symmetry

MDPI

Article

On Some New Trapezoidal Type Inequalities for Twice (p, q) Differentiable Convex Functions in Post-Quantum Calculus

Thanin Sitthiwirattham [1,*,†], Ghulam Murtaza [2,*,†], Muhammad Aamir Ali [3,*,†], Sotiris K. Ntouyas [4,5,†], Muhammad Adeel [2,†] and Jarunee Soontharanon [6,†]

1 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
2 Department of Mathematics, University of Management and Technology, Lahore 54700, Pakistan; F2019349078@umt.edu.pk
3 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China
4 Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece; sntouyas@uoi.gr
5 Nonlinear Analysis and Applied Mathematics (NAAM)—Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
6 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand; jarunee.s@sci.kmutnb.ac.th
* Correspondence: thanin_sit@dusit.ac.th (T.S.); ghulammurtaza@umt.edu.pk (G.M.); mahr.muhammad.aamir@gmail.com (M.A.A.)
† These authors contributed equally to this work.

Citation: Sitthiwirattham, T.; Murtaza, G.; Ali, M.A.; Ntouyas, S.K.; Adeel, M.; Soontharanon, J. On Some New Trapezoidal Type Inequalities for Twice (p, q) Differentiable Convex Functions in Post-Quantum Calculus. *Symmetry* **2021**, *13*, 1605. https://doi.org/10.3390/sym13091605

Academic Editor: Nicusor Minculete

Received: 20 July 2021
Accepted: 30 August 2021
Published: 1 September 2021

Abstract: Quantum information theory, an interdisciplinary field that includes computer science, information theory, philosophy, cryptography, and symmetry, has various applications for quantum calculus. Inequalities has a strong association with convex and symmetric convex functions. In this study, first we establish a (p, q)-integral identity involving the second (p, q)-derivative and then we used this result to prove some new trapezoidal type inequalities for twice (p, q)-differentiable convex functions. It is also shown that the newly established results are the refinements of some existing results in the field of integral inequalities. Analytic inequalities of this nature and especially the techniques involved have applications in various areas in which symmetry plays a prominent role.

Keywords: Hermite–Hadamard inequality; (p, q)-calculus; convex functions

1. Introduction

In convex functions theory, Hermite–Hadamard (HH) inequality is very important, and was discovered by C. Hermite and J. Hadamard independently (see, also [1,2], p. 137).

$$\Pi\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \Pi(\varkappa) d\varkappa \leq \frac{\Pi(\pi_1) + \Pi(\pi_2)}{2} \tag{1}$$

where Π is a convex function. In the case of concave mappings, the above inequality satisfies in reverse order.

On the other hand, several works in the field of q-analysis, beginning with Euler, have been implemented in order to master the mathematics that underpins quantum computing. The term q-calculus creates a link between mathematics and physics. It's used in combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other fields, as well as relativity theory, mechanics, and quantum theory [3,4]. In quantum information theory, it has many applications [5–7] and it not only has a link with the estimations calculus, but also to affine algebraic geometry including the famous Jacobian Conjecture [8,9]. Euler used the q-parameter in Newton's work on infinite series, which is why he is thought to be inventor of this important branch of mathematics.

The concept of q-calculus that is known to be calculus without limits was given by Jackson [10,11] for the first time in a proper way. The notions about the q-fractional integral and q-Riemann–Liouville fractional integral was given by Al-Salam [12] in 1996. Since the research increased gradually in this field, therefore Tariboon and Ntouyas [13] gave the idea about the ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. The notions about the ${}^{\pi_2}D_q$-difference operator and q^{π_2}-integral were given by Bermudo et al. [14] very recently in 2020. Sadjang [15] generalized the concept of q-calculus by introducing the concepts of (p,q)-calculus. Soontharanon et al. [16] introduced the concepts of fractional (p,q)-calculus later on. The (p,q)-variant of ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral was introduced by Tunç and Göv [17]. Recently, in 2021, Chu et al. introduced the notions of ${}^{\pi_2}D_{p,q}$ derivative and $(p,q)^{\pi_2}$-integral in [18].

Many integral inequalities for many sorts of functions have indeed been investigated employing quantum as well as post-quantum integrals. For example, the HH inequalities and their right–left estimates for convex and coordinated convex functions via ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_{π_1}, q^{π_2}-integrals were given by different authors in [19–27]. Noor et al. [28] used the pre-invexity to prove HH inequalities in the setup of q-calculus. Some parameterized q-integral inequalities for generalized quasi-convex functions established by Nwaeze et al. [29]. Khan et al. used the notions of Green functions to establish some new inequalities of HH type in [30]. Budak et al. [31], Ali et al. [32,33] and Vivas-Cortez et al. [34] proved some new boundaries for Simpson's and Newton's type inequalities for convex and coordinated convex functions in the setting of q-calculus. One can consult [35–37] for q-Ostrowski's inequalities for convex and coordinated convex functions. In [38], the authors generalized the results of [21] and proved HH type inequalities and their left estimates using ${}_{\pi_1}D_{p,q}$-difference operator and $(p,q)_{\pi_1}$-integral. Recently, in [39], the authors established the right estimates of HH type inequalities proved by Kunt et al. [38]. For (p,q)-Ostrowski type inequalities, one can consult [18]. The results proved in [14] were generalized in [40].

