+2)(n+3)}\frac{k}{n}\left(1-\frac{k}{n}\right)\left[\frac{3}{4}(n-6)+6\right] \\ &= \frac{1}{4(n+1)}\frac{k}{n}\left(1-\frac{k}{n}\right)\frac{3n+6}{(n+2)(n+3)} \\ &= \frac{3}{4(n+1)}\frac{k}{n}\left(1-\frac{k}{n}\right)\frac{1}{n+3} \\ &\le \frac{3}{4(n+1)^2}\frac{k}{n}\left(1-\frac{k}{n}\right). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \gamma(x) &= \sum_{k=0}^{n}\left(\mu_4^{F_k} + \mu_4^{G_k}\right)p_k(x) \\ &\le \frac{3}{4(n+1)^2}\sum_{k=0}^{n}\frac{k}{n}\left(1-\frac{k}{n}\right)\frac{k}{n}\left(1-\frac{k}{n}\right) \\ &= \frac{x(1-x)3(n-1)}{4n(n+1)^2} := \gamma'(x). \end{aligned}$$

From (3), we obtain

$$\delta(x) = \sum_{k=0}^{n}\left[\left(b^{F_k}-x\right)^2\mu_2^{F_k}(x)+\left(b^{G_k}-x\right)^2\mu_2^{G_k}(x)\right]p_{n,k}(x)$$
$$= \frac{2x(1-x)(2nx-2nx^2+5x^2-5x+1)}{n^2(n+1)}.$$

Again, making use of the second central moments of Bernstein and genuine Bernstein–Durrmeyer operators given by

$$B_n\left((e_1-xe_0)^2;x\right) = \frac{x(1-x)}{n},\;\; U_n\left((e_1-xe_0)^2;x\right) = \frac{2x(1-x)}{n+1},$$

the proof follows from Theorem 3. □

## 5. Conclusions

If we directly used the difference of the operators in the Voronovskaja setting, without taking into account the corresponding discrete operators, then we would obtain the following result.

Using the remainder of Taylor's formula again for $f'' \in C[0,1]$, with fixed $x$ and $t \in [0,1]$, given by

$$\begin{aligned} R_2(f;x,t) &= f(t)-f(x)-f'(x)(t-x)-\frac{f''(x)}{2}(t-x)^2 \\ &= \frac{(t-x)^2}{2}\left(f''(\xi)-f''(x)\right), \end{aligned}$$

where $\xi$ is a point between $x$ and $t$, we obtain that

$$\left|(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\left((e_1-xe_0)^p;x\right)\right|$$
$$\leq (U+V)\left(\frac{(e_1-xe_0)^2}{2}\omega\left(f'',\sqrt{|e_1-xe_0|}\right);x\right)$$

(see, e.g., [9] [1.7]). Therefore, we obtain

$$\left|(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\left((e_1-xe_0)^p;x\right)\right|$$
$$\leq \frac{\omega(f'',\delta)}{2}\left[(U+V)\left((e_1-xe_0)^2;x\right)+\frac{1}{\delta^2}(U+V)\left((e_1-xe_0)^4;x\right)\right],$$

from which, by choosing

$$\delta^2 = \delta^2(x) = (U+V)\left((e_1-xe_0)^4;x\right) \tag{13}$$

it readily follows that

$$\left|(U-V)(f;x)-\sum_{p=1}^{2}\frac{f^{(p)}(x)}{p!}(U-V)\left((e_1-xe_0)^p;x\right)\right|$$
$$\leq \frac{\omega(f'',\delta(x))}{2}\left[1+(U+V)\left((e_1-xe_0)^2;x\right)\right].$$

From this point of view, the argument of the modulus of continuity appearing in Theorem 3 is easier to evaluate than (13), since it is represented in terms of the point evaluation functional of the corresponding discrete operators, whereas in (13), the forth central moment of each operator must be calculated.

Differences of other pairs of operators will be considered in a forthcoming paper. The special case of the commutator AB-BA will be considered, due to its property of antisymmetry.

**Author Contributions:** The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** Project financed by Lucian Blaga University of Sibiu & Hasso Plattner Foundation research grants LBUS-IRG-2021-07.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest.

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