

Article

A Variety of Dynamic Steffensen-Type Inequalities on a General Time Scale

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Abstract: This work is motivated by the work of Josip Pečarić in 2013 and 1982 and the work of Srivastava in 2017. By the utilization of the diamond- α dynamic inequalities, which are defined as a linear mixture of the delta and nabla integrals, we present and prove very important generalized results of diamond- α Steffensen-type inequalities on a general time scale. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Steffensen's inequality; dynamic inequality; diamond- α dynamic integral; time scale



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1. Introduction

In 1982, Pečarić [1] speculated on the Steffensen inequality, presenting the following two hypotheses.

Theorem 1. Let $\hat{f}, \hat{g}, \hat{h} : [r_1, r_2] \rightarrow \mathbb{R}$ be integrable functions on $[r_1, r_2]$ such that \hat{f}/\hat{h} is nonincreasing and \hat{h} is non-negative. Further, let $0 \leq \hat{g}(t) \leq 1 \forall t \in [r_1, r_2]$. Then,

$$\int_{r_1}^{r_2} \hat{f}(t)\hat{g}(t)dt \leq \int_{r_1}^{r_1+\hat{\phi}} \hat{f}(t)dt, \quad (1)$$

where $\hat{\phi}$ is the solution of the equation

$$\int_{r_1}^{r_1+\hat{\phi}} \hat{h}(t)dt = \int_{r_1}^{r_2} \hat{h}(t)\hat{g}(t)dt.$$

We obtain the reverse of (1) if $\hat{f}(t)/\hat{h}(t)$ is nondecreasing.

Theorem 2. Let $\hat{f}, \hat{g}, \hat{h} : [r_1, r_2] \rightarrow \mathbb{R}$ be integrable functions on $[r_1, r_2]$ such that \hat{f}/\hat{h} is nonincreasing and \hat{h} is non-negative. Further, let $0 \leq \hat{g}(t) \leq 1 \forall t \in [r_1, r_2]$. Then,

$$\int_{r_2-\hat{\phi}}^{r_2} \hat{f}(t)dt \leq \int_{r_1}^{r_2} \hat{f}(t)\hat{g}(t)dt \quad (2)$$

where $\hat{\phi}$ gives us the solution of

$$\int_{r_2-\hat{\phi}}^{r_2} \hat{h}(t)dt = \int_{r_1}^{r_2} \hat{h}(t)\hat{g}(t)dt.$$

We obtain the reverse of (2) if $\hat{f}(t)/\hat{h}(t)$ is nondecreasing.

Wu and Srivastava in [2] acquired the accompanying result.

Theorem 3. Let $\hat{f}, \hat{g}, \hat{h} : [r_1, r_2] \rightarrow \mathbb{R}$ be integrable functions on $[r_1, r_2]$ such that \hat{f} is nonincreasing. Further, let $0 \leq \hat{g}(t) \leq \hat{h}(t) \forall t \in [r_1, r_2]$. Then,

$$\begin{aligned} \int_{r_2-\hat{\phi}}^{r_2} \hat{f}(t)\hat{h}(t)dt &\leq \int_{r_2-\hat{\phi}}^{r_2} (\hat{f}(t)\hat{h}(t) - [\hat{f}(t) - \hat{f}(r_2 - \hat{\phi})][\hat{h}(t) - \hat{g}(t)])dt \\ &\leq \int_{r_1}^{r_2} \hat{f}(t)\hat{g}(t)dt \\ &\leq \int_{r_1}^{r_1+\hat{\phi}} (\hat{f}(t)\hat{h}(t) - [\hat{f}(t) - \hat{f}(r_1 + \hat{\phi})][\hat{h}(t) - \hat{g}(t)])dt \\ &\leq \int_{r_1}^{r_1+\hat{\phi}} \hat{f}(t)\hat{h}(t)dt, \end{aligned}$$

where $\hat{\phi}$ is given by

$$\int_{r_1}^{r_1+\hat{\phi}} \hat{h}(t)dt = \int_{r_1}^{r_2} \hat{g}(t)dt = \int_{r_2-\hat{\phi}}^{r_2} \hat{h}(t)dt.$$

The following interesting findings were published in [3].

Theorem 4. Suppose the integrability of $\hat{g}, \hat{h}, \hat{f}, \psi : [r_1, r_2] \rightarrow \mathbb{R}$ such that \hat{f} is nonincreasing. Additionally, suppose that $0 \leq \hat{\psi}(t) \leq \hat{g}(t) \leq \hat{h}(t) - \hat{\psi}(t)$ for all $t \in [r_1, r_2]$. Then,

$$\int_{r_1}^{r_2} \hat{f}(t)\hat{g}(t)dt \leq \int_{r_1}^{r_1+\hat{\phi}} \hat{f}(t)\hat{h}(t)dt - \int_{r_1}^{r_2} |(\hat{f}(t) - \hat{f}(r_1 + \hat{\phi}))\psi(t)|dt,$$

where $\hat{\phi}$ is given by

$$\int_{r_1}^{r_1+\hat{\phi}} \hat{h}(t)dt = \int_{r_1}^{r_2} \hat{g}(t)dt.$$

Theorem 5. Under the hypotheses of Theorem 4,

$$\int_{r_2-\hat{\phi}}^{r_2} \hat{f}(t)\hat{h}(t)dt + \int_{r_1}^{r_2} |(\hat{f}(t) - \hat{f}(r_2 - \hat{\phi}))\hat{\psi}(t)|dt \leq \int_{r_1}^{r_2} \hat{f}(t)\hat{g}(t)dt,$$

where $\hat{\phi}$ is given by

$$\int_{r_2-\hat{\phi}}^{r_2} \hat{h}(t)dt = \int_{r_1}^{r_2} \hat{g}(t)dt.$$

The calculus of time scales with the intention to unify discrete and continuous analysis (see [4]) was proposed by Hilger [5]. For additional subtleties on time scales, we refer the reader to the book by Bohner and Peterson [6]. Additionally, understanding of diamond- α calculus on time scales is assumed, and we refer the interested reader to [7] for further details.