Inspired by the ongoing studies, we establish some new post-quantum trapezoidal type inequalities for (p,q)-differentiable convex functions through the (p,q)-integral. Furthermore, we prove that the newly established inequalities are the extensions of some already given inequalities.

The organization of this paper is as follows: In Section 2, a short explanation of the concepts of q-calculus and some associated works in this direction are given. In Section 3,, we review the notions of (p,q)-derivatives and integrals. In Section 4, the trapezoidal type inequalities for twice (p,q)-differentiable functions via (p,q)-integrals are presented. The relationship between the results provided here and comparable outcomes in the literature are also taken into account. Section 5 provides some findings as well as other study directions.

2. Quantum Derivatives and Integrals

In this portion, we recall a few known definitions and related inequalities in q-calculus. Set the following notation ([4]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}, \;\; q \in (0,1).$$

The q-Jackson integral of a mapping Π from 0 to π_2 is given by Jackson [11], which is defined as:

$$\int_0^{\pi_2} \Pi(\varkappa)\; d_q\varkappa \; = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n \Pi(\pi_2 q^n), \text{ where } 0 < q < 1 \tag{2}$$

provided that the sum converges absolutely. Moreover, over the interval $[\pi_1, \pi_2]$, he gave the following integral of a mapping Π:

$$\int_{\pi_1}^{\pi_2} \Pi(\varkappa)\ d_q\varkappa = \int_{0}^{\pi_2} \Pi(\varkappa)\ d_q\varkappa - \int_{0}^{\pi_1} \Pi(\varkappa)\ d_q\varkappa\ .$$

Definition 1 ([13])**.** *The q_{π_1}-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is defined as:*

$$_{\pi_1}D_q\Pi(\varkappa) = \frac{\Pi(\varkappa) - \Pi(q\varkappa + (1-q)\pi_1)}{(1-q)(\varkappa - \pi_1)},\ \varkappa \neq \pi_1. \tag{3}$$

For $\varkappa = \pi_1$, we state $_{\pi_1}D_q\Pi(\pi_1) = \lim_{\varkappa\to\pi_1}\ {}_{\pi_1}D_q\Pi(\varkappa)$ if it exists and it is finite.

Definition 2 ([14])**.** *The q^{π_2}-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$^{\pi_2}D_q\Pi(\varkappa) = \frac{\Pi(q\varkappa + (1-q)\pi_2) - \Pi(\varkappa)}{(1-q)(\pi_2 - \varkappa)},\ \varkappa \neq \pi_2. \tag{4}$$

For $\varkappa = \pi_2$, we state $^{\pi_2}D_q\Pi(\pi_2) = \lim_{\varkappa\to\pi_2}\ {}^{\pi_2}D_q\Pi(\varkappa)$ if it exists and it is finite.

Definition 3 ([13])**.** *The q_{π_1}-definite integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is defined as:*

$$\int_{\pi_1}^{\varkappa} \Pi(\tau)\ {}_{\pi_1}d_q\tau = (1-q)(\varkappa - \pi_1)\sum_{n=0}^{\infty} q^n\Pi(q^n\varkappa + (1-q^n)\pi_1),\ \varkappa \in [\pi_1, \pi_2]. \tag{5}$$

On the other side, the following concept of q-definite integral is stated by Bermudo et al. [14]:

Definition 4 ([14])**.** *The q^{π_2}-definite integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is given as:*

$$\int_{\varkappa}^{\pi_2} \Pi(\tau)\ {}^{\pi_2}d_q\tau = (1-q)(\pi_2 - \varkappa)\sum_{n=0}^{\infty} q^n\Pi(q^n\varkappa + (1-q^n)\pi_2),\ \varkappa \in [\pi_1, \pi_2]. \tag{6}$$

Remark 1. *If Π is a symmetric function, that is $\Pi(t) = f(\pi_1 + \pi_2 - t)$, then we have the following relation*

$$\int_{\pi_1}^{\pi_2} \Pi(t)\ {}_{\pi_1}d_qt = \int_{\pi_1}^{\pi_2} \Pi(t)\ {}^{\pi_2}d_qt.$$

3. Post-Quantum Derivatives and Integrals

In this section, we review some fundamental notions and notations of (p, q)-calculus. The $[n]_{p,q}$ is said to be (p, q)-integers and expressed as:

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}$$

with $0 < q < p \leq 1$. The $[n]_{p,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}!$ are called (p, q)-factorial and (p, q)-binomial, respectively, and expressed as:

$$[n]_{p,q}! = \prod_{k=1}^{n}[k]_{p,q},\ n \geq 1,\ [0]_{p,q}! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}! = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

Definition 5 ([15]). *The (p,q)-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$$D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa) - \Pi(q\varkappa)}{(p-q)\varkappa}, \ \varkappa \neq 0 \tag{7}$$

with $0 < q < p \leq 1$.

Definition 6 ([17]). *The $(p,q)_{\pi_1}$-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$${}_{\pi_1}D_{p,q}\Pi(\varkappa) = \frac{\Pi(p\varkappa + (1-p)\pi_1) - \Pi(q\varkappa + (1-q)\pi_1)}{(p-q)(\varkappa - \pi_1)}, \ \varkappa \neq \pi_1 \tag{8}$$

with $0 < q < p \leq 1$.

