

Article

Quark Self-Energy and Condensates in NJL Model with External Magnetic Field

Juan Liu ¹, Yilun Du ²  and Song Shi ^{1,*} 

¹ Department of Physics, National University of Defense Technology, Changsha 410000, China; ljuan9@mail.ustc.edu.cn

² Department of Physics and Technology, University of Bergen, 5020 Bergen, Norway; Yilun.Du@uib.no

* Correspondence: shisong15@nudt.edu.cn

Abstract: In a one-flavor NJL model with a finite temperature, chemical potential, and external magnetic field, the self-energy of the quark propagator contains more condensates besides the vacuum condensate. We use Fierz identity to identify the self-energy and propose a self-consistent analysis to simplify it. It turns out that these condensates are related to the chiral separation effect and spin magnetic moment.

Keywords: NJL model; external magnetic field; chiral symmetry breaking; dynamical mass; gap equations; self-energy

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1. Introduction

The study of QCD phase transition is important in theoretical physics and high-energy physics [1–7]. It is believed that in collisions of two particles with high energy, QCD matters will be produced in which the behaviors of the particles are dominated by the strong interaction. At the first moment of collision, these newborn particles form quark-gluon plasma (QGP) and, as time elapses, this evolves into hadrons. If the collisions of charged particles are noncentral, the produced QCD matter will be accompanied by an extremely strong magnetic field [8]. This field could highly affect QCD matter; therefore, studying the properties of QCD matter with magnetic fields is meaningful and important. So far, an interesting effect that comes from the magnetic field is known as ‘Magnetic Catalysis’ [9–15], it shows that the quark condensate, which is widely accepted as the order parameter of the phase transition between QGP and hadrons, are strengthened by magnetic field. Following these works, the lattice QCD has shown a new effect where at some temperatures, on the contrary, a strong magnetic field weakens the condensate, which is called ‘Inverse Magnetic Catalysis’ [16].

For qualitatively studying the phase transition of QCD matters, the NJL model is a useful and convenient tool because one can easily describe the mechanism of chiral symmetry broken and derive the dynamical mass [1,2,17–25]. To utilize this model, we usually apply mean field approximation to tackle the four-fermion interaction terms in the Lagrangian—for example, $(\bar{\psi}\psi)^2 \rightarrow 2\langle\bar{\psi}\psi\rangle(\bar{\psi}\psi) - \langle\bar{\psi}\psi\rangle^2$, $(i\bar{\psi}\gamma^5\vec{\tau}\psi)^2 \rightarrow 2\langle i\bar{\psi}\gamma^5\vec{\tau}\psi\rangle(i\bar{\psi}\gamma^5\vec{\tau}\psi) - \langle i\bar{\psi}\gamma^5\vec{\tau}\psi\rangle^2$. It is believed that this approximation is equivalent to the Dyson–Schwinger equations with contact interaction treatment from QCD, and the gap equation can be written as

$$\frac{\Sigma}{G} \int d^4x = i \int d^4x \langle x | \gamma^\mu \hat{S} \gamma_\mu | x \rangle, \tag{1}$$

$$\hat{S}^{-1} = \not{p} - m - \Sigma, \quad \Sigma = \sigma + i\gamma^5 \vec{\pi} \cdot \vec{\tau}, \quad \sigma = -\frac{G}{N_c} \langle \bar{\psi}\psi \rangle, \quad \vec{\pi} = -\frac{G}{N_c} \langle i\bar{\psi}\gamma^5\vec{\tau}\psi \rangle.$$

In the above equation, Σ represents the self-energy of quark propagator and the dynamical mass σ contained in Σ is generated by a nonperturbative effect, which will dynamically cause the chiral symmetry to break in specific situations. It is clear that in Equation (1), $\vec{\pi} = 0$ leads to $\Sigma = \sigma$; therefore, in most papers, the dynamical mass is studied directly rather than discussing a general form of the self-energy. However, in M. Asakawa and K. Yazaki's work [26], the authors firstly pointed out that the self-energy does not simply equal σ when chemical potential is not zero, and the actual self-energy should be written as $\Sigma = \sigma + a\gamma^0$. One can also refer to Klevansky's work [27] for a similar discussion.

We notice in [28] that the authors had considered more four-fermion interaction terms in the NJL model with the presence of magnetic field. In this article, we will rigorously prove that the existence of other four-fermion interaction terms is a necessary consequent of the self-consistency of the gap equations when the chemical potential and external magnetic field are not zero. Different from [28], we have included chemical potential and considered the effect of all Landau levels rather than the lowest Landau level in this paper.

In this paper, we propose a self-consistent analysis to study the self-energy in a one-flavor NJL model. Through this analysis, we prove that the self-energy has more components than Asakawa's assumption with the presence of chemical potential and external magnetic field. The purpose of this article is not to study the dynamical mass and phase transition but the properties of the other components in the self-energy under the influence of temperature, chemical potential, and external magnetic field. The structure of this article is as follows: in Section 2, we establish the Lagrangian with finite temperature, chemical potential, and external magnetic field, then use the Fierz identity, mean-field approximation, and self-consistent analysis to identify the self-energy and derive the gap equations; in Section 3, we solve the gap equations, respectively, in the chiral symmetry broken phase and chiral symmetry restored phase; in Section 4, we present the conclusion.

It is worth mentioning that the dynamical properties of the one-flavor NJL model and one-flavor QCD could be different. In one-flavor QCD, the quantum anomalies will prevent the existence of chiral symmetry [29], then there will be no phase transition between the chiral symmetry phase and chiral breaking phase. However, this is not the case we consider in this paper; here, the one-flavor NJL model cannot be treated as a simplified model of one-flavor QCD, it should be seen as a simplified model of the two-flavor NJL model instead. We will come back to this problem in the conclusion.

2. The Gap Equations

2.1. NJL Model and Mean-Field Approximation

At finite temperature, one can write the Lagrangian of the one-flavor NJL model with quarks with nonzero chemical potential and external magnetic field as

$$\mathcal{L} = \bar{\psi} \hat{D} \psi + G(\bar{\psi} \psi)^2 + \mu \bar{\psi} \gamma^0 \psi, \quad (2)$$

$$\hat{D} = \hat{p} + qeA, \quad \hat{p}_0 = -\frac{\partial}{\partial \tau}, \quad \hat{p}_i = i\partial_i, \quad (A_0, A_1, A_2, A_3) = (0, \frac{B}{2}x^2, -\frac{B}{2}x^1, 0).$$

where q is an arbitrary electric charge number. Generally in QCD matter, $q = \frac{2}{3}$ for the up quark and $q = -\frac{1}{3}$ for the down quark. Nevertheless, we do not assign any value to q until the numerical calculation in this paper.

