

Article

Existence Theory for Positive Iterative Solutions to a Type of Boundary Value Problem

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Abstract: We introduce some research results on a type of third-order boundary value problem for positive iterative solutions. The existence of solutions to these problems was proved using the monotone iterative technique. As an examination of the proposed method, an example to illustrate the effectiveness of our results was presented.

Keywords: successive iteration; positive solutions; third-order boundary value problem

1. Introduction

In the context of the intensification of research in various fields of applied mathematics and physics, it is necessary to discuss third-order differential equations in more detail. Third-order differential equation problems are fairly common; we often encounter them in the study of three-layer beams, the deflection of a curved beam having a constant or varying cross-section, gravity-driven flows, electromagnetic waves, and so on. For more on the applications of the third-order boundary value problems, see [1] and related topics.

We focus our attention on the third-order two-point boundary value problems. Some research has been undertaken into third-order two-point boundary value problems; see below for a summary.

The authors in [2] and the authors in [3] considered two different classes of third-order two-point boundary value problems respectively. They dealt with the problems in upper and lower solutions.

Li in [4] chose to discuss the existence of single and multiple positive solutions to a singular third-order two-point boundary value problem. Under certain assumptions, they have obtained some existence results by using Krasnoselskii's fixed point theorem.

Using the Leray-Schauder continuation principle, the authors obtained the existence of at least one positive solution for some third-order boundary value problems in [5].

The study focused on the upper and lower solution method because. The authors in [6] also thought that this method was interesting, so they developed it further. They discussed a class of third-order differential equations with mixed two-point boundary conditions. Some new existence results were obtained. Some applications were also presented.

Y. Sun, M. Zhao and S. Li were concerned with the existence of the monotone positive solution to a class of nonlinear third-order two-point boundary value problems in [7]. They gave us some suitable assumptions on f and their tool was also a fixed point theorem. They obtained the existence of at least one positive solution to the problem they were concerned with.

In [8], the authors focused on a kind of third-order equation. They studied various two-point boundary value problems for the equation they discussed. A variety of hypothetical conditions are taken into account. Then they obtained some rational results.

Again, we can study third-order boundary value problems in terms of giving the numerical solution. Ge and Zhang did this by using the barycentric rational interpolation collocation method in [9]. They focused on the matrix form of the problem they discussed.



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In the existing literature, most papers studied the existence of the solutions of various third-order boundary value problems by different types of techniques, for instance, see [10–16] and references therein. We focus our attention on computational methods. We undertook some preliminary work on this. For example, Ma, Du and Ge [17] and our papers [18,19] considered some boundary value problems with the p-Laplace operator. It is worth mentioning that we discussed the Pseudo-Symmetric solutions which are based on symmetry solutions to a kind of boundary value problem in [18]. We chose the monotone iterative technique. Then we obtained some existence results for the problem we were concerned with.

From our point of view, on the one hand, it is worthwhile and interesting to discuss these kinds of problems with computational methods, especially when the nonlinear term is involved explicitly with the all-order derivatives. It is worth emphasizing that the problems with all-order derivatives not only bring a lot of difficulties in the discussion but also have strong practical application value. For example, in mechanics, the problems with all-order derivatives can describe more precisely the vibration of beams; see [1] and related topics.

On the other hand, the monotone iteration technique is an interesting and effective procedure for investigating the existence of solutions to nonlinear problems. For more information on the monotone iterative method, one may refer to [10,20,21].

Motivated by the studies mentioned above, we will investigate the iteration and the existence of positive solutions to the following third-order two-point boundary value problem

$$\begin{aligned} u'''(t) + q(t)f(t, u(t), u'(t), u''(t)) &= 0, & 0 < t < 1, \\ u(0) = u'(1) = u''(0) &= 0, \end{aligned} \quad (1)$$

where $f(t, x, y, z) \in C([0, 1] \times [0, +\infty) \times \mathbb{R}^2 \rightarrow [0, +\infty))$, $q(t)$ is a nonnegative continuous function defined on $(0, 1)$, $q(t) \not\equiv 0$ on any subinterval of $(0, 1)$, and $\int_0^1 q(t)dt < +\infty$. Our main tool is a monotone iterative technique improved from the classical monotone iterative technique of Amann [10].