For $\varkappa = \pi_1$, we state ${}_{\pi_1}D_{p,q}\Pi(\pi_1) = \lim_{\varkappa \to \pi_1} {}_{\pi_1}D_{p,q}\Pi(\varkappa)$ if it exists and it is finite.

Definition 7 ([18]). *The $(p,q)^{\pi_2}$-derivative of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ is given as:*

$${}^{\pi_2}D_{p,q}\Pi(\varkappa) = \frac{\Pi(q\varkappa + (1-q)\pi_2) - \Pi(p\varkappa + (1-p)\pi_2)}{(p-q)(\pi_2 - \varkappa)}, \ \varkappa \neq \pi_2. \tag{9}$$

For $\varkappa = \pi_2$, we state ${}^{\pi_2}D_{p,q}\Pi(\pi_2) = \lim_{\varkappa \to \pi_2} {}^{\pi_2}D_{p,q}\Pi(\varkappa)$ if it exists and it is finite.

Remark 2. *It is clear that if we use $p = 1$ in (8) and (9), then the equalities (8) and (9) reduce to (3) and (4), respectively.*

Definition 8 ([17]). *The definite $(p,q)_{\pi_1}$-integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is stated as:*

$$\int_{\pi_1}^{\varkappa} \Pi(\tau) \ {}_{\pi_1}d_{p,q}\tau = (p-q)(\varkappa - \pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\varkappa + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_1 \right) \tag{10}$$

with $0 < q < p \leq 1$.

Definition 9 ([18]). *The definite $(p,q)^{\pi_2}$-integral of mapping $\Pi : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is stated as:*

$$\int_{\varkappa}^{\pi_2} \Pi(\tau) \ {}^{\pi_2}d_{p,q}\tau = (p-q)(\pi_2 - \varkappa) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}}\varkappa + \left(1 - \frac{q^n}{p^{n+1}}\right)\pi_2 \right) \tag{11}$$

with $0 < q < p \leq 1$.

Remark 3. *It is evident that if we pick $p = 1$ in (10) and (11), then the equalities (10) and (11) change into (5) and (6), respectively.*

Remark 4. *If we take $\pi_1 = 0$ and $\varkappa = \pi_2 = 1$ in (10), then we have*

$$\int_0^1 \Pi(\tau) \ {}_0d_{p,q}\tau = (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(\frac{q^n}{p^{n+1}} \right).$$

Similarly, by taking $\varkappa = \pi_1 = 0$ and $\pi_2 = 1$ in (11), then we obtain that

$$\int_0^1 \Pi(\tau) \ {}^1d_{p,q}\tau = (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Pi\left(1 - \frac{q^n}{p^{n+1}} \right).$$

In [38], Kunt et al. proved the following HH type inequalities for convex functions via $(p,q)_{\pi_1}$-integral:

Theorem 1. *For a convex mapping $\Pi:[\pi_1,\pi_2]\to\mathbb{R}$, which is differentiable on $[\pi_1,\pi_2]$, the following inequalities hold for $(p,q)_{\pi_1}$-integral:*

$$\Pi\left(\frac{q\pi_1+p\pi_2}{[2]_{p,q}}\right)\le\frac{1}{p(\pi_2-\pi_1)}\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\Pi(\varkappa)\ {}_{\pi_1}d_{p,q}\varkappa\le\frac{q\Pi(\pi_1)+p\Pi(\pi_2)}{[2]_{p,q}}\tag{12}$$

where $0<q<p\le 1$.

Lemma 1 ([40])**.** *We have the following equalities:*

$$\int_{\pi_1}^{\pi_2}(\pi_2-\varkappa)^{\alpha}\ {}^{\pi_2}d_{p,q}\varkappa=\frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

$$\int_{\pi_1}^{\pi_2}(\varkappa-\pi_1)^{\alpha}\ {}_{\pi_1}d_{p,q}\varkappa=\frac{(\pi_2-\pi_1)^{\alpha+1}}{[\alpha+1]_{p,q}}$$

where $\alpha\in\mathbb{R}-\{-1\}$.

Remark 5. *If Π is a symmetric function, that is $\Pi(t)=f(\pi_1+\pi_2-t)$, then we have following relation*

$$\int_{\pi_1}^{p\pi_2+(1-p)\pi_1}\Pi(t)\ {}_{\pi_1}d_{p,q}t=\int_{p\pi_1+(1-p)\pi_2}^{\pi_2}\Pi(t)\ {}^{\pi_2}d_{q}t.$$

4. Post-Quantum Trapezoidal Type Inequalities

In this section, we prove some new trapezoidal type inequalities for twice (p,q)-differentiable convex functions using the (p,q)-integrals.

Lemma 2. *Consider a mapping $\Pi:I=[\pi_1,\pi_2]\to\mathbb{R}$, which is twice (p,q)-differentiable and ${}^{\pi_2}D^2_{p,q}\Pi$ is continuous and integrable on I. Then, the following equality holds:*

$$\begin{aligned}&\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa\\ =\ &\frac{q^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\int_0^1 p\tau(1-q\tau)\ {}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2)d_{p,q}\tau,\end{aligned}\tag{13}$$

where $0<q<p\le 1$.