Recently, a massive range of dynamic inequalities on time scales have been investigated by using exclusive authors who have been inspired with the aid of a few applications (see [8–14]). Some authors found different results regarding fractional calculus on time scales to provide associated dynamic inequalities (see [15–18]).

We devote the remaining part of this section to the diamond- α calculus on time scales, and we refer the interested reader to [7] for further details.

If \mathbb{T} is a time scale, and ζ is a function that is delta and nabla differentiable on \mathbb{T} , then, for any $t \in \mathbb{T}$, the diamond- α dynamic derivative of ζ at t , denoted by $\zeta^{\diamond\alpha}(t)$, is defined by

$$\zeta^{\diamond\alpha}(t) = \alpha\zeta^\Delta(t) + (1 - \alpha)\zeta^\nabla(t), \quad 0 \leq \alpha \leq 1. \tag{3}$$

We conclude from the last relation that a function ζ is diamond- α differentiable if and only if it is both delta and nabla differentiable. For $\alpha = 1$, the diamond- α derivative boils down to a delta derivative, and for $\alpha = 0$ it boils down to a nabla derivative.

Assume $\zeta, g : \mathbb{T} \rightarrow \mathbb{R}$ are diamond- α differentiable functions at $t \in \mathbb{T}$, and let $k \in \mathbb{R}$. Then,

- (i) $(\zeta + \Xi)^{\diamond_\alpha}(t) = \zeta^{\diamond_\alpha}(t) + \Xi^{\diamond_\alpha}(t);$
- (ii) $(k\zeta)^{\diamond_\alpha}(t) = k\zeta^{\diamond_\alpha}(t);$
- (iii) $(\zeta\Xi)^{\diamond_\alpha}(t) = \zeta^{\diamond_\alpha}(t)\Xi(t) + \alpha\zeta^\sigma(t)\Xi^\Delta(t) + (1 - \alpha)\zeta^\rho(t)\Xi^\nabla(t).$

Let $\zeta : \mathbb{T} \rightarrow \mathbb{R}$ be a continuous function. Then, the definite diamond- α integral of ζ is defined by

$$\int_a^b \zeta(t) \diamond_{\alpha} t = \alpha \int_a^b \zeta(t) \Delta t + (1 - \alpha) \int_a^b \zeta(t) \nabla t, \quad 0 \leq \alpha \leq 1, \quad a, b \in \mathbb{T}. \tag{4}$$

Let $a, b, c \in \mathbb{T}, k \in \mathbb{R}$. Then,

- (i) $\int_a^b [\zeta(t) + \Xi(t)] \diamond_{\alpha} t = \int_a^b \zeta(t) \diamond_{\alpha} t + \int_a^b \Xi(t) \diamond_{\alpha} t;$
- (ii) $\int_a^b k\zeta(t) \diamond_{\alpha} t = k \int_a^b \zeta(t) \diamond_{\alpha} t;$
- (iii) $\int_a^b \zeta(t) \diamond_{\alpha} t = \int_a^c \zeta(t) \diamond_{\alpha} t + \int_c^b \zeta(t) \diamond_{\alpha} t;$
- (iv) $\int_a^b \zeta(t) \diamond_{\alpha} t = - \int_b^a \zeta(t) \diamond_{\alpha} t;$
- (v) $\int_a^a \zeta(t) \diamond_{\alpha} t = 0;$
- (vi) if $\zeta(t) \geq 0$ on $[a, b]_{\mathbb{T}}$, then $\int_a^b \zeta(t) \diamond_{\alpha} t \geq 0;$
- (vii) if $\zeta(t) \geq \Xi(t)$ on $[a, b]_{\mathbb{T}}$, then $\int_a^b \zeta(t) \diamond_{\alpha} t \geq \int_a^b \Xi(t) \diamond_{\alpha} t;$
- (viii) $\left| \int_a^b \zeta(t) \diamond_{\alpha} t \right| \leq \int_a^b |\zeta(t)| \diamond_{\alpha} t.$

Let ζ be a *diamond* - α differentiable function on $[a, b]_{\mathbb{T}}$. Then, ζ is increasing if $\zeta^{\diamond_\alpha}(t) > 0$, nondecreasing if $\zeta^{\diamond_\alpha}(t) \geq 0$, decreasing if $\zeta^{\diamond_\alpha}(t) < 0$, and nonincreasing if $\zeta^{\diamond_\alpha}(t) \leq 0$ on $[a, b]_{\mathbb{T}}$.

In this article, we explore new generalizations of the integral Steffensen inequality given in [1–3] via *diamond* - α integral on general time scale measure space. We also retrieve some of the integral inequalities known in the literature as special cases of our tests.