We aim to obtain the existence of positive solutions for the problem we were concerned with. At the same time, we focus upon trying to construct some successive iterative schemes for approximating the solutions. We would like to emphasize that here we do not require the existence of lower and upper solutions in our discussion. Happily, the starting point of the successive iterative schemes is a known constant function or a simple quadratic function. Finally, we will give a numerical example to illustrate the applicability of our results. Then we will analyze the calculation model, give computation detail process and draw figures of the results. We remark that knowledge of how to find the solutions is probably most important from a numerical and application standpoint.

2. Preliminaries

Before reaching the main conclusions, we need to present some background information and preparation. Firstly, we present here some general definitions from cone theory.

Definition 1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

- (i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$, and
- (ii) $u, -u \in P$ implies $u = 0$.

Definition 2. The map α is said to be concave on $[0, 1]$, if

$$\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in [0, 1]$ and $t \in [0, 1]$.

Secondly, let us do some preparatory work. Without loss of generality, the norm is defined as

$$\|u\| := \max \left\{ \max_{0 \leq t \leq 1} |u(t)|, \max_{0 \leq t \leq 1} |u'(t)|, \max_{0 \leq t \leq 1} |u''(t)| \right\}.$$

in the Banach space $E = C^2[0, 1]$. We introduce a symbol E_+ and define the cone $P \subset E$ that

$$E_+ = C^2_+[0, 1] = \{u \in E \mid u(t) \geq 0, t \in [0, 1]\},$$

$$P = \{u \in E \mid u(t) \geq 0, u \text{ is concave and nondecreasing on } [0, 1]\}.$$

Lemma 1. Let $g \in L^1[0, 1]$ then the boundary value problem

$$\begin{cases} u'''(t) + g(t) = 0, & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0. \end{cases}$$

has a unique solution $u(t) = \int_0^1 G(t, s)g(s)ds$, where

$$G(t, s) = \begin{cases} t(1-s) - \frac{1}{2}(t-s)^2, & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2}$$

It is obvious that $0 \leq G(t, s) \leq G(1, s) = \frac{1}{2}(1-s^2)$.

Define an operator $T : P \rightarrow E$ by

$$(Tu)(t) = \int_0^1 G(t, s)q(s)f(s, u(s), u'(s), u''(s))ds, \tag{3}$$

According to the solving process of Lemma 1, if u is a fixed point of T , then $Tu = u$ means that $u = u(t)$ is a solution to boundary value problem (1). On the contrary, if $u = u(t)$ is a solution to the boundary value problem (1), we can get $Tu = u$ which implies that u is a fixed point of T . So the original problem of the solution is transformed into the fix point problem.

Lemma 2. $T : P \rightarrow P$ defined by (2) is completely continuous.

Proof. From the definition of T , we deduce that for each $u \in P$, there is $Tu \in C^2[0, 1]$ which satisfies (2). Because $(Tu)''(t) \leq 0$ and $(Tu)'(t) \geq 0$, for $0 \leq t \leq 1$. Thus, Tu is concave and nondecreasing on $[0, 1]$. Then, $(Tu)(t) \geq 0, 0 \leq t \leq 1$. Hence, $T : P \rightarrow P$.

It is obvious that T is continuous. Let $\Omega \subset P$ be an bounded set. It is easy to prove that $T\Omega$ is bounded and equi-continuous. The Arzela-Ascoli theorem guarantees that $T\Omega$ is relatively compact, which means T is compact. Then, T is completely continuous. \square

3. The Existence and Iteration of Positive Solutions to (1)

With the preparatory work in hand, let us obtain our main results step by step.

Theorem 1. Assume there exists $a > 0$, such that the following (H_1) – (H_3) hold,

$$(H_1): f(t, x_1, y_1, z_1) \leq f(t, x_2, y_2, z_2) \\ \text{for any } 0 \leq t \leq 1, 0 \leq x_1 \leq x_2 \leq a, 0 \leq y_1 \leq y_2 \leq a, -a \leq z_2 \leq z_1 \leq 0;$$

$$(H_2): \max_{0 \leq t \leq 1} f(t, a, a, -a) \leq \frac{a}{A}, \text{ where } A = \int_0^1 q(s)ds;$$