Proof. Consider

$$\begin{aligned}&{}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2)\\ =\ &{}^{\pi_2}D_{p,q}\left[\frac{\Pi(q\tau\pi_1+(1-q\tau)\pi_2)-\Pi(p\tau\pi_1+(1-p\tau)\pi_2)}{(p-q)(\pi_2-\pi_1)\tau}\right]\\ =\ &\frac{p\Pi\big(q^2\tau\pi_1+(1-q^2\tau)\pi_2\big)-[2]_{p,q}\Pi(pq\tau+(1-pq\tau)\pi_2)+q\Pi\big(p^2\tau\pi_1+(1-p^2\tau)\pi_2\big)}{pq(p-q)^2(\pi_2-\pi_1)^2\tau^2}.\end{aligned}$$

Now, from Definition 9, we have

$$\begin{aligned}
&\int_0^1 p\tau(1-q\tau)\, {}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2)d_{p,q}\tau \\
=&\int_0^1 p\tau(1-q\tau)\left[\frac{p\Pi(q^2\tau\pi_1+(1-q^2\tau)\pi_2)-(p+q)\Pi(pq\tau +(1-pq\tau)\pi_2)+q\Pi(p^2\tau\pi_1+(1-p^2\tau)\pi_2)}{pq(p-q)^2(\pi_2-\pi_1)^2\tau^2}\right]d_{p,q}\tau \\
=&\frac{1}{q(p-q)^2(\pi_2-\pi_1)^2} \\
&\times\left[\begin{array}{c}\int_0^1 \frac{p\Pi\left(q^2\tau\pi_1+(1-q^2\tau)\pi_2\right)-(p+q)\Pi(pq\tau\pi_1+(1-pq\tau)\pi_2)+q\Pi\left(p^2\tau\pi_1+(1-p^2\tau)\pi_2\right)}{\tau}d_{p,q}\tau \\ -pq\int_0^1\Pi\left(q^2\tau\pi_1+(1-q^2\tau)\pi_2\right)d_{p,q}\tau+q[2]_{p,q}\int_0^1\Pi(pq\tau\pi_1+(1-pq\tau)\pi_2)d_{p,q}\tau \\ -q^2\int_0^1\Pi\left(p^2\tau\pi_1+(1-p^2\tau)\pi_2\right)d_{p,q}\tau\end{array}\right] \\
=&\frac{1}{q(p-q)^2(\pi_2-\pi_1)^2}\left[\begin{array}{c} p(p-q)\sum_{n=0}^{\infty}\Pi\left(\frac{q^{n+2}}{p^{n+1}}\pi_1+(1-\frac{q^{n+2}}{p^{n+1}})\pi_2\right) \\ -(p^2-q^2)\sum_{n=0}^{\infty}\Pi\left(\frac{q^{n+1}}{p^{n}}\pi_1+(1-\frac{q^{n+1}}{p^{n}})\pi_2\right) \\ +q(p-q)\sum_{n=0}^{\infty}\Pi\left(\frac{q^{n}}{p^{n-1}}\pi_1+(1-\frac{q^{n}}{p^{n-1}})\pi_2\right) \\ -pq(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^{n+2}}{p^{n+1}}\pi_1+(1-\frac{q^{n+2}}{p^{n+1}})\pi_2\right) \\ +q(p^2-q^2)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^{n+1}}{p^{n}}\pi_1+(1-\frac{q^{n+1}}{p^{n}})\pi_2\right) \\ -q^2(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\Pi\left(\frac{q^{n}}{p^{n+1}}\pi_1+(1-\frac{q^{n}}{p^{n+1}})\pi_2\right)\end{array}\right] \\
=&\frac{1}{q(p-q)^2(\pi_2-\pi_1)^2}\left[\begin{array}{c} p(p-q)(\Pi(\pi_2)-\Pi(q\pi_1+(1-q)\pi_2)) \\ +q(p-q)(\Pi(p\pi_1+(1-p)\pi_2)-\Pi(\pi_2)) \\ -\frac{p^3}{q(\pi_2-\pi_1)}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\, {}^{\pi_2}d_{p,q}\varkappa+\frac{p^2}{q}(p-q)\Pi(p\pi_1+(1-p)\pi_2) \\ +p(p-q)\Pi(q\pi_1+(1-q)\pi_2)-\frac{q^2}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\, {}^{\pi_2}d_{p,q}\varkappa \\ +\frac{p[2]_{p,q}}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\, {}^{\pi_2}d_{p,q}\varkappa-(p^2-q^2)\Pi(p\pi_1+(1-p)\pi_2)\end{array}\right]
\end{aligned}$$

$$=\frac{1}{q(\pi_2-\pi_1)^2}\Pi(\pi_2)+\frac{p}{q^2(\pi_2-\pi_1)^2}\Pi(p\pi_1+(1-p)\pi_2)-\frac{[2]_{p,q}}{q^2(\pi_2-\pi_1)^3}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\, {}^{\pi_2}d_{p,q}\varkappa. \tag{14}$$

Now, we have the identity (13) by multiplying both sides of (14) by $\frac{q^2(\pi_2-\pi_1)^2}{[2]_{p,q}}$, and the proof is complete. □

Remark 6. *In Lemma 2, If we set $p=1$, then we have*

$$\frac{\Pi(\pi_1)+q\Pi(\pi_2)}{[2]_q}-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(\varkappa)\, {}^{\pi_2}d_q\varkappa=\frac{q^2(\pi_2-\pi_1)^2}{[2]_q}\int_0^1\tau(1-q\tau)\, {}^{\pi_2}D_q^2\Pi(\tau\pi_1+(1-\tau)\pi_2)d_q\tau.$$

This is established by Ali et al. in [19].