2. Main Results

Next, we use the accompanying suppositions for the verifications of our primary outcomes:

- (S₁) $([r_1, r_2]_{\mathbb{T}}, \mathfrak{B}([r_1, r_2]_{\mathbb{T}}), \hat{\mu})$ is time scale measure space with a positive σ -finite measure on $\mathfrak{B}([r_1, r_2]_{\mathbb{T}})$.
- (S₂) $\zeta, Y, \Xi : [r_1, r_2]_{\mathbb{T}} \rightarrow \mathbb{R}$ is $\diamond_{\alpha} t$ -integrable functions on $[r_1, r_2]_{\mathbb{T}}$.
- (S₃) ζ/Ξ is nonincreasing and Ξ is non-negative.
- (S₄) $0 \leq Y(t) \leq 1$ for all $t \in [r_1, r_2]_{\mathbb{T}}$.
- (S₅) $\hat{\phi}$ is a real number.
- (S₆) ζ is nonincreasing.
- (S₇) $1 \leq Y(t) \leq \Xi(t)$ for all $t \in [r_1, r_2]_{\mathbb{T}}$.
- (S₈) $0 \leq \psi(t) \leq Y(t) \leq \Xi(t) - \psi(t)$ for all $t \in [r_1, r_2]_{\mathbb{T}}$.
- (S₉) $0 \leq M \leq Y(t) \leq 1 - M$ for all $t \in [r_1, r_2]_{\mathbb{T}}$.
- (S₁₀) $0 \leq \psi(t) \leq Y(t) \leq 1 - \psi(t)$ for all $t \in [r_1, r_2]_{\mathbb{T}}$.

$\hat{\phi}$ is the solution of the equations listed below:

- (S₁₁) $\int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} \Xi(t) Y(t) \diamond_{\alpha} t.$
- (S₁₂) $\int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} \Xi(t) Y(t) \diamond_{\alpha} t.$
- (S₁₃) $\int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t = \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t.$
- (S₁₄) $\int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t.$
- (S₁₅) $\int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t.$

Presently, we are prepared to state and explain the principle results that have had more effect effect from the literature.

Theorem 6. Let S_1, S_2, S_3, S_4 and S_{11} be satisfied. Then,

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \leq \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \zeta(t)\diamond_{\alpha}t. \tag{5}$$

We obtain the reverse of (5) if ζ/Ξ is nondecreasing.

Proof.

$$\begin{aligned} & \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \zeta(t)\diamond_{\alpha}t - \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \\ &= \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \Xi(t)[1 - Y(t)]\frac{\zeta(t)}{\Xi(t)}\diamond_{\alpha}t - \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \\ &\geq \frac{\zeta(r_1 + \hat{\phi})}{\Xi(r_1 + \hat{\phi})} \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \Xi(t)[1 - Y(t)]\diamond_{\alpha}t - \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \\ &= \frac{\zeta(r_1 + \hat{\phi})}{\Xi(r_1 + \hat{\phi})} \left[\int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \Xi(t)\diamond_{\alpha}t - \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \Xi(t)Y(t)\diamond_{\alpha}t \right] - \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \\ &= \frac{\zeta(r_1 + \hat{\phi})}{\Xi(r_1 + \hat{\phi})} \left[\int_{[r_1, r_2]_{\mathbb{T}}} \Xi(t)Y(t)\diamond_{\alpha}t - \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \Xi(t)Y(t)\diamond_{\alpha}t \right] - \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \\ &= \frac{\zeta(r_1 + \hat{\phi})}{\Xi(r_1 + \hat{\phi})} \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t)Y(t)\diamond_{\alpha}t - \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \\ &= \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t)Y(t) \left(\frac{\zeta(r_1 + \hat{\phi})}{\Xi(r_1 + \hat{\phi})} - \frac{\zeta(t)}{\Xi(t)} \right) \diamond_{\alpha}t \geq 0. \end{aligned}$$

The proof is complete. \square

Corollary 1. Delta version obtained from Theorem 6 by taking $\alpha = 1$

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t \leq \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \zeta(t)\Delta t.$$

Corollary 2. Nabla version obtained from Theorem 6 by taking $\alpha = 0$

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\nabla t \leq \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \zeta(t)\nabla t.$$

Remark 1. In case of $\mathbb{T} = \mathbb{R}$ in Corollary 1, we recollect [1] (Theorem 1).

Theorem 7. Assumptions S_1, S_2, S_3, S_4 and S_{12} imply

$$\int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t)\diamond_{\alpha}t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t \tag{6}$$

We obtain the reverse of (6) if ζ/Ξ is nondecreasing.