To extract the dynamical mass from the Lagrangian of Equation (2), we need to apply mean-field approximation to the four-fermion interaction term $(\bar{\psi} \psi)^2$. However, according to Asakawa's work [26], a nonzero dynamical mass ($\langle \bar{\psi} \psi \rangle \neq 0$), with the presence of nonzero chemical potential, will induce a $U(1)$ charge condensate ($\langle \bar{\psi} \gamma^0 \psi \rangle \neq 0$). Furthermore, we notice that, with chemical potential and external magnetic field, the dynamical mass can induce more condensates besides $\langle \bar{\psi} \gamma^0 \psi \rangle$. To introduce these condensates, firstly,

we need to transform the four-fermion interaction term to its Fierz identity in the Lagrangian [27], which is

$$\mathcal{F}[(\bar{\psi}\psi)^2] = \frac{1}{4N_c} [(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma^\mu\psi)^2 - (i\bar{\psi}\gamma^5\psi)^2 - (\bar{\psi}\gamma^5\gamma^\mu\psi)^2 + \frac{1}{2}(\bar{\psi}\sigma^{\mu\nu}\psi)^2]. \quad (3)$$

Strictly speaking, the fermion field in the Fierz identity should be a tensor product of spinor space, flavor space (if it is a two-flavor NJL model), and color space; therefore, the crossing matrices, referring to Equations (B6) and (B7) in [27], must have Pauli matrices (for two-flavor NJL) and Gell-Mann matrices (for color space) beside the Dirac matrices in Equation (3). Except Dirac matrices, the other two kinds of matrices are neglected in our paper for simplicity, because, according to our minimum condensates ansatz in the following discussion, their existence does not cause new condensates; further, through the mean-field approximation, the four-fermion terms that couple with nonidentity Pauli matrices and Gell-Mann matrices are zero. For more discussions on the Fierz identity, one can also refer to [30,31], in which the relation between strong interaction and four-fermion interaction terms is also discussed.

It is generally believed that $(\bar{\psi}\psi)^2$ and $\mathcal{F}[(\bar{\psi}\psi)^2]$ are dynamically equivalent, but the mean-field approximation could break such equivalence. Applying the approximation to $(\bar{\psi}\psi)^2$ gives only the vacuum condensate, while for $\mathcal{F}[(\bar{\psi}\psi)^2]$, according to Equation (3), it could give $\langle\bar{\psi}\Gamma\psi\rangle$ ($\Gamma \in \{I, \gamma^\mu, \gamma^5, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$)—16 kinds of possible condensates. If chemical potential is nonzero, in the case of $\mathcal{F}[(\bar{\psi}\psi)^2]$, one can prove that $\langle\bar{\psi}\gamma^0\psi\rangle \neq 0$ as long as $\langle\bar{\psi}\psi\rangle \neq 0$; this is where the $U(1)$ charge condensate comes from. In this paper, we apply the mean-field approximation to $\mathcal{F}[(\bar{\psi}\psi)^2]$ instead of $(\bar{\psi}\psi)^2$ to find more condensates.

The general equation of the mean-field approximation to any four-fermion term $(\bar{\psi}\Gamma\psi)^2$ is

$$(\bar{\psi}\Gamma\psi)^2 = (\bar{\psi}\Gamma\psi - \langle\bar{\psi}\Gamma\psi\rangle)^2 + 2\langle\bar{\psi}\Gamma\psi\rangle(\bar{\psi}\Gamma\psi) - \langle\bar{\psi}\Gamma\psi\rangle^2 \approx 2\langle\bar{\psi}\Gamma\psi\rangle(\bar{\psi}\Gamma\psi) - \langle\bar{\psi}\Gamma\psi\rangle^2. \quad (4)$$

In the Lagrangian, $(\bar{\psi}\Gamma\psi)^2$ normally multiplies a coupling constant— g , for example—thus, the mean-field is defined as $\xi = -2g\langle\bar{\psi}\Gamma\psi\rangle$, and Equation (4) is equivalent to

$$g(\bar{\psi}\Gamma\psi)^2 \approx -\xi\bar{\psi}\Gamma\psi - \frac{\xi^2}{4g}. \quad (5)$$

Applying the approximation to each term in $\mathcal{F}[(\bar{\psi}\psi)^2]$, there are

$$G\mathcal{F}[(\bar{\psi}\psi)^2] \approx -\bar{\psi}\Sigma \otimes 1_c\psi + \mathcal{L}_M, \quad (6)$$

$$\Sigma = \sigma + a\gamma^0 + b\gamma^5\gamma^3 + c\sigma^{12}, \quad \mathcal{L}_M = -\frac{N_c}{G}(\sigma^2 + a^2 - b^2 + c^2),$$

where σ , a , b , and c are mean-fields, defined as

$$\sigma = -\frac{G}{2N_c}\langle\bar{\psi}\psi\rangle, \quad a = -\frac{G}{2N_c}\langle\bar{\psi}\gamma^0\psi\rangle, \quad b = \frac{G}{2N_c}\langle\bar{\psi}\gamma^5\gamma^3\psi\rangle, \quad c = -\frac{G}{2N_c}\langle\bar{\psi}\sigma^{12}\psi\rangle. \quad (7)$$

Accordingly, the Lagrangian evolves into

$$\mathcal{L}' = \bar{\psi}(\hat{D} - \Sigma + \mu\gamma^0)\psi + \mathcal{L}_M. \quad (8)$$

Someone might get confused by the exposition above—as to why in Equation (6) there are only 4 kinds of condensates (or mean-fields) rather than 16 kinds of them. To answer the question, we need a proof and an ansatz.

Firstly, we assume in the chiral symmetry broken phase (broken phase for short) that there is only one kind of condensate, $\langle \bar{\psi}\Gamma\psi \rangle \neq 0$, if and only if $\Gamma = I_4$. Then, the propagator is

$$\hat{S}_0 = \frac{1}{\hat{D} - \sigma + \mu\gamma^0}. \quad (9)$$

As $\langle \bar{\psi}\Gamma\psi \rangle \propto \text{Tr}(\hat{S}_0\Gamma)$, the other condensates can be deduced from \hat{S}_0 . One can complete the deduction through the method in [20] and the assistance of symbolic computation; then, a qualitative result is

$$\langle \bar{\psi}\gamma^0\psi \rangle, \langle \bar{\psi}\gamma^5\gamma^3\psi \rangle, \langle \bar{\psi}\sigma^{12}\psi \rangle \propto \sigma, \quad \langle \bar{\psi}\Gamma\psi \rangle = 0 \text{ for } \Gamma \neq I_4, \gamma^0, \gamma^5\gamma^3, \sigma^{12}. \quad (10)$$

The operator ‘Tr’ means taking trace on the Hilbert space of the quark wave function, including the coordinate/momentum space, spinor space, and color space.

The result of Equation (10) suggests that the assumption $\langle \bar{\psi}\Gamma\psi \rangle \neq 0$ if and only if $\Gamma = I_4$, in the broken phase, is not self-consistent. In order to keep the self-consistency, we have to assume $\langle \bar{\psi}\psi \rangle, \langle \bar{\psi}\gamma^0\psi \rangle, \langle \bar{\psi}\gamma^5\gamma^3\psi \rangle, \langle \bar{\psi}\sigma^{12}\psi \rangle \neq 0$ at least, which leads to Equations (6) and (7).

Rigorously speaking, the proof above is not complete, as the self-energy Σ in Equation (6) is minimally self-consistent only. A complete proof—which is complex and has hard to deduce computable gap equations—should assume Σ has all 16 kinds of condensates, and then prove that some of them equal 0. To avoid such complexity, we propose an ansatz that the minimally self-consistent Σ is adequate for the quark propagator in our case.