Remark 7. *In Lemma 2, If we set $p = 1$ and later take the limit as $q \to 1^-$, then we have*

$$\frac{\Pi(\pi_1) + \Pi(\pi_2)}{2} - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \Pi(\varkappa)d\varkappa = \frac{(\pi_2 - \pi_1)^2}{2}\int_0^1 \tau(1-\tau)\Pi''(\tau\pi_1 + (1-\tau)\pi_2)d\tau.$$

This is established by Alomari et al. in [41].

Theorem 2. *Consider the assumptions in Lemma 2 are valid. If $|{}^{\pi_2}D^2_{p,q}\Pi|$ is convex on I, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{p\Pi(p\pi_1 + (1-p)\pi_2) + q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{\pi_2 - \pi_1} \int_{p^2\pi_1 + (1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\
\le\ & \frac{p^3 q^2 (\pi_2 - \pi_1)^2}{[2]_{p,q}[3]_{p,q}[4]_{p,q}} \left[p\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right| + (p^2 - p + q^2)\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right| \right],
\end{aligned}$$

where $0 < q < p \le 1$.

Proof. Taking modulus of (13) and applying the convexity of $\left|{}^{\pi_2}D^2_{p,q}\Pi\right|$, we obtain

$$\begin{aligned}
&\left| \frac{p\Pi(p\pi_1 + (1-p)\pi_2) + q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{\pi_2 - \pi_1} \int_{p^2\pi_1 + (1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\
\le\ & \frac{pq^2(\pi_2 - \pi_1)^2}{[2]_{p,q}} \int_0^1 \tau(1 - q\tau)\left| {}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1 + (1-\tau)\pi_2) \right| d_{p,q}\tau \\
\le\ & \frac{pq^2(\pi_2 - \pi_1)^2}{[2]_{p,q}} \int_0^1 \tau(1 - q\tau)\left[\tau\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right| + (1-\tau)\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right| \right] d_{p,q}\tau \\
\le\ & \frac{pq^2(\pi_2 - \pi_1)^2}{[2]_{p,q}} \left[\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right| \int_0^1 \tau^2(1 - q\tau)d_{p,q}\tau + \left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right| \int_0^1 \tau(1 - q\tau)(1-\tau)d_{p,q}\tau \right] \\
=\ & \frac{pq^2(\pi_2 - \pi_1)^2}{[2]_{p,q}[3]_{p,q}[4]_{p,q}} \left[p^3\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right| + p^2(p^2 - p + q^2)\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right| \right]
\end{aligned}$$

and the proof is completed. □

Remark 8. *In Theorem 2, if we set $p = 1$, then we obtain [19], Theorem 4.*

Remark 9. *In Theorem 2, If we set $p = 1$ and later take the limit as $q \to 1^-$, then we obtain [42], Proposition 2.*

Theorem 3. *Consider the assumptions in Lemma 2 as valid. If $|{}^{\pi_2}D^2_{p,q}\Pi|^r$, $r \ge 1$, is convex on I, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{p\Pi(p\pi_1 + (1-p)\pi_2) + q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{\pi_2 - \pi_1} \int_{p^2\pi_1 + (1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right| \\
\le\ & \frac{p^3 q^2 (\pi_2 - \pi_1)^2}{[2]_{p,q}^{2-\frac{1}{r}}[3]_{p,q}[4]_{p,q}^{\frac{1}{r}}} \left[p\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right|^r + (p^2 - p + q^2)\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right|^r \right]^{\frac{1}{r}},
\end{aligned}$$

where $0 < q < p \leq 1$.

Proof. Taking modulus of (13) and applying the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa\right| \\
\leq\ & \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\int_0^1\tau^2(1-q\tau)\left|{}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2)\right|d_{p,q}\tau \\
\leq\ & \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^1\tau(1-q\tau)d_{p,q}\tau\right]^{1-\frac{1}{r}}\left[\int_0^1\tau(1-q\tau)\left|{}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2)\right|^r d_{p,q}\tau\right]^{\frac{1}{r}}.
\end{aligned}$$