Proof.

$$\begin{aligned}
 & \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t) \diamond_{\alpha} t - \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\
 &= \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) [1 - Y(t)] \frac{\zeta(t)}{\Xi(t)} \diamond_{\alpha} t - \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\
 &\leq \frac{\zeta(r_2 - \hat{\phi})}{\Xi(r_2 - \hat{\phi})} \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) [1 - Y(t)] \diamond_{\alpha} t - \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\
 &= \frac{\zeta(r_2 - \hat{\phi})}{\Xi(r_2 - \hat{\phi})} \left[\int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t - \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) Y(t) \diamond_{\alpha} t \right] - \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\
 &= \frac{\zeta(r_2 - \hat{\phi})}{\Xi(r_2 - \hat{\phi})} \left[\int_{[r_1, r_2]_{\mathbb{T}}} \Xi(t) Y(t) \diamond_{\alpha} t - \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) Y(t) \diamond_{\alpha} t \right] - \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\
 &= \frac{\zeta(r_2 - \hat{\phi})}{\Xi(r_2 - \hat{\phi})} \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \Xi(t) Y(t) \diamond_{\alpha} t - \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\
 &= \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} \Xi(t) Y(t) \left(\frac{\zeta(r_2 - \hat{\phi})}{\Xi(r_2 - \hat{\phi})} - \frac{\zeta(t)}{\Xi(t)} \right) \diamond_{\alpha} t \leq 0.
 \end{aligned}$$

□

Corollary 3. Delta version obtained from Theorem 7 by taking $\alpha = 1$

$$\int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t) \Delta t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \Delta t. \tag{7}$$

Corollary 4. Nabla version obtained from Theorem 7 by taking $\alpha = 0$

$$\int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t) \nabla t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \nabla t. \tag{8}$$

Remark 2. In Corollary 8 and $\mathbb{T} = \mathbb{R}$, we recapture [1] (Theorem 2).

We will need the following lemma to prove the subsequent results.

Lemma 1. Let S_1, S_2, S_5 hold, such that

$$\int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t = \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t.$$

Then,

$$\begin{aligned}
 \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t &= \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\phi})] [\Xi(t) - Y(t)]) \diamond_{\alpha} t \\
 &\quad + \int_{[r_1+\hat{\phi}, r_2]_{\mathbb{T}}} [\zeta(t) - \zeta(r_1 + \hat{\phi})] Y(t) \diamond_{\alpha} t,
 \end{aligned} \tag{9}$$

and

$$\begin{aligned}
 \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t &= \int_{[r_1, r_2-\hat{\phi}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2 - \hat{\phi})] Y(t) \diamond_{\alpha} t \\
 &\quad + \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\phi})] [\Xi(t) - Y(t)]) \diamond_{\alpha} t.
 \end{aligned} \tag{10}$$

Proof. The suppositions of the Lemma imply that

$$r_1 \leq r_1 + \hat{\phi} \leq r_2 \quad \text{and} \quad r_1 \leq r_2 - \hat{\phi} \leq r_2.$$

Now we have proved (9), we see that

$$\begin{aligned}
 & \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} (\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\rho})][\Xi(t) - Y(t)]) \diamond_{\alpha} t - \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t \\
 &= \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} (\zeta(t)\Xi(t) - \zeta(t)Y(t) - [\zeta(t) - \zeta(r_1 + \hat{\rho})][\Xi(t) - Y(t)]) \diamond_{\alpha} t \\
 & \quad + \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t - \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t \\
 &= \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} \zeta(r_1 + \hat{\rho})[\Xi(t) - Y(t)] \diamond_{\alpha} t - \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t \\
 &= \zeta(r_1 + \hat{\rho}) \left(\int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t - \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t \right) - \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t. \tag{11}
 \end{aligned}$$

Since

$$\int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t,$$

we have

$$\begin{aligned}
 & \zeta(r_1 + \hat{\rho}) \left(\int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} \Xi(t) \diamond_{\alpha} t - \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t \right) - \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t \\
 &= \zeta(r_1 + \hat{\rho}) \left(\int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t - \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t \right) - \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t \\
 &= \zeta(r_1 + \hat{\rho}) \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t - \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t \\
 &= \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} [\zeta(r_1 + \hat{\rho}) - \zeta(t)]Y(t) \diamond_{\alpha} t. \tag{12}
 \end{aligned}$$

Combination of (11) and (12) led to the required integral identity (9) asserted by the Lemma. The integral identity (16) can be proved similarly. The proof is complete. \square

Corollary 5. Delta version obtained from Lemma 1 by taking $\alpha = 1$

$$\begin{aligned}
 \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t &= \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} (\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\rho})][\Xi(t) - Y(t)])\Delta t \\
 & \quad + \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} [\zeta(t) - \zeta(r_1 + \hat{\rho})]Y(t)\Delta t, \tag{13}
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t &= \int_{[r_1, r_2 - \hat{\rho}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2 - \hat{\rho})]Y(t)\Delta t \\
 & \quad + \int_{[r_2 - \hat{\rho}, r_2]_{\mathbb{T}}} (\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\rho})][\Xi(t) - Y(t)])\Delta t, \tag{14}
 \end{aligned}$$

such that

$$\int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} \Xi(t)\Delta t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t)\Delta t = \int_{[r_2 - \hat{\rho}, r_2]_{\mathbb{T}}} \Xi(t)\Delta t.$$

Corollary 6. Nabla version obtained from Lemma 1 by taking $\alpha = 0$

$$\begin{aligned}
 \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t &= \int_{[r_1, r_1 + \hat{\rho}]_{\mathbb{T}}} (\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\rho})][\Xi(t) - Y(t)])\Delta t \\
 & \quad + \int_{[r_1 + \hat{\rho}, r_2]_{\mathbb{T}}} [\zeta(t) - \zeta(r_1 + \hat{\rho})]Y(t)\Delta t, \tag{15}
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\nabla t &= \int_{[r_1, r_2 - \hat{\rho}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2 - \hat{\rho})]Y(t)\Delta t \\
 & \quad + \int_{[r_2 - \hat{\rho}, r_2]_{\mathbb{T}}} (\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\rho})][\Xi(t) - Y(t)])\nabla t. \tag{16}
 \end{aligned}$$

such that

$$\int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \Xi(t) \nabla t = \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \nabla t = \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \Xi(t) \nabla t.$$

