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A Modified Krasnosel'skiĭ–Mann Iterative Algorithm for Approximating Fixed Points of Enriched Nonexpansive Mappings

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Abstract: For approximating the fixed points of enriched nonexpansive mappings in Hilbert spaces, we consider a modified Krasnosel'skiĭ–Mann algorithm for which we prove a strong convergence theorem. We also empirically compare the rate of convergence of the modified Krasnosel'skiĭ–Mann algorithm and of the simple Krasnosel'skiĭ fixed point algorithm. Based on the numerical experiments reported in the paper we conclude that, for the class of enriched nonexpansive mappings, it is more convenient to work with the simple Krasnosel'skiĭ fixed point algorithm than with the modified Krasnosel'skiĭ–Mann algorithm.

Keywords: Hilbert space; enriched nonexpansive mapping; fixed point; modified Krasnosel'skiĭ–Mann algorithm; strong convergence

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1. Introduction and Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Suppose that C is a nonempty closed and convex subset of H. A mapping $T : C \to C$ is nonexpansive if it satisfies the following symmetric contractive type condition:

$$\|Tx - Ty\| \leq \|x - y\| \tag{1}$$

for all $x, y \in C$.

A point $x \in C$ is called a fixed point of T provided $Tx = x$. We denote by $Fix\,(T)$ the set of fixed points of T, that is, $Fix\,(T) := \{x \in C : Tx = x\}$.

The problem of existence and approximation of fixed points of nonexpansive mappings is important as it has important applications in various areas of research, and many problems can be regarded as fixed point problems for appropriate nonexpansive mappings: convex feasibility problems, convex optimization problems, monotone variational inequalities, image recovery, signal processing, and so on.

However, the study of fixed points of nonexpansive mappings is not a trivial task. Indeed, if C is a closed nonempty subset of a Banach space X and $T : C \to C$ is nonexpansive, it is known that T may not have a fixed point or it may have many fixed points, and third, it may happen that, even if T has a unique fixed point, the Picard iteration $\{x_n = T^n x_0\}$ may fail to converge to such a fixed point. One of the simplest examples of such a map is $Tx = 1 - x$ on $[0, 1]$ with the usual norm, which gives, for $x_0 = 1$, say, $x_{2n} = 1$ and $x_{2n+1} = 0$. In addition, rotation about the origin of the unit disk in the plane is another example of nonexpansive mapping having a unique fixed point while $\{x_n = T^n x_0\}(x_0 \neq 0)$ does not converge.

These aspects made the study of nonexpansive mappings one of the major and most active research areas of nonlinear analysis since the mid-1960s, see, for example, the monographs [1–4] and references therein.

One way to obtain convergent iterative schemes for the approximation of fixed points of nonexpansive mappings is mainly due to Mann [5] and to Krasnosel'skiǐ [6], who considered, instead of Picard iteration (which does not converge, in general, for nonexpansive mappings), an explicit averaged iteration of the form

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, n \geq 0, \tag{2}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. This simple but powerful algorithm is usually called the Mann iteration, or the Krasnosel'skiǐ–Mann iteration, or simply Krasnosel'skiǐ iteration, in the case $\lambda_n = \lambda$ (constant), to acknowledge the pioneering results in [5,6].

Although the Krasnosel'skiǐ–Mann algorithm defined by (2) provides a unified framework for different algorithms, it however has only weak convergence under certain conditions, see for example [7]. Thus, in order to achieve the convergence in norm of the iterates, it is necessary to impose additional conditions on the considered operators or on the space (demicompactness, continuity, compactness) or to operate some modifications of the algorithm itself, such as in [8–11] etc.

Let H be a real Hilbert space and let $T : H \to H$ be a nonlinear mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. Yao, Zhou and Liou [12] introduced the following modified Krasnosel'skiǐ–Mann iterative algorithm:

$$\begin{cases} y_n = (1-\alpha_n)x_n; \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, n \geq 0. \end{cases} \tag{3}$$

where $x_0 \in H$ is given, and proves that $\{x_n\}$ converges strongly to a fixed point of the nonexpansive mapping T.