Now, using the convexity of $\left|{}^{\pi_2}D^2_{p,q}\Pi\right|^r$, we have

$$\begin{aligned}
&\left|\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa\right| \\
\leq\ & \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^1\tau(1-q\tau)d_{p,q}\tau\right]^{1-\frac{1}{r}} \\
& \times\left[\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r\int_0^1\tau^2(1-q\tau)d_{p,q}\tau+\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r\int_0^1\tau(1-q\tau)(1-\tau)\ d_{p,q}\tau\right]^{\frac{1}{r}} \\
\leq\ & \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\frac{p^2}{[2]_{p,q}[3]_{p,q}}\right]^{1-\frac{1}{r}} \\
& \times\left[\frac{p^3}{[3]_{p,q}[4]_{p,q}}\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r+\frac{p^2(p^2-p+q^2)}{[3]_{p,q}[4]_{p,q}}\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)\right|^r\right]^{\frac{1}{r}} \\
=\ & \frac{p^3q^2(\pi_2-\pi_1)^2}{[2]^{2-\frac{1}{r}}_{p,q}[3]_{p,q}[4]^{\frac{1}{r}}_{p,q}}\left[p\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r+(p^2-p+q^2)\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)\right|^r\right]^{\frac{1}{r}}
\end{aligned}$$

which completes the proof. □

Remark 10. *In Theorem 3, if we set* $p=1$*, then we obtain [19], Theorem 5.*

Remark 11. *In Theorem 3, if we set* $p=1$ *and later take the limit as* $q\to 1^-$*, then we have*

$$\left|\frac{\Pi(\pi_1)+\Pi(\pi_2)}{2}-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa\right|\leq\frac{2^{-\frac{1}{r}}(\pi_2-\pi_1)^2}{12}\left[\left|\Pi''(\pi_1)\right|^r+\left|\Pi''(\pi_2)\right|^r\right]^{\frac{1}{r}}.$$

Theorem 4. *Consider the assumptions in Lemma 2 are valid. If* $|{}^{\pi_2}D^2_{p,q}\Pi|^r$ *is convex on* I *for some* $r>1$*, then we have the following inequality:*

$$\left|\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa\right|$$

$$\leq \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}} z^{\frac{1}{s}} \left[\frac{\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right|^r + ([2]_{p,q}-1)\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right|^r}{[2]_{p,q}} \right]^{\frac{1}{r}},$$

where $z = (p-q)\sum_{n=0}^{\infty}(\frac{q^n}{p^{n+1}})^{s+1}(1-(\frac{q}{p})^{n+1})^s$, $0 < q < p \leq 1$ *and* $s = r/(r-1)$.

Proof. Taking the modulus of (13) and applying the Hölder's inequality, we have

$$\left| \frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right|$$
$$\leq \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}} \left[\int_0^1 (\tau(1-q\tau))^s d_{p,q}\tau \right]^{\frac{1}{s}} \left[\int_0^1 \left| {}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2) \right|^r d_{p,q}\tau \right]^{\frac{1}{r}}.$$

Now, using the convexity of $|{}^{\pi_2}D^2_{p,q}\Pi|^r$, we have

$$\left| \frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}} - \frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2} \Pi(\varkappa)\ {}^{\pi_2}d_{p,q}\varkappa \right|$$
$$\leq \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}} \left[\int_0^1 (\tau(1-q\tau)^s d_{p,q}\tau \right]^{\frac{1}{s}}$$
$$\times \left[\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right|^r \int_0^1 \tau\ d_{p,q}\tau + \left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right|^r \int_0^1 (1-\tau)\ d_{p,q}\tau \right]^{\frac{1}{r}}$$
$$= \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}} z^{\frac{1}{s}} \left[\frac{\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_1) \right|^r + ([2]_{p,q}-1)\left| {}^{\pi_2}D^2_{p,q}\Pi(\pi_2) \right|^r}{[2]_{p,q}} \right]^{\frac{1}{r}},$$

where

$$z = \int_0^1 (\tau(1-q\tau)^s d_{p,q}\tau = (p-q)\sum_{n=0}^{\infty}(\frac{q^n}{p^{n+1}})^{s+1}\left[1-(\frac{q}{p})^{n+1}\right]^s.$$

Hence, the proof is completed. □

Remark 12. *In Theorem 4, if we set* $p = 1$*, then we obtain [19], Theorem 6.*

Remark 13. *In Theorem 4, if we set* $p = 1$ *and later take the limit as* $q \to 1^-$*, then we have*

$$\left| \frac{\Pi(\pi_1)+\Pi(\pi_2)}{2} - \frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa \right|$$
$$\leq \frac{(\pi_2-\pi_1)^2}{2}(\mathcal{B}(s+1,s+1))^{1/s}\left[\frac{|\Pi''(\pi_1)|^r+|\Pi''(\pi_2)|^r}{2}\right]^{\frac{1}{r}},$$

where $\mathcal{B}(s+1,s+1) = z = \int_0^1 (\tau(1-\tau)^s d\tau$ *is the famous Euler's beta function.*

Theorem 5. *With the assumptions of Theorem 4, we have the following inequality*

$$\left|\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa\right|$$
$$\leq \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left(\frac{1}{[s+1]_{p,q}}\right)^{\frac{1}{s}}\left[z_1\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r+z_2\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)\right|^r\right]^{\frac{1}{r}},$$

where $z_1=(p-q)\sum_{n=0}^{\infty}\frac{q^{2n}}{p^{2n+2}}\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r$ *and* $z_2=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}(1-\frac{q^n}{p^{n+1}})\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r$.