$$(H_3): f(t, 0, 0, 0) \neq 0 \text{ for } 0 \leq t \leq 1.$$

then the boundary value problem (1) has at least one positive, nondecreasing and concave solution w^* or v^* with $\|w^*\| \leq a$ and $\|v^*\| \leq a$. There are some conclusions of w^* and v^* as follows

$$w^* = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} T^n w_0, \quad (w^*)' = \lim_{n \rightarrow \infty} (w_n)' = \lim_{n \rightarrow \infty} (T^n w_0)', \\ (w^*)'' = \lim_{n \rightarrow \infty} (w_n)'' = \lim_{n \rightarrow \infty} (T^n w_0)'', \text{ where } w_0(t) = at(1 - \frac{t}{2}), \quad 0 \leq t \leq 1,$$

and

$$v^* = \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} T^n v_0, \quad (v^*)' = \lim_{n \rightarrow \infty} (v_n)' = \lim_{n \rightarrow \infty} (T^n v_0)',$$

$$(v^*)'' = \lim_{n \rightarrow \infty} (v_n)'' = \lim_{n \rightarrow \infty} (T^n v_0)'', \quad \text{where } v_0(t) = 0, \quad 0 \leq t \leq 1,$$

where $(Tu)(t)$ is defined by (3).

The iterative schemes in this theorem are $w_0(t) = at(1 - \frac{t}{2})$, $w_{n+1} = Tw_n = T^n w_0$, $n = 0, 1, 2, \dots$ which starts off with a known simple quadratic function and $v_0(t) = 0$, $v_{n+1} = Tv_n = T^n v_0$, $n = 0, 1, 2, \dots$ which starts off with the zero function.

Proof. We denote $\overline{P}_a = \{u \in P \mid \|u\| \leq a\}$. Firstly, we will prove that $T : \overline{P}_a \rightarrow \overline{P}_a$.

If $u \in \overline{P}_a$, by (H_1) and (H_2) we have

$$0 \leq f(t, u(t), u'(t), u''(t)) \leq f(t, a, a, -a) \leq \max_{0 \leq t \leq 1} f(t, a, a, -a) \leq \frac{a}{A}, \quad \text{for } 0 \leq t \leq 1.$$

Since

$$(Tu)(1) \leq \frac{a}{A} \frac{1}{2} \int_0^1 q(s) ds < a, \quad (Tu)'(0) \leq \frac{a}{A} \int_0^1 q(s) ds = a,$$

$$-(Tu)''(1) \leq \frac{a}{A} \int_0^1 q(s) ds = a.$$

Hence, we assert that $T : \overline{P}_a \rightarrow \overline{P}_a$.

By (H_1) we have, for any $u_i \in P$ ($i = 1, 2$) with $u_1 \leq u_2$, $u_1' \leq u_2'$ and $u_1'' \geq u_2''$. Then by (2) and (3), we can easily get $Tu_1 \leq Tu_2$, $(Tu_1)' \leq (Tu_2)'$ and $(Tu_1)'' \geq (Tu_2)''$.

Let $w_0(t) = at(1 - \frac{t}{2})$, $0 \leq t \leq 1$, then $w_0(t) \in \overline{P}_a$. Let $w_1 = Tw_0$, then $w_1 \in \overline{P}_a$. We denote $w_{n+1} = Tw_n$, $n = 0, 1, 2, \dots$. Then we have $w_n \subseteq \overline{P}_a$, $n = 1, 2, \dots$. Due to the complete continuity of T , $\{w_n\}_{n=1}^\infty$ is asserted a sequentially compact set.

Because

$$w_1(t) = Tw_0(t) = \int_0^1 G(t, s)q(s)f(s, w_0(s), w_0'(s), w_0''(s))ds \leq at(1 - \frac{t}{2}) = w_0(t), \quad 0 \leq t \leq 1,$$

$$w_1'(t) = (Tw_0)'(t)$$

$$= \int_0^t (s - t)q(s)f(s, w_0(s), w_0'(s), w_0''(s))ds + \int_0^1 (1 - s)q(s)f(s, w_0(s), w_0'(s), w_0''(s))ds$$

$$\leq a(1 - t) = w_0'(t), \quad 0 \leq t \leq 1,$$

and

$$w_1''(t) = (Tw_0)''(t) = - \int_0^t q(s)f(s, w_0(s), w_0'(s), w_0''(s))ds \geq -a = w_0''(t), \quad 0 \leq t \leq 1.$$