Note that, in the particular case $\alpha_n \equiv 0$ and $\beta_n \equiv \lambda$ (constant), the modified Krasnosel'skiǐ–Mann iterative algorithm (3) reduces to the Krasnosel'skiǐ algorithm (2).

On the other hand, the author [13], see also [14], introduced and studied the class of enriched nonexpansive mapping as a generalization of the nonexpansive mappings.

Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an *enriched nonexpansive* (or *b-enriched nonexpansive*) if there exists $b \in [0, \infty)$ such that

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x, y \in X. \tag{4}$$

We note that condition (4) is also symmetric and that the class of enriched nonexpansive mappings includes all nonexpansive mappings, which are obtained for $b = 0$ in (4). In [13] it was also shown (see Example 2.1) that there exist other important classes of mappings, e.g., Lipschitzian and generalized pseudocontractive mappings, which are not nonexpansive but are enriched nonexpansive. Results on the existence and approximation of fixed points of enriched nonexpansive mappings by means of the Krasnosel'skiǐ iteration were also established, for which both weak and strong convergence are provided.

However, the strong convergence result obtained in [13] for the Krasnosel'skiǐ iteration (Theorem 1 below) is tributary to the additional property of *demicompactness* of the enriched nonexpansive mapping T. To state it here, we need the following concept.

Definition 1 ([15]). *Let H be a Hilbert space and C a subset of H. A mapping $T : C \to H$ is called demicompact if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent.*

Theorem 1 ([13]). *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be a b-enriched nonexpansive and demicompact mapping. Then the set $Fix\,(T)$ of fixed points of T is a* *nonempty convex set and there exists $\lambda \in (0,1)$ such that, for any given $x_0 \in C$, the Krasnosel'skiǐ iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n,\ n \geq 0, \tag{5}$$

converges strongly to a fixed point of T.

Starting from these facts, our aim in this paper is to achieve strong convergence by considering the modified Krasnosel'skiǐ–Mann iterative algorithm (3) to approximate fixed points of enriched nonexpansive mappings which are not necessarily nonexpansive. We also compare numerically the rate of convergence of the modified Krasnosel'skiǐ–Mann iterative algorithm (3) to the one of the Krasnosel'skiǐ algorithm used in [13].

2. Strong Convergence of the Modified Krasnosel'skiǐ–Mann Algorithm

To prove the main result of this paper, we need some auxiliary results collected in the next lemmas.

Lemma 1. *Let H be a real Hilbert space. Then the following identity holds:*

$$\|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2, \forall x,y \in H.$$

Lemma 2 (Browder's demiclosedness principle, [16])**.** *Let C be a nonempty closed convex of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed on C, i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\|x_n - Tx_n\| \to 0$, then $x \in Fix\,(T)$.*

Lemma 3 ([17])**.** *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to}\delta_n \leq 0$ *or* $\sum_{n=0}^{\infty}\gamma_n\delta_n < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 2. *Let H be a real Hilbert space and let $T : H \to H$ be an enriched nonexpansive mapping with $Fix\,(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers in $[0,1]$. Assume the following conditions are satisfied:*

(C_1) $\lim\limits_{n\to\infty}\alpha_n = 0$;
(C_2) $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$;
(C_3) $\beta_n \in [a,b] \subset (0,1)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the modified Krasnosel'skiǐ–Mann iterative algorithm

$$\begin{cases} y_n = (1-\alpha_n)x_n; \\ x_{n+1} = (1-\mu\beta_n)y_n + \mu\beta_n T y_n, n \geq 0 \end{cases} \tag{6}$$

converge strongly to a fixed point of T, where $x_0 \in H$ and $\mu \in (0,1]$ is some constant.