2.2. The Gap Equations

So far, we have enunciated the mean-field approximation and probable condensates; the next step is to obtain the gap equations from the Lagrangian \mathcal{L}' . Firstly, the partition function of \mathcal{L}' is

$$\begin{aligned} \mathcal{Z} &= \int D\bar{\psi}D\psi \exp\left(\int_0^\beta d\tau \int d\vec{x} \mathcal{L}'\right) = e^{-\beta\mathcal{J}}, \\ \mathcal{J} &= -\mathcal{L}_M \int d\vec{x} + TN_c \text{Tr} \ln \hat{S}, \quad \hat{S} = \frac{1}{\hat{D} - \Sigma + \mu\gamma^0}. \end{aligned} \quad (11)$$

For a thermal equilibrium system, its grand potential \mathcal{J} should be at a local minimum with fixed temperature, chemical potential, and external magnetic field, and the equations to identify the minimum are

$$\frac{\delta\mathcal{J}}{\delta\sigma} = 0, \quad \frac{\delta\mathcal{J}}{\delta a} = 0, \quad \frac{\delta\mathcal{J}}{\delta b} = 0, \quad \frac{\delta\mathcal{J}}{\delta c} = 0. \quad (12)$$

These are the elementary gap equations. As σ, a, b are constant fields, the variations in Equation (12) are equivalent to partial differentials; then, the gap equations can be written more explicitly as

$$\begin{aligned} \frac{2}{G}\sigma \int d\vec{x} &= -T \text{tr} \sum_{m=-\infty}^{+\infty} \int \langle m, \vec{x} | \hat{S} | m, \vec{x} \rangle d\vec{x}, & \frac{2}{G}a \int d\vec{x} &= -T \text{tr} \sum_{m=-\infty}^{+\infty} \int \langle m, \vec{x} | \hat{S}\gamma^0 | m, \vec{x} \rangle d\vec{x}, \\ \frac{2}{G}b \int d\vec{x} &= T \text{tr} \sum_{m=-\infty}^{+\infty} \int \langle m, \vec{x} | \hat{S}\gamma^5\gamma^3 | m, \vec{x} \rangle d\vec{x}, & \frac{2}{G}c \int d\vec{x} &= -T \text{tr} \sum_{m=-\infty}^{+\infty} \int \langle m, \vec{x} | \hat{S}\sigma^{12} | m, \vec{x} \rangle d\vec{x}, \end{aligned} \quad (13)$$

where the operator ‘tr’ means taking the trace of the gamma matrices only, and ‘ m ’ is the quantum number of the eigenstate of $\frac{\partial}{\partial\tau}$ at finite temperature T . The state $|m, \vec{x}\rangle$ has the properties of

$$\langle l, \vec{y} | m, \vec{x} \rangle = \delta_{lm}\delta(\vec{x} - \vec{y}), \quad \frac{\partial}{\partial\tau} |m, \vec{x}\rangle = i\lambda_m |m, \vec{x}\rangle, \quad \lambda_m = (2m + 1)\pi T, \quad l, m \in \mathbb{Z}. \quad (14)$$

To proceed, we have to tackle $\sum_m \int \langle m, \vec{x} | \hat{S} | m, \vec{x} \rangle d\vec{x}$, in which \hat{S} is rewritten as

$$\hat{S} = \frac{1}{(\hat{\mathcal{D}}' + \tilde{\Sigma})(\hat{\mathcal{D}}' - \Sigma')} (\hat{\mathcal{D}}' + \tilde{\Sigma}),$$

$$\hat{\mathcal{D}}' = \hat{p}'_0 \gamma^0 + \hat{D}_i \gamma^i, \quad \hat{p}'_0 = \hat{p}_0 + \mu_r, \quad \mu_r = \mu - a, \quad (15)$$

$$\Sigma' = \sigma + b\gamma^5 \gamma^3 + c\sigma^{12}, \quad \tilde{\Sigma} = \sigma + b\gamma^5 \gamma^3 - c\sigma^{12}.$$

Clearly, the $U(1)$ charge condensate $\langle \bar{\psi} \gamma^0 \psi \rangle$ (or a) serves as a renormalization parameter to μ ; therefore, $\mu_r = \mu - a$ is the chemical potential of real physical significance and will be treated as a free parameter in the following discussion.

According to Equation (15), the fermion propagator is

$$\hat{S} = \frac{1}{\hat{p}'_0{}^2 - \hat{D}_1^2 - \hat{D}_2^2 - \hat{p}'_3{}^2 - \sigma^2 - b^2 + c^2 - 2b\hat{p}'_3\gamma^5 + 2c\hat{p}'_3\gamma^5\gamma^0 + 2(c\hat{p}'_0 - \sigma b)\gamma^5\gamma^3 - qeB\sigma^{12}} (\hat{\mathcal{D}}' + \tilde{\Sigma}). \quad (16)$$

Referring to our previous work [20], by replacing $|m, \vec{x}\rangle$ with another set of complete states $|m, p_3; n, \alpha\rangle$, one can quantize $(\hat{D}_1^2 + \hat{D}_2^2)$ in the denominator of \hat{S} to the Landau levels $(2n + 1)|q|eB$ ($n = 0, 1, 2, \dots$). A simple introduction of $|m, p_3; n, \alpha\rangle$ is shown in Appendix A. From the properties of $|m, p_3; n, \alpha\rangle$, one can prove that

$$\sum_m \int \langle m, \vec{x} | \hat{S} | m, \vec{x} \rangle d\vec{x} = \sum_m \sum_{n=0}^{+\infty} \int dp_3 d\alpha \langle m, p_3; n, \alpha | \hat{S} | m, p_3; n, \alpha \rangle = \frac{|q|eB}{2\pi} \sum_m \int S_{\text{eff}} \frac{dp_3}{2\pi} \int d\vec{x}, \quad (17)$$

$$S_{\text{eff}} = f_1 I_4 + f_2 \gamma^0 + f_3 \gamma^5 \gamma^3 + f_4 \sigma^{12}.$$

$f_1, f_2, f_3,$ and f_4 are functions of $(m, p_3, eB, T, \mu_r, \sigma, b, c)$ —their explicit expressions are shown in Appendix B. Now, the gap equations are

$$\frac{\sigma}{2G} = -\frac{|q|eB}{2\pi} T \sum_m \int f_1 \frac{dp_3}{2\pi}, \quad \frac{a}{2G} = -\frac{|q|eB}{2\pi} T \sum_m \int f_2 \frac{dp_3}{2\pi},$$

$$\frac{b}{2G} = \frac{|q|eB}{2\pi} T \sum_m \int f_3 \frac{dp_3}{2\pi}, \quad \frac{c}{2G} = -\frac{|q|eB}{2\pi} T \sum_m \int f_4 \frac{dp_3}{2\pi}. \quad (18)$$

For succinctness of the following formulae, we take \sum_m as $\sum_{m=-\infty}^{+\infty}$, and \sum_n as $\sum_{n=0}^{+\infty}$.

From Appendix B, we know $f_1, f_2, f_3,$ and f_4 are functions of (b_q, c_q) rather than (b, c) , and the gap equations of (b, c) can be rewritten as

$$\frac{b_q}{2G} = \frac{|q|eB}{2\pi} T \sum_m \int f'_3 \frac{dp_3}{2\pi}, \quad \frac{c_q}{2G} = -\frac{|q|eB}{2\pi} T \sum_m \int f'_4 \frac{dp_3}{2\pi}, \quad (19)$$

where f'_3 and f'_4 are also functions of (b_q, c_q) . Clearly, if we can solve the gap equations of (σ, a, b_q, c_q) , the results are independent of the sign of q , which implies that in the broken phase, the quarks with opposite electric charges have opposite condensates (b, c) . The physical significance of condensates (b, c) will be discussed in the last section.