Theorem 8. Suppose that S_1, S_2, S_6, S_7 and S_{13} give

$$\begin{aligned} \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t) \Xi(t) \diamond_{\alpha} t &\leq \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\phi})] [\Xi(t) - Y(t)]) \diamond_{\alpha} t \\ &\leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \\ &\leq \int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\phi})] [\Xi(t) - Y(t)]) \diamond_{\alpha} t \\ &\leq \int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \zeta(t) \Xi(t) \diamond_{\alpha} t. \end{aligned}$$

Proof. In perspective of the considerations that the function ζ is nonincreasing on $[r_1, r_2]$ and $0 \leq Y(t) \leq \Xi(t)$ for all $t \in [r_1, r_2]$, we infer that

$$\int_{[r_1, r_2 - \hat{\phi}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2 - \hat{\phi})] Y(t) \diamond_{\alpha} t \geq 0, \tag{17}$$

and

$$\int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} [\zeta(r_2 - \hat{\phi}) - \zeta(t)] [\Xi(t) - Y(t)] \diamond_{\alpha} t \geq 0. \tag{18}$$

Using (9), (17) and (18), we find that

$$\begin{aligned} \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t) \Xi(t) \diamond_{\alpha} t &\leq \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\phi})] [\Xi(t) - Y(t)]) \diamond_{\alpha} t \\ &\leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t. \end{aligned} \tag{19}$$

$$\begin{aligned} \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t &\leq \int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\phi})] [\Xi(t) - Y(t)]) \diamond_{\alpha} t \\ &\leq \int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \zeta(t) \Xi(t) \diamond_{\alpha} t. \end{aligned} \tag{20}$$

The confirmation is finished by joining the integral inequalities (19) and (20). \square

Corollary 7. Delta version obtained from Theorem 8 by taking $\alpha = 1$

$$\begin{aligned} \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} \zeta(t) \Xi(t) \Delta t &\leq \int_{[r_2 - \hat{\phi}, r_2]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\phi})] [\Xi(t) - Y(t)]) \Delta t \\ &\leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \Delta t \\ &\leq \int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} (\zeta(t) \Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\phi})] [\Xi(t) - Y(t)]) \Delta t \\ &\leq \int_{[r_1, r_1 + \hat{\phi}]_{\mathbb{T}}} \zeta(t) \Xi(t) \Delta t. \end{aligned}$$

Corollary 8. Nabla version obtained from Theorem 8 by taking $\alpha = 0$

$$\begin{aligned} \int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\Xi(t)\nabla t &\leq \int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \left(\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_2 - \hat{\rho})] [\Xi(t) - Y(t)] \right) \nabla t \\ &\leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\nabla t \\ &\leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \left(\zeta(t)\Xi(t) - [\zeta(t) - \zeta(r_1 + \hat{\rho})] [\Xi(t) - Y(t)] \right) \nabla t \\ &\leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t)\Xi(t)\nabla t. \end{aligned}$$

Remark 3. We can reclaim [2] (Theorem 1) in Corollary 7 and $\mathbb{T} = \mathbb{R}$.

Theorem 9. Assume that S_1, S_2, S_6, S_8 and S_{13} are fulfilled. Then,

$$\begin{aligned} &\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\Xi(t)\diamond_{\alpha} t + \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(t) - \zeta(r_2 - \hat{\rho})] \psi(t) \right| \diamond_{\alpha} t \\ &\leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha} t \\ &\leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t)\Xi(t)\diamond_{\alpha} t - \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(t) - \zeta(r_1 + \hat{\rho})] \psi(t) \right| \diamond_{\alpha} t, \end{aligned} \tag{21}$$

Proof. Clearly, function ζ is nonincreasing on $[r_1, r_2]$ and $0 \leq \psi(t) \leq Y(t) \leq \Xi(t) - \psi(t)$ for all $t \in [r_1, r_2]$; so, we obtain

$$\begin{aligned} &\int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_1 + \hat{\rho})] [\Xi(t) - Y(t)] \diamond_{\alpha} t + \int_{[r_1+\hat{\rho}, r_2]_{\mathbb{T}}} [\zeta(r_1 + \hat{\rho}) - \zeta(t)] Y(t) \diamond_{\alpha} t \\ &= \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} |\zeta(t) - \zeta(r_1 + \hat{\rho})| [\Xi(t) - Y(t)] \diamond_{\alpha} t + \int_{[r_1+\hat{\rho}, r_2]_{\mathbb{T}}} |\zeta(r_1 + \hat{\rho}) - \zeta(t)| Y(t) \diamond_{\alpha} t \\ &\geq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} |\zeta(t) - \zeta(r_1 + \hat{\rho})| \psi(t) \diamond_{\alpha} t + \int_{[r_1+\hat{\rho}, r_2]_{\mathbb{T}}} |\zeta(r_1 + \hat{\rho}) - \zeta(t)| \psi(t) \diamond_{\alpha} t \\ &\geq \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(t) - \zeta(r_1 + \hat{\rho})] \psi(t) \right| \diamond_{\alpha} t. \end{aligned}$$