We obtain that

$$w_2(t) = Tw_1(t) \leq Tw_0(t) = w_1(t), \quad w_2'(t) = (Tw_1)'(t) \leq (Tw_0)'(t) = w_1'(t),$$

$$w_2''(t) = (Tw_1)''(t) \geq (Tw_0)''(t) = w_1''(t), \quad 0 \leq t \leq 1.$$

Compute continuously according to the iteration rule and by the induction, then

$$w_{n+1} \leq w_n, \quad w_{n+1}'(t) \leq w_n'(t), \quad w_{n+1}''(t) \geq w_n''(t), \quad 0 \leq t \leq 1, \quad n = 0, 1, 2, \dots$$

Then we can show there exists $w^* \in \overline{P}_a$ that satisfies $w_n \rightarrow w^*$. By applying the continuity of T and the iteration of $w_{n+1} = Tw_n$ in combination, we obtain $Tw^* = w^*$.

The other iteration direction is start off with the zero function. We choose $v_0(t) = 0$, $0 \leq t \leq 1$, then $v_0(t) \in \overline{P}_a$. Let $v_1 = Tv_0$, then $v_1 \in \overline{P}_a$. We denote $v_{n+1} = Tv_n$, $n = 0, 1, 2, \dots$. Then we have $v_n \subseteq \overline{P}_a$, $n = 1, 2, \dots$. Since T is completely continuous, we assert that $\{v_n\}_{n=1}^\infty$ is a sequentially compact set.

Since $v_1 = Tv_0 \in \overline{P}_a$, we have $v_1(t) = Tv_0(t) \geq 0$, $v_1'(t) = Tv_0'(t) \geq 0$, $v_1''(t) = (Tv_0)''(t) \leq 0$, for $0 \leq t \leq 1$. Then $v_2(t) \geq v_1(t)$, $v_2'(t) \geq v_1'(t)$, $v_2''(t) \leq v_1''(t)$, for $0 \leq t \leq 1$. By an induction argument similar to the above we obtain

$$v_{n+1} \geq v_n, \quad v'_{n+1}(t) \geq v'_n(t), \quad v''_{n+1}(t) \leq v''_n(t), \quad 0 \leq t \leq 1, \quad n = 0, 1, 2, \dots$$

Hence there exists $v^* \in \overline{P}_a$ such that $v_n \rightarrow v^*$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

If $f(t, 0, 0, 0) \neq 0$, $0 \leq t \leq 1$, then the zero function is not the solution of (1). Thus $\max_{0 \leq t \leq 1} |v^*(t)| > 0$, we have $v^* > 0$, for $0 < t < 1$.

From the previous discussion, if we find the fixed point of T in P , we get the solution to (1). Hence, we can conclude that that the boundary value problem (1) has at least one positive nondecreasing and concave solution w^* or v^* .

Furthermore, if $\lim_{n \rightarrow \infty} w_n \neq \lim_{n \rightarrow \infty} v_n$, then w^* and v^* are different. We obtain problem (1) has two positive nondecreasing and concave solutions. The other case is $\lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} v_n$, then $w^* = v^*$ which means the two values coincide and problem (1) has a positive nondecreasing and concave solution. Anyway, the problem (1) has at least one positive nondecreasing and concave solution.

The proof is completed. \square

The following corollary follows easily.

Corollary 1. Assume that (H_1) and (H_3) hold, and there exists $a > 0$, such that $(H_4): \lim_{\ell \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, \ell, a, -a)}{\ell} \leq \frac{1}{A}$, (particularly, $\lim_{\ell \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, \ell, a, -a)}{\ell} = 0$).

Then the boundary value problem (1) has at least one positive, nondecreasing and concave solution w^* or v^* . Consequently, the relevant conclusions about w^* and v^* in Theorem 1 hold.

4. Example

Example 1. Let $q(t) = 1$, we consider the following boundary value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \quad 0 < t < 1, \tag{4}$$

$$u(0) = u'(1) = u''(0) = 0, \tag{5}$$

where $f(t, x, y, z) = \frac{1}{4}t^2 + \frac{1}{4}x + \frac{1}{8}y^2 - \frac{1}{4}z$. Choose $a = 2$, and we have $A = 1$.