Normally, if one wants to solve the gap equations in Equation (18), one has to use a numerical method to solve the 4 equations simultaneously. However, in previous discussions, we have mentioned that a is a renormalization parameter to μ , and we tend to treat μ_r as a free variable; thus, the equation about a can be isolated from the gap equations. One can solve the equations of (σ, b, c) first, then use the results to determine a afterwards.

2.3. The Coupling Constant and Cut-Off Parameter

In the limit of zero chemical potential, $a, b,$ and c equal 0, the gap equations degrade into the equation about $\sigma,$

$$\frac{\sigma}{2G} = -\frac{|q|eB}{4\pi} \sigma T \sum_m \int dp_3 \sum_n \frac{2 - \delta_{0n}}{(-i\lambda_m)^2 - \omega_n^2}, \quad \omega_n = \sqrt{\sigma^2 + 2n|q|eB + p_3^2}. \quad (20)$$

To proceed, we employ the formula below to transform the sum of m into the integral of p_0 [32],

$$2\pi iT \sum_m f(p_0 = -i\lambda_m + \mu) = -\int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} \frac{f(p_0)}{e^{\beta(p_0-\mu)} + 1} dp_0 - \int_{-i\infty}^{i\infty} \frac{f(p_0)}{e^{\beta(\mu-p_0)} + 1} dp_0 \quad (21)$$

$$+ \oint_C \frac{f(p_0)}{e^{\beta(p_0-\mu)} + 1} dp_0 + \int_{-i\infty}^{i\infty} f(p_0) dp_0,$$

where the ‘ C ’ in the contour integral represents a specific contour in the complex plane of $p_0,$ shown in Figure 1; then, Equation (20) becomes

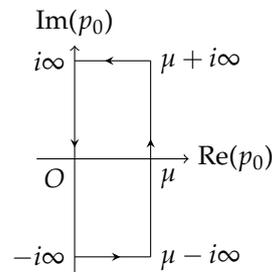


Figure 1. The contour C in the complex plane of $p_0.$

$$\frac{\sigma}{2G} = \frac{|q|eB}{16\pi^3} \sigma \int dp_0 dp_3 \sum_n \frac{2 - \delta_{0n}}{p_0^2 + \omega_n^2} - \frac{|q|eB}{8\pi^2} \sigma \sum_n (2 - \delta_{0n}) \int dp_3 \frac{1}{\omega_n} \frac{1}{e^{\beta\omega_n} + 1} \quad (22)$$

$$= \frac{|q|eB}{16\pi^2} \sigma \int_0^{+\infty} \frac{e^{-\sigma^2 s}}{s} \coth(|q|eBs) ds - \frac{|q|eB}{8\pi^2} \sigma \sum_n (2 - \delta_{0n}) \int dp_3 \frac{1}{\omega_n} \frac{1}{e^{\beta\omega_n} + 1}.$$

In Equation (22), the integral of the proper time ‘ s ’ is divergent; therefore, we need a cut-off to the lower limit of the integral of ‘ s ’, which is

$$\int_0^{+\infty} ds \rightarrow \int_{1/\Lambda^2}^{+\infty} ds. \quad (23)$$

In order to identify the values of the coupling constant G and the cut-off parameter $\Lambda,$ we need a gap equation at zero temperature and zero magnetic field, which can be simplified from Equation (22),

$$\frac{8\pi^2}{G} = \int_{1/\Lambda^2}^{+\infty} \frac{e^{-\sigma^2 s}}{s^2} ds, \quad (24)$$

As a reminder, this is the gap equation of one-flavor quark.

We already know that in the two-flavor NJL model [27,33], with g as the coupling constant of the four-fermion interaction term, the gap equation is

$$\frac{\pi^2}{N_c g} = \int_{1/\Lambda'^2}^{+\infty} \frac{e^{-\sigma'^2 s}}{s^2} ds, \quad g \approx 3.20388\text{GeV}^{-2}, \quad \Lambda' \approx 1.08631\text{GeV}, \quad (25)$$

In this equation, the dynamical mass σ' is

$$\sigma' = -2g\langle\bar{\psi}\psi\rangle = -2g(\langle\bar{u}u\rangle + \langle\bar{d}d\rangle) \approx -4g\langle\bar{u}u\rangle, \quad (26)$$

while, in this article, the dynamical mass σ is

$$\sigma = -\frac{G}{2N_c}\langle\bar{\psi}\psi\rangle = -\frac{G}{2N_c}\langle\bar{u}u\rangle. \quad (27)$$

Both σ and σ' are dynamical masses of up quark and down quark; thus, there should be $\sigma = \sigma'$. Then, from Equations (26) and (27), one can derive $G = 8N_c g$. The cut-off parameters Λ and Λ' are both determined by the pion decay constant; therefore, $\Lambda = \Lambda'$. In conclusion, the values of the coupling constant and the cut-off parameter in one-flavor gap equation, Equation (24), are

$$G \approx 76.89312\text{GeV}^{-2}, \quad \Lambda \approx 1.08631\text{GeV}. \quad (28)$$

3. The Solutions of the Gap Equations

3.1. The Approximate Solutions in Chiral Symmetry Broken Phase

In the broken phase, $\sigma, a, b, c \neq 0$, theoretically, one needs to solve the gap equations of (σ, b, c) in Equation (18), but the divergent parts in the integrals of the ' f 's (or f_1, f_2, f_3 , and f_4) cannot be separated out simply and properly such as in Equation (22)—therefore, a direct numerical iteration method is not available. Through some numerical experiments, we notice that in the solutions of the gap equations, (b, c) are much smaller than σ , one can set $b, c = 0$ in the ' f 's of Equation (18) for approximate solutions, and then the gap equations are simplified to

$$\begin{aligned} \frac{8\pi^2}{G}\sigma &= |q|eB\sigma \int_{1/\Lambda^2}^{+\infty} \frac{e^{-\sigma^2 s}}{s} \coth(|q|eBs) ds \\ &\quad - |q|eB\sigma \sum_n (2 - \delta_{0n}) \int dp_3 \frac{1}{\omega_n} \left[\frac{1}{e^{\beta(\omega_n - \mu_r)} + 1} + \frac{1}{e^{\beta(\omega_n + \mu_r)} + 1} \right], \\ \frac{8\pi^2}{G}b &= qeB \int \left[\frac{1}{e^{\beta(\omega_0 - \mu_r)} + 1} - \frac{1}{e^{\beta(\omega_0 + \mu_r)} + 1} \right] dp_3, \\ \frac{8\pi^2}{G}c &= -qeB\sigma \int_{1/\Lambda^2}^{+\infty} \frac{e^{-\sigma^2 s}}{s} ds + qeB\sigma \int \frac{1}{\omega_0} \left[\frac{1}{e^{\beta(\omega_0 - \mu_r)} + 1} + \frac{1}{e^{\beta(\omega_0 + \mu_r)} + 1} \right] dp_3. \end{aligned} \quad (29)$$

To derive the above equations, we employed the formula of Equation (21). One should notice that in the simplified gap equations of (b, c) , we write q instead of $|q|$ since $q = |q| \text{sgn}(q)$, which has explicitly shown that the directions of mean-fields (b, c) depend on the sign of the electric number. Incidentally, if one uses \hat{S}_0 of Equation (9) to deduce $\langle\bar{\psi}\psi\rangle$, $\langle\bar{\psi}\gamma^5\gamma^3\psi\rangle$, and $\langle\bar{\psi}\sigma^{12}\psi\rangle$, the derived equations are equivalent to Equation (29).