Additionally,

$$\begin{aligned} &\int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_1 + \hat{\rho})] [\Xi(t) - Y(t)] \diamond_{\alpha} t + \int_{[r_1+\hat{\rho}, r_2]_{\mathbb{T}}} [\zeta(r_1 + \hat{\rho}) - \zeta(t)] Y(t) \diamond_{\alpha} t \\ &\geq \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(t) - \zeta(r_1 + \hat{\rho})] \psi(t) \right| \diamond_{\alpha} t. \end{aligned} \tag{22}$$

Similarly, we find that

$$\begin{aligned} &\int_{[r_1, r_2-\hat{\rho}]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2 - \hat{\rho})] Y(t) \diamond_{\alpha} t + \int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} [\zeta(r_2 - \hat{\rho}) - \zeta(t)] [\Xi(t) - Y(t)] \diamond_{\alpha} t \\ &\geq \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(t) - \zeta(r_2 - \hat{\rho})] \psi(t) \right| \diamond_{\alpha} t. \end{aligned} \tag{23}$$

By combining (9), (16), and (22), (23), we arrive at the inequality (21), asserted by Theorem 9. \square

Corollary 9. Delta version obtained from Theorem 9 by taking $\alpha = 1$

$$\begin{aligned} & \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(i)\Xi(i)\Delta i + \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(i) - \zeta(r_2 - \hat{\phi})] \psi(i) \right| \Delta i \\ & \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(i)Y(i)\Delta i \\ & \leq \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \zeta(i)\Xi(i)\Delta i - \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(i) - \zeta(r_1 + \hat{\phi})] \psi(i) \right| \Delta i, \end{aligned}$$

Corollary 10. Nabla version obtained from Theorem 9 by taking $\alpha = 0$

$$\begin{aligned} & \int_{[r_2-\hat{\phi}, r_2]_{\mathbb{T}}} \zeta(i)\Xi(i)\nabla i + \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(i) - \zeta(r_2 - \hat{\phi})] \psi(i) \right| \nabla i \\ & \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(i)Y(i)\nabla i \\ & \leq \int_{[r_1, r_1+\hat{\phi}]_{\mathbb{T}}} \zeta(i)\Xi(i)\nabla i - \int_{[r_1, r_2]_{\mathbb{T}}} \left| [\zeta(i) - \zeta(r_1 + \hat{\phi})] \psi(i) \right| \nabla i, \end{aligned}$$

Remark 4. If we take $\mathbb{T} = \mathbb{R}$, in Corollary 9, we recapture [2] (Theorem 2).

Theorem 10. Let S_1, S_2, S_6, S_9 be satisfied, and

$$0 \leq \hat{\phi}_1 \leq \int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i \leq \hat{\phi}_2 \leq r_2 - r_1.$$

Then,

$$\begin{aligned} & \int_{[r_2-\hat{\phi}_1, r_2]_{\mathbb{T}}} \zeta(i) \diamond_{\alpha} i + \zeta(r_2) \left(\int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i - \hat{\phi}_1 \right) \\ & + M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(i) - f \left(r_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i \right) \right| \diamond_{\alpha} i \\ & \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(i)Y(i) \diamond_{\alpha} i \\ & \leq \int_{[r_1, r_1+\hat{\phi}_2]_{\mathbb{T}}} \zeta(i) \diamond_{\alpha} i - \zeta(r_2) \left(\hat{\phi}_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i \right) \\ & - M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(i) - f \left(r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i \right) \right| \diamond_{\alpha} i. \end{aligned} \tag{24}$$

Proof. By using straightforward calculations, we have

$$\begin{aligned} & \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(i)Y(i) \diamond_{\alpha} i - \int_{[r_1, r_1+\hat{\phi}_2]_{\mathbb{T}}} \zeta(i) \diamond_{\alpha} i + \zeta(r_2) \left(\hat{\phi}_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i \right) \\ & = \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(i)Y(i) \diamond_{\alpha} i - \int_{[r_1, r_1+\hat{\phi}_2]_{\mathbb{T}}} \zeta(i) \diamond_{\alpha} i + \int_{[r_1, r_1+\hat{\phi}_2]_{\mathbb{T}}} \zeta(r_2) \diamond_{\alpha} i - \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(r_2)Y(i) \diamond_{\alpha} i \\ & = \int_{[r_1, r_2]_{\mathbb{T}}} [\zeta(i) - \zeta(r_2)]Y(i) \diamond_{\alpha} i - \int_{[r_1, r_1+\hat{\phi}_2]_{\mathbb{T}}} [\zeta(i) - \zeta(r_2)] \diamond_{\alpha} i \\ & \leq \int_{[r_1, r_2]_{\mathbb{T}}} [\zeta(i) - \zeta(r_2)]Y(i) \diamond_{\alpha} i - \int_{[r_1, r_1+\int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i]} [\zeta(i) - \zeta(r_2)] \diamond_{\alpha} i, \end{aligned} \tag{25}$$

where we used the theorem’s hypotheses

$$r_1 \leq r_1 + \hat{\phi}_1 \leq r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(i) \diamond_{\alpha} i \leq r_1 + \hat{\phi}_2 \leq r_2$$

and

$$\zeta(i) - \zeta(r_2) \geq 0 \quad \text{for all } i \in [r_1, r_2].$$

The function $\zeta(t) - \zeta(r_2)$ is nonincreasing and integrable on $[r_1, r_2]$ and, by applying Theorem 9 with $\Xi(t) = 1$, $\psi(t) = M$ and $\zeta(t)$ replaced by $\zeta(t) - \zeta(r_2)$,

$$\begin{aligned} & \int_{[r_1, r_2]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2)]Y(t) \diamond_{\alpha} t - \int_{[r_1, r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t]} [\zeta(t) - \zeta(r_2)] \diamond_{\alpha} t \\ & \leq -M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f\left(r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t\right) \right| \diamond_{\alpha} t. \end{aligned} \tag{26}$$