Proof. Taking modulus of (13) and applying the Hölder's inequality, we have

$$\left|\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa\right|$$
$$\leq \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left[\int_0^1\tau^s d_{p,q}\tau\right]^{\frac{1}{s}}\left[\int_0^1(1-q\tau)^r\left|{}^{\pi_2}D^2_{p,q}\Pi(\tau\pi_1+(1-\tau)\pi_2)\right|^r d_{p,q}\tau\right]^{\frac{1}{r}}.$$

Now, using the convexity of $|{}^{\pi_2}D^2_{p,q}\Pi|^r$, we have

$$\left|\frac{p\Pi(p\pi_1+(1-p)\pi_2)+q\Pi(\pi_2)}{[2]_{p,q}}-\frac{1}{\pi_2-\pi_1}\int_{p^2\pi_1+(1-p^2)\pi_2}^{\pi_2}\Pi(\varkappa)\,{}^{\pi_2}d_{p,q}\varkappa\right|$$
$$\leq \frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left(\frac{1}{[s+1]_{p,q}}\right)^{\frac{1}{s}}$$
$$\times\left[\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r\int_0^1\tau(1-q\tau)^r\,d_{p,q}\tau+\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)\right|^r\int_0^1(1-\tau)(1-q\tau)^r\,d_{p,q}\tau\right]^{\frac{1}{r}}$$
$$=\frac{pq^2(\pi_2-\pi_1)^2}{[2]_{p,q}}\left(\frac{1}{[s+1]_{p,q}}\right)^{\frac{1}{s}}\left[z_1\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_1)\right|^r+z_2\left|{}^{\pi_2}D^2_{p,q}\Pi(\pi_2)\right|^r\right]^{\frac{1}{r}},$$

where
$$z_1=(p-q)\sum_{n=0}^{\infty}\left(\frac{q^n}{p^{n+1}}\right)^2\left(1-\frac{q^{n+1}}{p^{n+1}}\right)^r$$
and
$$z_2=(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}(1-\frac{q^n}{p^{n+1}})\left(1-(\frac{q}{p})^{n+1}\right)^r.$$

Hence, the proof is completed. □

Remark 14. *In Theorem 5, if we set $p=1$, then we obtain [19], Theorem 7.*

Remark 15. *In Theorem 5, if we put $p=1$ and later take the limit as $q\to 1^-$, then we have*

$$\left|\frac{\Pi(\pi_1)+\Pi(\pi_2)}{2}-\frac{1}{\pi_2-\pi_1}\int_{\pi_1}^{\pi_2}\Pi(\varkappa)d\varkappa\right|$$
$$\leq \frac{(\pi_2-\pi_1)^2}{2}\left(\frac{1}{s+1}\right)^{\frac{1}{s}}\left(\frac{1}{(r+1)(r+2)}\right)^{\frac{1}{r}}\left[(r+2)|\Pi''(\pi_1)|^r+|\Pi''(\pi_2)|^r\right]^{\frac{1}{r}}.$$

5. Conclusions

In this work, we established some new trapezoidal type (p,q)-integral inequalities for twice (p,q)-differentiable convex functions. We deduce that the findings proved in this work are naturally universal and contribute into the theory of inequalities, as well as applications for determining the uniqueness of solutions in quantum boundary value problems, quantum mechanics, and special relativity theory. The findings of this study can be applied to quantum information theory and symmetry. Results for the case of symmetric functions can be obtained by applying the concepts in Remarks 1 and 5, which will be studied in future work. As a future direction, one can find similar inequalities for coordinated convex functions.

Author Contributions: Conceptualization, T.S., S.K.N. and M.A.A.; methodology, T.S., S.K.N. and M.A.A.; formal analysis, T.S., S.K.N., M.A.A., G.M., M.A. and J.S.; funding acquisition, T.S. and J.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-63-KNOW-21.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University: Melbourne, Australia, 2000.
2. Pećarixcx, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: Boston, MA, USA, 1992.
3. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Springer: Basel, Switzerland, 2012.
4. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2001.
5. Benatti, F.; Fannes, M.; Floreanini, R.; Petritis, D. *Quantum Information, Computation and Cryptography: An Introductory Survey of Theory, Technology and Experiments*; Springer: Berlin/Heidelberg, Germany, 2010.
6. Bokulich, A.; Jaeger, G. *Philosophy of Quantum Information Theory and Entaglement*, Cambridge University Press: Cambridge, UK, 2010.
7. Szczęśniak, D.; Kais, S. Gap states and valley-spin filtering in transition metal dichalcogenide monolayers. *Phys. Rev. B* **2020**, *101*, 115423. [CrossRef]
8. Belov, A.; Bokut, L.; Rowen, L.; Yu, J.T. The Jacobian Conjecture, Together with Specht and Burnside-Type Problems. In *Automorphisms in Birational and Affine Geometry*; Springer Proceedings in Mathematics and Statistics, 79; Springer: Berlin/Heidelberg, Germany, 2014; pp. 249–285.
9. Ya, A.; Kanel-Belov; Kontsevich, M.L. The Jacobian conjecture is stably equivalent to the Dixmier conjecture. *Mosc. Math. J.* **2007**, *7*, 209–218.
10. Ernst, T. *The History of Q-Calculus and New Method*; Department of Mathematics, Uppsala University: Uppsala, Sweden, 2000.
11. Jackson, F.H. On a q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
12. Al-Salam, W. Some Fractional q-Integrals and q-Derivatives. *Proc. Edinb. Math. Soc.* **1966**, *15*, 135–140. [CrossRef]
13. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *2013*, 282. [CrossRef]
14. Bermudo, S.; Kórus, P.; Valdés, J.N. On q-Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [CrossRef]
15. Sadjang, P.N. On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas. *Results Math.* **2018**, *73*, 1–21.
16. Soontharanon, J.; Sitthiwirattham, T. On Fractional (p,q)-Calculus. *Adv. Differ. Equ.* **2020**, *2020*, 35. [CrossRef]
17. Tunç, M.; Göv, E. Some integral inequalities via (p,q)-calculus on finite intervals. *RGMIA Res. Rep. Coll.* **2016**, *19*, 95.
18. Chu, Y.M.; Awan, M.U.; Talib, S.; Noor, M.A.; Noor, K.I. New post quantum analogues of Ostrowski-type inequalities using new definitions of left–right (p,q)-derivatives and definite integrals. *Adv. Differ. Equ.* **2020**, *2020*, 634. [CrossRef]
19. Ali, M.A.; Budak, H.; Abbas, M.; Chu, Y.-M. Quantum Hermite—Hadamard-type inequalities for functions with convex absolute values of second q^b-derivatives. *Adv. Differ. Equ.* **2021**, *2021*, 7. [CrossRef]
20. Ali, M.A.; Alp, N.; Budak, H.; Chu, Y.-M.; Zhang, Z. On some new quantum midpoint type inequalities for twice quantum differentiable convex functions. *Open Math.* **2021**, *19*, 427–439. [CrossRef]
21. Alp, N.; Sarikaya, M.Z.; Kunt, M.; İşcan, İ. q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ.-Sci.* **2018**, *30*, 193–203. [CrossRef]