By setting $q = 1$, the numerical results of Equation (29) are shown in Figures 2–7. One should notice that in Figures 6 and 7, we have drawn $(-c)-\mu_r$ relation instead of $c-\mu_r$ relation for the comparison with $\sigma-\mu_r$ relation. In these figures, the results of chiral symmetry restored phase (restored phase for short) are also included, in which $\sigma, c = 0$ and b is proportional to μ_r ; these results will be discussed in the next subsection. It is clear in Figures 2, 3, 6 and 7 that when μ_r passes a specific point (the phase transition point), σ and c have a transition from a nonzero value to zero, which indicates it is a first-order phase transition, while in Figures 4 and 5, the mean-field b also has a transition when μ_r passes the same point, but the gap between two phases is ambiguous.

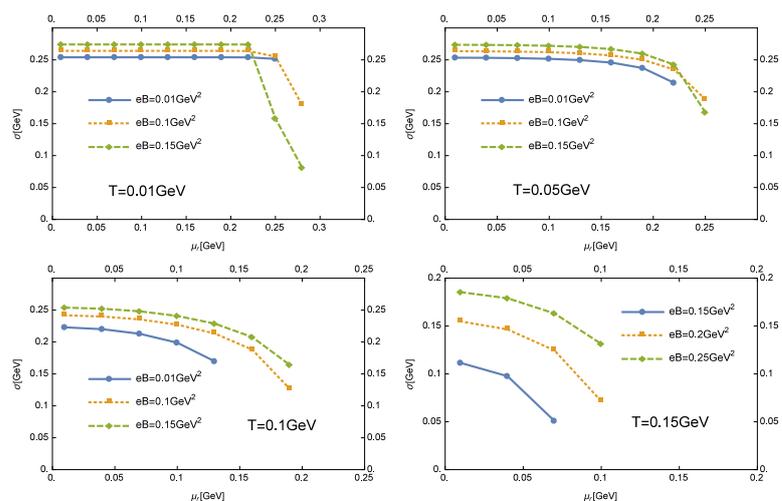


Figure 2. In these four subfigures, we fix temperature at 0.01, 0.05, 0.1, and 0.15 GeV separately and plot the σ - μ_r relations in each subfigure with different magnitudes of the external magnetic field.

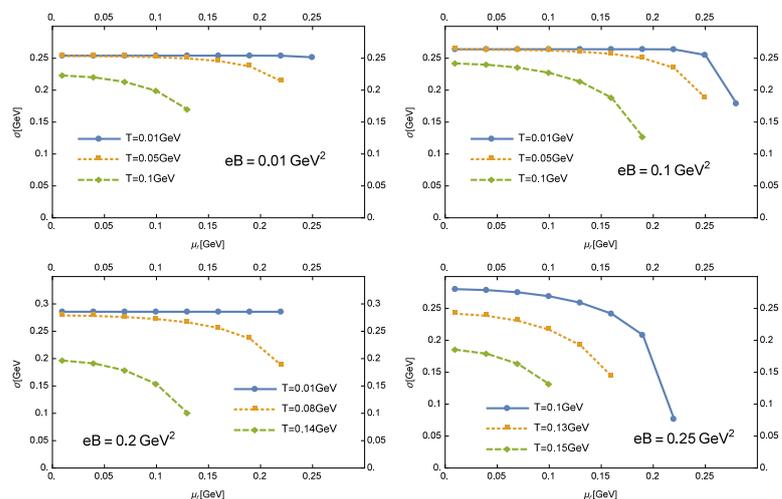


Figure 3. In these four subfigures, we fix the external magnetic field at 0.01, 0.1, 0.2, and 0.25 GeV^2 separately and plot the σ - μ_r relations in each subfigure with different temperatures.

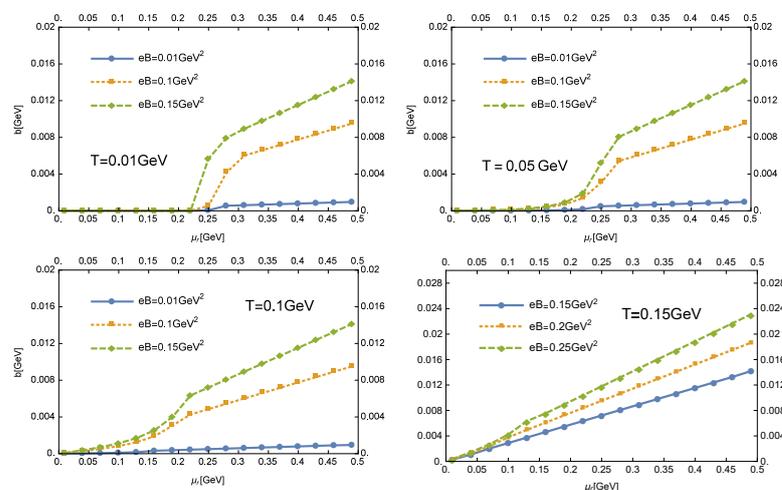


Figure 4. In these four subfigures, we fix temperature at 0.01, 0.05, 0.1, and 0.15 GeV separately and plot the b - μ_r relations in each subfigure with different magnitudes of the external magnetic field.

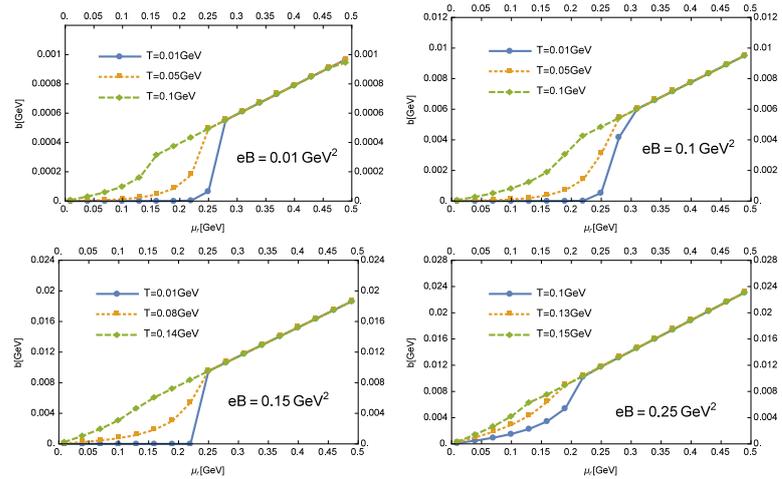


Figure 5. In these four subfigures, we fix external magnetic field at 0.01, 0.1, 0.2, and 0.25 GeV² separately and plot the $b-\mu_T$ relations in each subfigure with different temperatures.