Proof. As T is enriched nonexpansive, it follows by Equation (4) that there exists a constant $b \in [0,\infty)$, such that

$$\|b(x-y) + Tx - Ty\| \leq (b+1)\|x-y\|, \forall x,y \in C.$$

By using $b = \frac{1}{\mu} - 1$, it follows that $\mu \in (0,1]$ and the previous inequality becomes

$$\|(1-\mu)(x-y) + \mu Tx - \mu Ty\| \leq \|x-y\|, \forall x, y \in C. \quad (7)$$

Denote $T_\mu x = (1-\mu)x + \mu Tx$. Then Inequality (7) shows that

$$\|T_\mu x - T_\mu y\| \leq \|x-y\|, \forall x, y \in C,$$

i.e., that the averaged operator T_μ is nonexpansive.

In view of the hypotheses we have

$$Fix\,(T_\mu) = Fix\,(T) \neq \emptyset.$$

In order to prove the theorem, let us observe that the sequence $\{x_n\}_{n=0}^{\infty}$ given by Equation (6), that is,

$$\begin{cases} y_n = (1-\alpha_n)x_n; \\ x_{n+1} = (1-\mu\beta_n)y_n + \mu\beta_n Ty_n, n \geq 0 \end{cases}$$

is actually the modified Krasnosel'skiĭ–Mann iterative algorithm (3) corresponding to the averaged operator T_μ.

Claim 1. Sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $p \in Fix\,(T_\mu)$. As T_μ is nonexpansive, for any $x \in H$ we have

$$\langle T_\mu x - p, T_\mu x - p\rangle = \|T_\mu x - p\|^2 \leq \|x-y\|^2 = \langle x-p, x-p\rangle$$

which implies

$$\langle T_\mu x - p, T_\mu x - p\rangle \leq \langle x-p, x - T_\mu x\rangle + \langle x-p, T_\mu x - p\rangle$$

$$\iff \langle T_\mu x - p, T_\mu x - x\rangle \leq \langle x-p, x - T_\mu x\rangle$$

$$\iff \langle T_\mu x - p, T_\mu x - x\rangle + \langle x-p, T_\mu x - x\rangle \leq \langle x-p, x - T_\mu x\rangle$$

and therefore

$$\|T_\mu x - x\|^2 \leq 2\langle x-p, x - T_\mu x\rangle. \quad (8)$$

Using Equation (6), we have

$$\|x_{n+1} - p\|^2 = \|(1-\beta_n)y_n + \beta_n T_\mu y_n\|^2 = \|(y_n - p) - \beta_n(y_n - T_\mu y_n)\|^2$$

which, by Lemma 1, yields

$$\|x_{n+1} - p\|^2 = \|y_n - p\|^2 - 2\beta_n\langle y_n - T_\mu y_n, y_n - p\rangle + \beta_n^2\|y_n - T_\mu y_n\|^2$$

and from this identity, by using Inequality (8), one obtains

$$\|x_{n+1} - p\|^2 \leq \|y_n - p\|^2 - \beta_n\|y_n - T_\mu y_n\|^2 + \beta_n^2\|y_n - T_\mu y_n\|^2$$

that is,

$$\|x_{n+1} - p\|^2 \leq \|y_n - p\|^2 - \beta_n(1-\beta_n)\|y_n - T_\mu y_n\|^2 \quad (9)$$

which implies

$$\|x_{n+1} - p\|^2 \leq \|y_n - p\|^2. \quad (10)$$

Now, by Equation (6), we have

$$\|y_n - p\| = \|(1-\alpha_n)x_n - p\| = \|(1-\alpha_n)(x_n - p) - \alpha_n p\|$$

$$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\| \leq \max\{\|x_n - p\|, \|p\|\},$$

from which we can show easily by induction that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\},$$

which proves that sequences $\{x_n\}$ and $\{y_n\}$ are bounded, and so Claim 1 is proved.