From (25) and (26), we obtain

$$\begin{aligned} & \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t - \int_{[r_1, r_1 + \hat{\phi}_2]_{\mathbb{T}}} \zeta(t) \diamond_{\alpha} t + \zeta(r_2) (\hat{\phi}_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t) \\ & \leq -M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f\left(r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t\right) \right| \diamond_{\alpha} t, \end{aligned} \tag{27}$$

which is the right-hand side inequality in (24).

Similarly, one can show that

$$\begin{aligned} & \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t) \diamond_{\alpha} t - \int_{[r_2 - \hat{\phi}_1, r_2]_{\mathbb{T}}} \zeta(t) \diamond_{\alpha} t + \zeta(r_2) \left(\int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t - \hat{\phi}_2 \right) \\ & \geq \int_{[r_1, r_2]_{\mathbb{T}}} [\zeta(t) - \zeta(r_2)]Y(t) \diamond_{\alpha} t + \int_{[r_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t, r_2]} [\zeta(r_2) - \zeta(t)] \diamond_{\alpha} t \\ & \geq M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f\left(r_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \diamond_{\alpha} t\right) \right| \diamond_{\alpha} t, \end{aligned} \tag{28}$$

which is the left-hand side inequality in (24). \square

Corollary 11. Delta version obtained from Theorem 10 by taking $\alpha = 1$

$$\begin{aligned} & \int_{[r_2 - \hat{\phi}_1, r_2]_{\mathbb{T}}} \zeta(t)\Delta t + \zeta(r_2) \left(\int_{[r_1, r_2]_{\mathbb{T}}} Y(t)\Delta t - \hat{\phi}_1 \right) \\ & + M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f\left(r_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t)\Delta t\right) \right| \Delta t \\ & \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t \\ & \leq \int_{[r_1, r_1 + \hat{\phi}_2]_{\mathbb{T}}} \zeta(t)\Delta t - \zeta(r_2) \left(\hat{\phi}_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t)\Delta t \right) \\ & - M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f\left(r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(t)\Delta t\right) \right| \Delta t, \end{aligned}$$

such that

$$0 \leq \hat{\phi}_1 \leq \int_{[r_1, r_2]_{\mathbb{T}}} Y(t)\Delta t \leq \hat{\phi}_2 \leq r_2 - r_1.$$

Corollary 12. Nabla version obtained from Theorem 10 by taking $\alpha = 0$

$$\begin{aligned} & \int_{[r_2-\hat{\rho}_1, r_2]_{\mathbb{T}}} \zeta(t) \nabla t + \zeta(r_2) \left(\int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \nabla t - \hat{\rho}_1 \right) \\ & + M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f \left(r_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \nabla t \right) \right| \nabla t \\ & \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \nabla t \\ & \leq \int_{[r_1, r_1+\hat{\rho}_2]_{\mathbb{T}}} \zeta(t) \nabla t - \zeta(r_2) \left(\hat{\rho}_2 - \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \nabla t \right) \\ & - M \int_{[r_1, r_2]_{\mathbb{T}}} \left| \zeta(t) - f \left(r_1 + \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \nabla t \right) \right| \nabla t, \end{aligned}$$

such that

$$0 \leq \hat{\rho}_1 \leq \int_{[r_1, r_2]_{\mathbb{T}}} Y(t) \nabla t \leq \hat{\rho}_2 \leq r_2 - r_1.$$

Remark 5. Ref. [2] (Theorem 3) can be obtained if $\mathbb{T} = \mathbb{R}$ in Corollary 11.

Theorem 11. If S_1, S_2, S_6, S_7 and S_{14} hold. Then,

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t) \Xi(t) \diamond_{\alpha} t - \int_{[r_1, r_2]_{\mathbb{T}}} |(\zeta(t) - \zeta(r_1 + \hat{\rho})) \psi(t)| \diamond_{\alpha} t. \tag{29}$$

Proof. This proof is similar to the proof of the right-hand side inequality in Theorem 9. \square

Corollary 13. Delta version obtained from Theorem by taking $\alpha = 1$

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \Delta t \leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t) \Xi(t) \Delta t - \int_{[r_1, r_2]_{\mathbb{T}}} |(\zeta(t) - \zeta(r_1 + \hat{\rho})) \psi(t)| \Delta t.$$

Corollary 14. Nabla version obtained from Theorem by taking $\alpha = 0$

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \nabla t \leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t) \Xi(t) \nabla t - \int_{[r_1, r_2]_{\mathbb{T}}} |(\zeta(t) - \zeta(r_1 + \hat{\rho})) \psi(t)| \nabla t.$$

Remark 6. If we take $\mathbb{T} = \mathbb{R}$, in Corollary 13, we recapture [3] (Theorem 2.12).