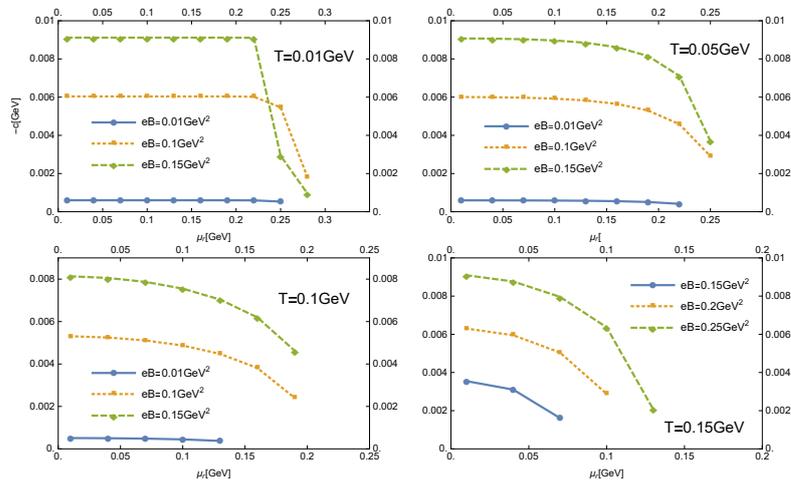


Figure 6. In these four subfigures, we fix temperature at 0.01, 0.05, 0.1, and 0.15 GeV separately and plot the $(-c)-\mu_T$ relations in each subfigure with different magnitudes of the external magnetic field.

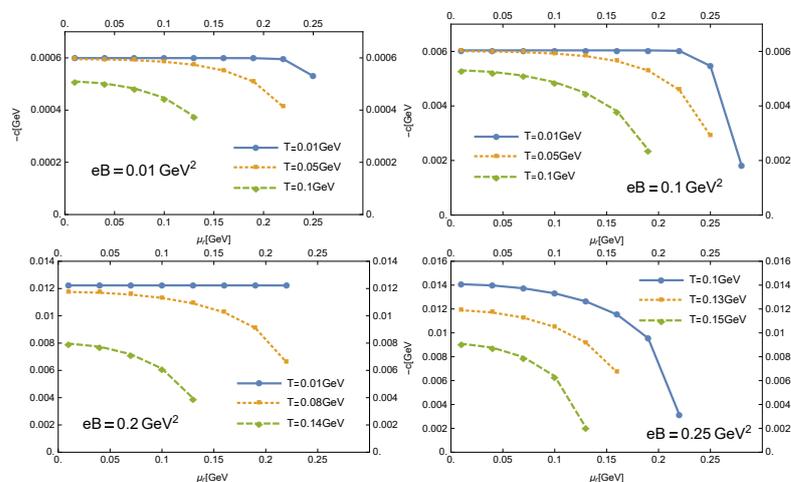


Figure 7. In these four subfigures, we fix external magnetic field at 0.01, 0.1, 0.2, and 0.25 GeV² separately and plot the $(-c)-\mu_T$ relations in each subfigure with different temperatures.

3.2. The Solutions in Chiral Symmetry Restored Phase

In the chiral symmetry restored phase, $\sigma = 0$, the expression of f_1 is

$$f_1 = -\frac{p_0'^2 - p_3^2 + b^2 - c^2}{F_0(p_0', p_3, |q|eB, 0, b, c)} \frac{c_q}{2} - \sum_{n=0}^{+\infty} \frac{2 - \delta_{0n}}{2} \frac{2p_0'bc}{F_n(p_0', p_3, |q|eB, 0, b, c)}, \quad (30)$$

As f_1 belongs to the gap equation of σ , which equals 0 at present, we need to prove that $\sum_m \int f_1 dp_3 = 0$. Through analyzing Equation (30), one can conclude that $c = 0$ ensures $\sum_m \int f_1 dp_3 = 0$ since f_1 is proportional to c . We also notice that the expression of f_4 ,

$$f_4 = \frac{p_0'bc}{F_0(p_0', p_3, |q|eB, 0, b, c)} + c \sum_{n=0}^{+\infty} \frac{2 - \delta_{0n}}{2} \frac{p_0'^2 - p_3^2 + 2n|q|eB + b^2 - c^2}{F_n(p_0', p_3, |q|eB, 0, b, c)}, \quad (31)$$

is proportional to c as well, and $c \propto \sum_m \int f_4 dp_3$; therefore, $c = 0$ is one of the valid solutions. However, how can we be sure that $c = 0$ is the only valid solution in this case? The reason is as below, assuming $c \neq 0$, the mean-fields (b, c) , which are 2 variables, must adapt to 3 gap equations of (σ, b, c) , and clearly, the 3 equations are independent of each other, one cannot solve the equations when the amount of independent equations outnumbers the amount of their variables, while in the case of $c = 0$, the equations of (σ, c) become identities, leaving 1 equation with 1 variable (the equation of b); thus, $c = 0$ is the only valid solution in the restored phase.

Since we have $\sigma, c = 0$, the only gap equation left is

$$\frac{b}{2G} = \frac{|q|eB}{2\pi} T \sum_m \int f_3 \frac{dp_3}{2\pi} = \frac{qeB\mu_r}{8\pi^2}. \quad (32)$$

Clearly, in the chiral restored phase, the condensate $\langle \bar{\psi}\gamma^5\gamma^3\psi \rangle$, independent of temperature, is proportional to the strength of magnetic field and chemical potential; the detailed discussion of this result will be argued in the last section.

4. Conclusions and Remarks

In this paper, we have studied the self-energy of NJL model in the presence of temperature, chemical potential, and external magnetic field, it turns out that when chemical potential is nonzero, the self-energy no longer equals dynamical mass. In order to establish the correct gap equations in the frame of the NJL model, we employ the Fierz identity to introduce more condensates; then, by a proof and an ansatz, we reduce the condensates to (σ, a, b, c) .

In the self-energy (6), σ , the vacuum condensate, is the dynamical mass, which is a positive real number depending on temperature, chemical potential, and external magnetic field in the broken phase, while in the restored phase, it is constantly zero. The properties of σ in different situations and taking it as the order parameter of the phase transition have been widely studied for decades, there is no new result that can be added to the properties of σ from this paper. The role of σ in our case is a benchmark, which can be used to verify that after introducing other condensates, the qualitative properties of the phase transition do not deviate from the classical results. Besides the vacuum condensate, the mean-field a , or $\langle \bar{\psi}\gamma^0\psi \rangle$, is thought of as a $U(1)$ charge condensate, which is absorbed by the bare chemical potential μ of the original Lagrangian Equation (2). As a result, $\mu_r = \mu - a$ is the real chemical potential of physical significance in the dynamical process and μ_r is to μ as the running coupling constant is to bare coupling constant. In this article, we do not study a since μ_r is treated as a free variable, and, in the absence of bare chemical potential, the mean-field a is constantly zero.