So by Theorem 1, the boundary value problem (4) and (5) has at least one positive, nondecreasing and concave solution w^* or v^* with $\|w^*\| \leq 2$ and $\|v^*\| \leq 2$, such that

$$w^* = \lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} T^n w_0, \quad (w^*)' = \lim_{n \rightarrow \infty} (w_n)' = \lim_{n \rightarrow \infty} (T^n w_0)',$$

$$(w^*)'' = \lim_{n \rightarrow \infty} (w_n)'' = \lim_{n \rightarrow \infty} (T^n w_0)'', \quad \text{where } w_0(t) = 2t - t^2, \quad 0 \leq t \leq 1,$$

and

$$v^* = \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} T^n v_0, \quad (v^*)' = \lim_{n \rightarrow \infty} (v_n)' = \lim_{n \rightarrow \infty} (T^n v_0)',$$

$$(v^*)'' = \lim_{n \rightarrow \infty} (v_n)'' = \lim_{n \rightarrow \infty} (T^n v_0)'', \quad \text{where } v_0(t) = 0, \quad 0 \leq t \leq 1,$$

where $(Tu)(t)$ is defined by (3).

For $n = 0, 1, 2, \dots$ The two iterative schemes are

$$\begin{aligned}
 w_0(t) &= 2t - t^2, \quad 0 \leq t \leq 1, \\
 w_1(t) &= -\frac{1}{120}t^5 + \frac{1}{48}t^4 - \frac{1}{6}t^3 + \frac{11}{24}t, \quad 0 \leq t \leq 1, \\
 w_2(t) &= -\frac{1}{4561920}t^{11} + \frac{1}{829440}t^{10} - \frac{1}{82944}t^9 + \frac{1}{26880}t^8 - \frac{73}{483840}t^7 - \frac{11}{138240}t^6 \\
 &\quad - \frac{5}{2304}t^5 - \frac{35}{2304}t^4 - \frac{121}{27648}t^3 + \frac{24989}{290304}t, \quad 0 \leq t \leq 1, \\
 w_3(t) &= -\frac{14620753605427200}{9649t^{19}} + \frac{1271369878732800}{2201t^{18}} - \frac{23t^{21}}{823t^{17}} + \frac{271t^{20}}{5503t^{16}} \\
 &\quad - \frac{18666021401395200}{982422179020800} + \frac{74912366592000}{7912366592000} + \frac{299649466368000}{299649466368000} \\
 &\quad - \dots \dots \dots - \frac{807311t^6}{11466178560} - \frac{632025743t^5}{128421199872} - \frac{8153t^4}{6967296} - \frac{624450121t^3}{4045267795968} \\
 &\quad + \frac{5622836449329950699t}{186846874227965952000}, \quad 0 \leq t \leq 1,
 \end{aligned}$$

$$\begin{aligned}
 w_4(t) &= -\frac{t^{47}}{314515486692275039735441994547200000} + \frac{t^{46}}{13383637731586171903635829555200000} \\
 &\quad - \frac{118852522029411983100766008115200000}{173t^{45}} + \frac{29581072149542538016190650908672000}{571t^{44}} \\
 &\quad - \dots \dots \dots - \frac{31616289735913447234529323165770588601t^3}{1675764211620543699419355421020782592000000} \\
 &\quad + \frac{160511358248153446347381073284028009201404114366497t}{6780718732903555345790847898231140300789645312000000}, \quad 0 \leq t \leq 1,
 \end{aligned}$$

$$w_{n+1}(t) = (Tw_n)(t) = \int_0^1 G(t,s) \left(\frac{1}{4}s^2 + \frac{1}{4}w_n(s) + \frac{1}{8}w_n^2(s) - \frac{1}{4}w_n''(s) \right) ds, \quad 0 \leq t \leq 1.$$