Corollary 15. Hypotheses S_1, S_2, S_3, S_{10} and S_{11} yield

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \diamond_{\alpha} t \leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t) \diamond_{\alpha} t - \int_{[r_1, r_2]_{\mathbb{T}}} \left| \left(\frac{\zeta(t)}{\Xi(t)} - \frac{\zeta(r_1 + \hat{\rho})}{\Xi(r_1 + \hat{\rho})} \right) \Xi(t) \psi(t) \right| \diamond_{\alpha} t. \tag{30}$$

Proof. Insert $Y(t) \mapsto \Xi(t)Y(t), \zeta(t) \mapsto \zeta(t)/\Xi(t)$ and $\psi(t) \mapsto \Xi(t)\psi(t)$ in Theorem 11. \square

Corollary 16. Delta version obtained in Corollary 15 by taking $\alpha = 1$

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \Delta t \leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t) \Delta t - \int_{[r_1, r_2]_{\mathbb{T}}} \left| \left(\frac{\zeta(t)}{\Xi(t)} - \frac{\zeta(r_1 + \hat{\rho})}{\Xi(r_1 + \hat{\rho})} \right) \Xi(t) \psi(t) \right| \Delta t.$$

Corollary 17. Nabla version obtained in Corollary 15 by taking $\alpha = 0$

$$\int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t) Y(t) \nabla t \leq \int_{[r_1, r_1+\hat{\rho}]_{\mathbb{T}}} \zeta(t) \nabla t - \int_{[r_1, r_2]_{\mathbb{T}}} \left| \left(\frac{\zeta(t)}{\Xi(t)} - \frac{\zeta(r_1 + \hat{\rho})}{\Xi(r_1 + \hat{\rho})} \right) \Xi(t) \psi(t) \right| \nabla t.$$

Remark 7. Ref. [3] (Corollary 2.3) can be recovered with the help of $\mathbb{T} = \mathbb{R}$, in Corollary 16.

Theorem 12. If S_1, S_2, S_6, S_7 and S_{15} hold, then

$$\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\Xi(t)\diamond_{\alpha}t + \int_{[r_1, r_2]_{\mathbb{T}}} |(\zeta(t) - \zeta(r_2 - \hat{\rho}))\psi(t)|\diamond_{\alpha}t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t. \quad (31)$$

Proof. Carry out the same proof of the left-hand side inequality in Theorem 9. \square

Corollary 18. Delta version obtained from Theorem 12 by taking $\alpha = 1$

$$\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\Xi(t)\Delta t + \int_{[r_1, r_2]_{\mathbb{T}}} |(\zeta(t) - \zeta(r_2 - \hat{\rho}))\psi(t)|\Delta t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t.$$

Corollary 19. Nabla version obtained from Theorem 12 by taking $\alpha = 0$

$$\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\Xi(t)\nabla t + \int_{[r_1, r_2]_{\mathbb{T}}} |(\zeta(t) - \zeta(r_2 - \hat{\rho}))\psi(t)|\nabla t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\nabla t.$$

Remark 8. If we take $\mathbb{T} = \mathbb{R}$, in Corollary 18, we recapture [3] (Theorem 2.13).

Corollary 20. Let S_1, S_2, S_3, S_9 and S_{12} , be fulfilled. Then,

$$\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\diamond_{\alpha}t + \int_{[r_1, r_2]_{\mathbb{T}}} \left| \left(\frac{\zeta(t)}{\Xi(t)} - \frac{\zeta(r_2 - \hat{\rho})}{\Xi(r_2 - \hat{\rho})} \right) \Xi(t)\psi(t) \right| \diamond_{\alpha}t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\diamond_{\alpha}t. \quad (32)$$

Proof. Proof can be completed by taking $Y(t) \mapsto \Xi(t)Y(t)$, $\zeta(t) \mapsto \zeta(t)/\Xi(t)$ and $\psi(t) \mapsto \Xi(t)\psi(t)$ in Theorem 12. \square

Corollary 21. Delta version obtained from Corollary 20 by taking $\alpha = 1$

$$\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\Delta t + \int_{[r_1, r_2]_{\mathbb{T}}} \left| \left(\frac{\zeta(t)}{\Xi(t)} - \frac{\zeta(r_2 - \hat{\rho})}{\Xi(r_2 - \hat{\rho})} \right) \Xi(t)\psi(t) \right| \Delta t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\Delta t.$$

Corollary 22. Nabla version obtained from Corollary 20 by taking $\alpha = 0$

$$\int_{[r_2-\hat{\rho}, r_2]_{\mathbb{T}}} \zeta(t)\nabla t + \int_{[r_1, r_2]_{\mathbb{T}}} \left| \left(\frac{\zeta(t)}{\Xi(t)} - \frac{\zeta(r_2 - \hat{\rho})}{\Xi(r_2 - \hat{\rho})} \right) \Xi(t)\psi(t) \right| \nabla t \leq \int_{[r_1, r_2]_{\mathbb{T}}} \zeta(t)Y(t)\nabla t.$$

Remark 9. By letting $\mathbb{T} = \mathbb{R}$, in Corollary 21, we recapture [3] (Corollary 2.4).

3. Conclusions

In this article, we explore new generalizations of the integral Steffensen inequality given in [1–3] by the utilization of the diamond- α dynamic inequalities which are used in various problems involving symmetry. We generalize a number of those inequalities to a general time scale measure space. In addition to this, in order to obtain some new inequalities as special cases, we also extend our inequalities to a discrete and constant calculus.

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