The mean-field b , or $\langle \bar{\psi}\gamma^5\gamma^3\psi \rangle$, represents a condensate of chiral current; it is induced by chemical potential and external magnetic field simultaneously. Generally, the chiral current has three directions, but due to the external magnetic field, the $SO(3)$ symmetry of space is broken, and the induced chiral current is parallel or antiparallel to the direction of

magnetic field. Unlike the dynamical mass, the mean-field b is a nonzero quantity in chiral broken and restored phase as long as μ and eB are not zero. In the chiral broken phase, the chiral current is a small quantity compared with σ , while in the chiral restored phase, it is proportional to μ_r and qeB ; actually, this is the chiral separation effect (CSE for short) proposed by M. Metlitski and A. Zhitnitsky [34] and others' works [35–38]. Although chiral current has physical significance, in the chiral broken phase, its magnitude is too small to have an obvious effect, but in the restored phase, due to CSE, stronger magnetic field or higher chemical potential (denser QCD matter) may cause observable effects. It is worth noting that CSE is discovered through chiral anomaly in the work of Metlitski and Zhitnitsky, while in our case, CSE is a result of self-consistence of the condensates—there must be some mathematical correlation behind the coincidence. Besides CSE, in a series of Tatsumi's works [39–41], the authors used the chiral current to study ferromagnetism in nuclear matter and QCD matter, it is the parameter to describe 'spin polarization' in their articles. Different from our case, their works do not involve external magnetic fields; therefore, the chiral current only exists in color super conductivity. The chiral current does not present in the work of [28]; instead, it is $\bar{\psi}\gamma^0\gamma^3\psi$ in their work. Clearly, the self-consistent analysis does not prevent $\bar{\psi}\gamma^0\gamma^3\psi$ appearing in the Lagrangian, but one should be cautious that $\bar{\psi}\gamma^0\gamma^3\psi$ could stimulate other condensates (or four-fermion interaction terms) in the Lagrangian due to the self-consistency of the gap equations.

For the mean-field c , or $\langle\bar{\psi}\sigma^{12}\psi\rangle$, we name it 'spin current'. Like the chiral current, it is also induced by chemical potential and external magnetic field. Due to the magnetic field, the spin current is parallel or antiparallel to the direction of magnetic field as the chiral current does; therefore, the current has no other directions such as $\langle\bar{\psi}\sigma^{23}\psi\rangle$ and $\langle\bar{\psi}\sigma^{31}\psi\rangle$. Actually, 'spin current' is not an accurate description to the mean-field c since $\langle\bar{\psi}\sigma^{12}\psi\rangle$ is not exactly the angular momentum of spin. Through Noether's theorem, one can easily find out that for the Dirac field, its spin angular momentum is contained in the term of $\langle\bar{\psi}\gamma^0\sigma^{12}\psi\rangle$ rather than $\langle\bar{\psi}\sigma^{12}\psi\rangle$. The physical significance of $\langle\bar{\psi}\sigma^{12}\psi\rangle$ is elucidated in Sakurai's textbook [42]. It is proved that the vector current, $j_\mu = e\bar{\psi}\gamma_\mu\psi$, can be decomposed into $j_\mu^{(1)}$ and $j_\mu^{(2)}$ when the electron couples to a classical vector potential A_μ (Gordon decomposition), and $j_\mu^{(2)}$ couples to A_μ ; $j_\mu^{(2)}A^\mu = \frac{e}{2mc}(\frac{1}{2}F_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}\psi)$ can account for the spin magnetic moment interaction with the gyromagnetic ratio $g = 2$ in the nonrelativistic limit. It is clear that magnetic moment is inversely proportional to the mass of fermion; therefore, one cannot define the magnetic moment for a massless fermion. This statement is consistent with the property of the spin current in this article, because $c = 0$ in the restored phase. The spin current is a small quantity compared with σ in the broken phase, but $\int_V c d\vec{x}$ relates to the magnetic moment of the whole system, which could give rise to considerable and observable effects in the experiments.

In the introduction, we mentioned that the one-flavor QCD is different from the one-flavor NJL model due to quantum anomalies; therefore, in one-flavor QCD, there should be no chiral symmetry phase. As a consequence, if one wishes to study the properties of QCD matters in one-flavor by using the one-flavor NJL model, the nonexistence of chiral symmetry should be taken into account. However, in this paper, we do not intend to study QCD matters in one-flavor; the employment of the one-flavor NJL model here is for simplicity, it can be treated as a degraded version of two-flavor NJL and used to keep the gap equations simple. Through mean-field approximation, the mechanisms of chiral symmetry breaking of one-flavor NJL and two-flavor NJL are the same. Further, as one can notice, we define q as an arbitrary electric charge number at the beginning, with which one can easily extend the formulae to the two-flavor NJL model. The extension is not necessary, since from one-flavor to two-flavor, the qualitative results are the same except for some quantitative differences. Nonetheless, if one wishes to study two-flavor NJL with external

magnetic field, the gap equation of quark condensate, by consulting Equation (18), can be written as

$$\frac{\sigma}{2G} = - \sum_{f=u,d} \frac{|q_f|eB}{2\pi} T \sum_m \int f_1(|q_f|eB) \frac{dp_3}{2\pi}, \quad q_u = \frac{2}{3}, \quad q_d = -\frac{1}{3}. \quad (33)$$

The main purpose of this work is to prove the existence of new condensates besides the quark condensate; therefore, other effects such as the ‘Inverse Magnetic Catalysis’ are not involved. In order to reproduce ‘Inverse Magnetic Catalysis’ with the NJL model, one has to consider the quantum fluctuation effect to the coupling constant, which requires that G is a function of eB [43]. Although the new condensates, Equation (10), could bring new observable effects, they do not help to induce the ‘Inverse Magnetic Catalysis’ without considering the magnetic field dependence of G , because in the mean-field approximation, the modifications of the quark condensate caused by these new condensates are negligible. However, magnetic field dependence of the coupling constant can also be considered in these new condensates and gap equations Equation (18) (or Equation (29)) in following works.

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Appendix A. The Properties of $|m, p_3; n, \alpha\rangle$

$|m, p_3; n, \alpha\rangle$ is the eigenstate of \hat{p}_0, \hat{p}_3 , and $(\hat{D}_1^2 + \hat{D}_2^2)$,

$$|m, p_3; n, \alpha\rangle = |m\rangle \otimes |n, \alpha\rangle \otimes |p_3\rangle, \quad (A1)$$

$$\hat{p}_0|p_0\rangle = -i\lambda_m|m\rangle, \quad \hat{p}_3|p_3\rangle = p_3|p_3\rangle, \quad (\hat{D}_1^2 + \hat{D}_2^2)|n, \alpha\rangle = (2n + 1)|q|eB|n, \alpha\rangle.$$

In order to express $|n, \alpha\rangle$ in an analytic form, we define a state in the subspace spanned by $|x_1\rangle \otimes |x_2\rangle$,

$$|D_1, p\rangle = \sqrt{\frac{2}{|q|eB}} |p\rangle \otimes |X\rangle, \quad X = \frac{2}{qeB}(p + D_1) \quad (A2)$$

$$\hat{D}_1|D_1, p\rangle = D_1|D_1, p\rangle, \quad \hat{p}_1|p\rangle = -p|p\rangle, \quad \hat{x}_2|X\rangle = -\frac{2}{qeB}(p + D_1)|X\rangle.$$

Clearly, $|D_1, p\rangle$ is normalized and complete,

$$\langle D_1, p|D'_1, p'\rangle = \delta(D_1 - D'_1)\delta(p - p'), \quad \int dD_1 dp |D_1, p\rangle \langle D_1, p| = I. \quad (A3)$$

In the $|D_1, p\rangle$ representation, $|n, \alpha\rangle$ is expressed as

$$\langle D_1, p|n, \alpha\rangle = c_n e^{ia(p + \frac{D_1}{2})} h_n(\sqrt{\frac{2}{|q|eB}} D_1), \quad c_n = \left(\frac{1}{n! 2\pi \sqrt{|q|eB\pi}} \right)^{\frac{1}{2}}, \quad (A4)$$

where $h_n(z)$ is the solution of Weber differential equation; it is an even-function of z , and its properties of completeness and orthogonality are

$$\int h_n(z)h_m(z) dz = n!\sqrt{2\pi}\delta_{nm}, \quad \sum_{n=0}^{+\infty} \frac{1}{n!\sqrt{2\pi}}h_n(x)h_n(y) = \delta(x - y), \tag{A5}$$