which starts off with a known simple quadratic function, and

$$\begin{aligned}
 v_0(t) &= 0, \\
 v_1(t) &= -\frac{1}{240}t^5 + \frac{1}{48}t, \quad 0 \leq t \leq 1, \\
 v_2(t) &= -\frac{1}{18247680}t^{11} + \frac{1}{322560}t^8 + \frac{1}{1935360}t^7 - \frac{1}{5760}t^6 - \frac{1}{240}t^5 - \frac{1}{4068}t^4 \\
 &\quad - \frac{1}{110592}t^3 + \frac{33011}{1451520}t, \quad 0 \leq t \leq 1, \\
 v_3(t) &= -\frac{233932057686835200}{251t^{17}} + \frac{1830002098176000}{41t^{16}} + \frac{10666297943654400}{55247t^{15}} - \frac{31188005683200}{3919t^{14}} \\
 &\quad - \frac{318377558016000}{898948399104000} + \frac{23372658376704000}{59513713459200} \\
 &\quad - \dots \dots \dots - \frac{132359t^4}{557383680} - \frac{1089726121t^3}{101131694899200} + \frac{251944745719415081t}{10990992601645056000}, \quad 0 \leq t \leq 1,
 \end{aligned}$$

$$v_4(t) = - \frac{t^{47}}{80515964593222410172273150604083200000} + \frac{t^{44}}{295810721495425380161906509086720000}$$

$$+ \frac{t^{43}}{1691155758433176951246880411484160000} - \frac{t^{42}}{4855457244999072498555499315200000}$$

$$+ \dots - \frac{63476154895620723780956748770236561t^3}{5798492081731985119098115643670528000000}$$

$$+ \frac{49776339527441954834684459870069552303067862246322909t}{2169829994529137710653071327433964896252686499840000000}, \quad 0 \leq t \leq 1,$$

$$\dots$$

$$v_{n+1}(t) = (Tv_n)(t) = \int_0^1 G(t,s) \left(\frac{1}{4}s^2 + \frac{1}{4}w_n(s) + \frac{1}{8}w_n^2(s) - \frac{1}{4}w_n''(s) \right) ds, \quad 0 \leq t \leq 1.$$

which starts off with the zero function.

In Figures 1 and 2 below, we will illustrate the first several elements in the schemes. The figures demonstrate the iterative process.

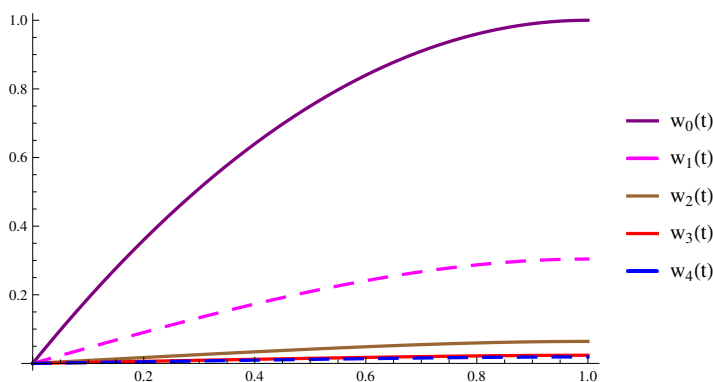


Figure 1. Curves of successive iterations.

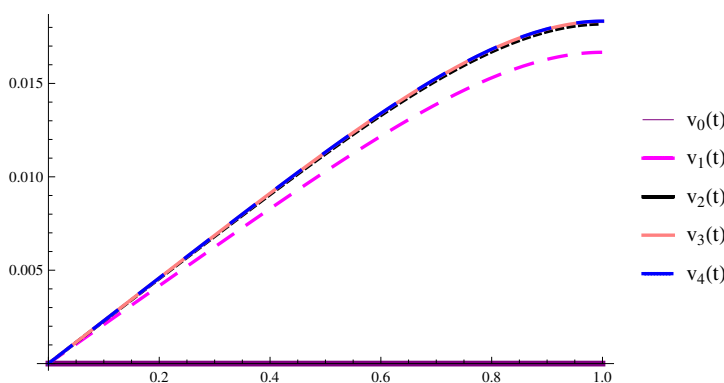


Figure 2. Curves of successive iterations.

and

$$\|w_1 - w_0\| = 2.0000000000000000, \quad \|v_1 - v_0\| = 0.083333333333333333,$$

$$\|w_2 - w_1\| = 0.657289255401234568, \quad \|v_2 - v_1\| = 0.0076774691358024691,$$

$$\|w_3 - w_2\| = 0.144307032230180722, \quad \|v_3 - v_2\| = 0.0007056973781479649,$$

$$\|w_4 - w_3\| = 0.020648460036251130, \quad \|v_4 - v_3\| = 0.0000666371356413746.$$

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