Claim 2. For $p \in Fix\,(T_\mu)$, there exists $M \geq 0$ such that

$$\|x_{n+1} - p\|^2 - \|x_n - p\|^2 + k\|x_{n+1} - x_n\|^2 \leq M\alpha_n,\ n \geq 0. \tag{11}$$

Indeed, by Equation (6), we have

$$y_n - T_\mu y_n = \frac{1}{\beta_n}(y_n - x_{n+1}) \tag{12}$$

which, by Equation (9), yields

$$\|y_n - p\|^2 \leq \|y_n - p\|^2 - \frac{1-\beta_n}{\beta_n}\|y_n - x_{n+1}\|^2. \tag{13}$$

Having in view that $0 < a < \beta_n < b < 1$, it follows that $\frac{1-\beta_n}{\beta_n} > \frac{1-b}{b} := k$ and $k < \frac{1}{2}$, and hence by Equation (13), we obtain

$$\|x_{n+1} - p\|^2 = \|y_n - p\|^2 \leq \|y_n - p\|^2 - k\|y_n - x_{n+1}\|^2 = \|x_n - p - \alpha_n x_n\|^2$$
$$-k\|x_n - x_{n+1} - \alpha_n x_n\|^2$$

By using Lemma 1, the previous inequality implies

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|x_n - p\|^2 - 2\alpha_n\langle x_n, x_n - p\rangle + \alpha_n^2\|x_n\|^2 - k\|x_n - x_{n+1}\|^2 \\ &+2k\alpha_n\langle x_n, x_n - x_{n+1}\rangle - k\alpha_n^2\|x_n\|^2 = \|x_n - p\|^2 - k\|x_n - x_{n+1}\|^2 \\ &+ \alpha_n\Big(-2\langle x_n, x_n - p\rangle + 2k\alpha_n\langle x_n, x_n - x_{n+1}\rangle + (1-k)\alpha_n\|x_n\|^2\Big)\end{aligned} \tag{14}$$

As $\{x_n\}$ is bounded, there exists $M \geq 0$ such that, for $n \geq 0$,

$$-2\langle x_n, x_n - p\rangle + 2k\alpha_n\langle x_n, x_n - x_{n+1}\rangle + (1-k)\alpha_n\|x_n\|^2 \leq M,$$

which by using Equation (14) proves Claim 2.

As $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point, say, $q \in H$.

Claim 3. $\{x_n\}$ and $\{y_n\}$ converge strongly to q.

We discuss the following two cases.

Case 1. The sequence $\{\|x_n - q\|\}$ is nonincreasing.

Then $\|x_n - p\|$ is convergent and therefore

$$\|x_{n+1} - q\|^2 - \|x_n - q\|^2 \to 0,\ \text{as } n \to \infty$$

which, by Equation (11) and assumption (C_1), implies

$$\|x_n - x_{n+1}\|^2 \to 0,\ \text{as } n \to \infty. \tag{15}$$

Now, by Equation (6) and (C_1), we deduce that

$$\|y_n - x_n\| \leq \alpha_n\|x_n\| \to 0,\ \text{as } n \to \infty$$

which, by Equation (16), proves that

$$\|y_n - T_\mu y_n\| \to 0, \text{ as } n \to \infty.$$

Now, using the nonexpansiveness of T_μ, one obtains

$$\|x_n - T_\mu x_n\| \le \|x_n - y_n\| + \|y_n - T_\mu y_n\| + \|T_\mu y_n - T_\mu x_n\|$$

$$\le 2\|x_n - y_n\| + \|y_n - T_\mu y_n\|$$

which shows that $\{x_n - T_\mu x_n\}$ converges to 0 as $n \to \infty$.

As $I - T_\mu$ is demiclosed, it follows by Lemma 2 that $q \in Fix\,(T_\mu)$. This means that $\{x_n\}$ and $\{y_n\}$ converge weakly $q \in Fix\,(T_\mu)$.

By using Equation (10) and Lemma 1, we have

$$\|x_{n+1} - q\|^2 \le \|y_n - q\|^2 = \|(1-\alpha_n)(x_n - q)) - \alpha_n q\|^2$$

$$\le (1-\alpha_n)^2\|x_n - q\|^2 - 2\alpha_n\langle y_n - q, q\rangle \le (1-\alpha_n)\|x_n - q\|^2 - 2\alpha_n\langle y_n - q, q\rangle. \tag{16}$$

Now, as $y_n \rightharpoonup q$, it follows that $\lim_{n\to\infty}\langle y_n - q, q\rangle = 0$ and hence, applying Lemma 3 to Equation (16), we obtain that $x_n \to q$ (and also $y_n \to q$), as claimed.