Through these properties, one can prove that $|n, \alpha\rangle$ is complete and orthogonal,

$$\langle m, \alpha' | n, \alpha \rangle = \int dD_1 dp \langle m, \alpha' | D_1, p \rangle \langle D_1, p | n, \alpha \rangle = \delta_{mn}\delta(\alpha' - \alpha), \tag{A6}$$

$$\sum_{n=0}^{+\infty} \int d\alpha |n, \alpha\rangle \langle n, \alpha| = \sum_{n=0}^{+\infty} \int d\alpha dD_1 dp dD'_1 dp' |D_1, p\rangle \langle D_1, p | n, \alpha\rangle \langle n, \alpha | D'_1, p'\rangle \langle D'_1, p' | = I.$$

One can also prove that

$$\begin{aligned} \int \langle n, \alpha | n, \alpha \rangle d\alpha &= \int dx_1 dx_2 dD_1 dp dD'_1 dp' d\alpha \langle n, \alpha | D_1, p \rangle \langle D_1, p | x_1, x_2 \rangle \langle x_1, x_2 | D'_1, p' \rangle \langle D'_1, p' | n, \alpha \rangle \\ &= \frac{1}{2\pi} \int dx_1 dx_2 \int dD_1 dp dD'_1 dp' \left(\frac{2}{|q|eB}\right)^{\frac{3}{2}} \frac{1}{n!\sqrt{2\pi}} h_n\left(\sqrt{\frac{2}{|q|eB}} D_1\right) h_n\left(\sqrt{\frac{2}{|q|eB}} D'_1\right) \\ &\quad \times e^{ix_1(p-p')} \delta\left(p + \frac{D_1}{2} - p' - \frac{D'_1}{2}\right) \delta\left(x_2 - \frac{2}{qeB}(p' + D'_1)\right) \delta\left(x_2 - \frac{2}{qeB}(p + D_1)\right) \\ &= \frac{|q|eB}{2\pi} \int dx_1 dx_2. \end{aligned} \tag{A7}$$

For $|m, p_3\rangle$, it has

$$\langle m, p_3 | m, p_3 \rangle = \frac{1}{2\pi} \int dx_3. \tag{A8}$$

According to Equations (A7) and (A8), for $|m, p_3; n, \alpha\rangle$, there is

$$\int \langle m, p_3; n, \alpha | m, p_3; n, \alpha \rangle d\alpha = \frac{|q|eB}{(2\pi)^2} \int d\vec{x}. \tag{A9}$$

At last, for the convenience of deducing $\langle m, p_3; n, \alpha | \hat{S} | m, p_3; n, \alpha \rangle$, the equations below are useful:

$$\langle m, p_3; n, \alpha | \hat{D}_{1,2} | m, p_3; n, \alpha \rangle = 0, \quad \int \langle m, p_3; n, \alpha | \hat{p}_3 | m, p_3; n, \alpha \rangle dp_3 = 0. \tag{A10}$$

Appendix B. The Expressions of f_1, f_2, f_3, f_4

To express f_1, f_2, f_3 , and f_4 (the 'f's for short) in succinct forms, we define a function F_n as

$$F_n(p'_0, p_3, |q|eB, \sigma, b, c) = (p'^2_0 - p^2_3 - 2n|q|eB - \sigma^2 - b^2 + c^2)^2 - 4[(p'_0c - \sigma b)^2 + (b^2 - c^2)p^2_3]. \tag{A11}$$

In $F_n, p'_0 = -i\lambda_m + \mu_r, (-i\lambda_m)$ is the eigenvalue of \hat{p}_0 .

In the process of deducing the gap equations, we notice that some results depend on the sign of electric charge number q ; therefore, it is convenient to generalize the formulae by introducing two new parameters:

$$b_q = \text{sgn}(q)b, \quad c_q = \text{sgn}(q)c. \tag{A12}$$

Regarding the new parameters, one can easily prove that the equations below are valid,

$$b_q^2 = b^2, \quad c_q^2 = c^2, \quad b_q c_q = bc, \quad F_n(p'_0, p_3, |q|eB, \sigma, b_q, c_q) = F_n(p'_0, p_3, |q|eB, \sigma, b, c), \quad (\text{A13})$$

these equations are useful to express the 'f's succinctly.

Deducing the 'f's requires some tedious but basic algebraic operations, one may employ programs of symbolic computation, Mathematica programs for example, to ease the work. The formulae of f_1 and f_2 are

$$f_1 = \frac{(p'_0 + b_q)^2 - p_3^2 - \sigma^2 + c^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{\sigma}{2} + \sigma \sum_{n=1}^{+\infty} \frac{p_0'^2 - p_3^2 - 2n|q|eB - \sigma^2 + b^2 + c^2}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)} - \frac{(p'_0 + b_q)^2 - p_3^2 + \sigma^2 - c^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{c_q}{2} - \sum_{n=1}^{+\infty} \frac{2p'_0 bc}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)}, \quad (\text{A14})$$

$$f_2 = \frac{p_0'^2 - p_3^2 - (\sigma + c_q)^2 - b^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{p'_0}{2} + p'_0 \sum_{n=1}^{+\infty} \frac{p_0'^2 - p_3^2 - 2n|q|eB - \sigma^2 - b^2 - c^2}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)} + \frac{p_0'^2 + p_3^2 + (\sigma + c_q)^2 - b^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{b_q}{2} + \sum_{n=1}^{+\infty} \frac{2\sigma bc}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)}, \quad (\text{A15})$$

Due to the sign problem of q , it is better to define and deduce the formulae of f'_3 and f'_4 than write down the formulae of f_3 and f_4 directly,

$$f_3 = \text{sgn}(q)f'_3, \quad f_4 = \text{sgn}(q)f'_4, \quad (\text{A16})$$

for f'_3 and f'_4 , there are

$$f'_3 = \frac{p_0'^2 - p_3^2 - (\sigma + c_q)^2 - b^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{p'_0}{2} - \sum_{n=1}^{+\infty} \frac{2p'_0 \sigma c_q}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)} + \frac{p_0'^2 + p_3^2 + (\sigma + c_q)^2 - b^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{b_q}{2} + b_q \sum_{n=1}^{+\infty} \frac{p_0'^2 + p_3^2 - 2n|q|eB + \sigma^2 - b^2 + c^2}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)}, \quad (\text{A17})$$

$$f'_4 = -\frac{(p'_0 + b_q)^2 - p_3^2 - \sigma^2 + c^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{\sigma}{2} - \sum_{n=1}^{+\infty} \frac{2p'_0 \sigma b_q}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)} + \frac{(p'_0 + b_q)^2 - p_3^2 + \sigma^2 - c^2}{F_0(p'_0, p_3, |q|eB, \sigma, b, c)} \frac{c_q}{2} + c_q \sum_{n=1}^{+\infty} \frac{p_0'^2 - p_3^2 + 2n|q|eB + \sigma^2 + b^2 - c^2}{F_n(p'_0, p_3, |q|eB, \sigma, b, c)}. \quad (\text{A18})$$

The advantage of defining f'_3 and f'_4 has been discussed in the main text.

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