22. Alp, N.; Sarikaya, M.Z. Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral. *Appl. Math. E-Notes* **2020**, *20*, 341–356.
23. Budak, H. Some trapezoid and midpoint type inequalities for newly defined quantum integrals. *Proyecciones* **2021**, *40*, 199–215. [CrossRef]
24. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* **2020**, *186*, 899–910. [CrossRef]
25. Liu, W.; Hefeng, Z. Some quantum estimates of Hermite-Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2016**, *7*, 501–522.
26. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [CrossRef]
27. Seksan, J.; Tariboon, J.; Ntouyas, D.K.; Nonlaopon, K. On q-Hermite-Hadamard inequalities for differentiable convex functions. *Mathematics* **2019**, *7*, 632.
28. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **2015**, *269*, 242–251. [CrossRef]
29. Nwaeze, E.R.; Tameru, A.M. New parameterized quantum integral inequalities via η-quasiconvexity. *Adv. Differ. Equ.* **2019**, *2019*, 425. [CrossRef]
30. Khan, M.A.; Noor, M.; Nwaeze, E.R.; Chu, Y.-M. Quantum Hermite–Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 99. [CrossRef]
31. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* **2020**, *44*, 378–390. [CrossRef]
32. Ali, M.A.; Budak, H.; Zhang, Z.; Yildrim, H. Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus. *Math. Meth. Appl. Sci.* **2021**, *44*, 4515–4540. [CrossRef]
33. Ali, M.A.; Abbas, M.; Budak, H.; Agarwal, P.; Murtaza, G.; Chu, Y.-M. New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions. *Adv. Differ. Equ.* **2021**, *2021*, 64. [CrossRef]
34. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Sial, I.B.; Zhang, Z. Some New Newton's Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus. *Symmetry* **2020**, *12*, 1476. [CrossRef]
35. Ali, M.A.; Chu, Y.-M.; Budak, H.; Akkurt, A.; Yildrim, H. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. *Adv. Differ. Equ.* **2021**, *2021*, 25. [CrossRef]
36. Ali, M.A.; Budak, H.; Akkurt, A.; Chu, Y.-M. Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus. *Open Math.* **2021**, *19*, 427–439. [CrossRef]
37. Budak, H.; Ali, M.A.; Alp, N.; Chu, Y.-M. Quantum Ostrowski type integral inequalities. *J. Math. Inequal.* **2021**, in press.
38. Kunt, M.; İşcan, İ.; Alp, N.; Sarikaya, M.Z. (p,q)-Hermite-Hadamard inequalities and (p,q)-estimates for midpoint inequalities via convex quasi-convex functions. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **2018**, *112*, 969–992. [CrossRef]
39. Latif, M.A.; Kunt, M.; Dragomir, S.S.; İşcan, İ. Post-quantum trapezoid type inequalities. *AIMS Math.* **2020**, *5*, 4011–4026. [CrossRef]
40. Vivas-Cortez, M.; Ali, M.A.; Budak, H.; Kalsoom, H.; Agarwal, P. Some New Hermite—Hadamard and Related Inequalities for Convex Functions via (p,q)-Integral. *Entropy* **2021**, *23*, 828. [CrossRef] [PubMed]
41. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex. *RGMIA Res. Rep. Coll.* **2009**, 12. [CrossRef]
42. Sarikaya, M.Z.; Aktan, N. On the generalization of some integral inequalities and their applications. *Math. Comput. Model.* **2011**, *54*, 2175–2182. [CrossRef]