Case 2. The sequence $\{\|x_n - q\|\}$ is not nonincreasing.

Denote $\Gamma_n = \|x_n - q\|^2$ and consider the function $\tau : \mathbb{N} \to \mathbb{N}$ defined for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N} : k \le n,\, \Gamma_k \le \Gamma_{k+1}\}.$$

By definition, τ is a nondecreasing sequence such that

$$\tau(n) \to \infty \text{ as } n \to \infty \text{ and } \tau(n) \le \tau(n+1),\, n \ge n_0.$$

Now, by Equation (11), we have

$$\|x_{\tau(n)+1} - q\|^2 - \|x_{\tau(n)} - q\|^2 + k\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \le M\alpha_{\tau(n)},\, n \ge n_0,$$

which, by letting $n \to \infty$, implies

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\|^2 \le \frac{M}{k}\cdot\lim_{n\to\infty}\alpha_{\tau(n)} = 0$$

which implies

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0 \text{ as } n \to \infty.$$

On the other hand, for all $n \ge n_0$, we have

$$0 \le \|x_{\tau(n)+1} - q\|^2 - \|x_{\tau(n)} - q\|^2 \le \alpha_{\tau(n)}[2\langle q - y_{\tau(n)}, q\rangle - \|x_{\tau(n)} - q\|^2]$$

from which we deduce that

$$\|x_{\tau(n)+1} - q\|^2 \le 2\langle q - y_{\tau(n)}, q\rangle$$

and, as $\{y_n\}$ converges weakly to q (and hence $\{y_{\tau(n)}\}$ converges, too), this shows that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - q\| = 0.$$

Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$$

Let us note that, for $n \geq n_0$, one has $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because for j satisfying $\tau(n) \leq j < n$, we have $\Gamma_j > \Gamma_{j+1}$. This implies that, for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\},$$

and this implies

$$\lim_{n\to\infty} \Gamma_n = 0,$$

which shows that $\{x_n\}$ converges strongly to q (and therefore, $\{y_n\}$ converges strongly to q, too). □

The next Corollary is the main result (Theorem 1) in [12].

Corollary 1. *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $Fix\,(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers in $[0,1]$. Assume the following conditions are satisfied:*

(C_1) $\lim_{n\to\infty} \alpha_n = 0$; (C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (C_3) $\beta_n \in [a,c] \subset (0,1)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the modified Krasnosel'skiǐ–Mann iterative algorithm

$$\begin{cases} y_n = (1-\alpha_n)x_n; \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, n \geq 0 \end{cases} \tag{17}$$

converges strongly to a fixed point of T, where $x_0 \in H$.

Proof. As T is nonexpansive, T is 0-enriched nonexpansive and therefore the constant μ defined in the proof of Theorem 2 equals 1. So, the algorithm (6) reduces to Equation (17) and conclusion follows by Theorem 2. □

Remark 1. *To avoid the assumption $Fix\,(T) \neq \emptyset$ in Theorem 2, we can merge it with Theorem 1 and consider T defined on a bounded closed subset of H.*

Theorem 3. *Let H be a real Hilbert space, C a bounded closed convex subset of H, and $T : C \to C$ a b-enriched nonexpansive mapping. Then $Fix\,(T)$ is a nonempty convex set and the sequences $\{x_n\}$ and $\{y_n\}$ generated by the modified Krasnosel'skiǐ–Mann iterative algorithm*

$$\begin{cases} y_n = (1-\alpha_n)x_n; \\ x_{n+1} = (1-\mu\beta_n)y_n + \mu\beta_n T y_n, n \geq 0 \end{cases} \tag{18}$$

converge strongly to a fixed point of T, where $x_0 \in H$ and $\mu \in (0,1]$ is some constant and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[0,1]$, satisfying the conditions

(C_1) $\lim_{n\to\infty} \alpha_n = 0$;

(C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C_3) $\beta_n \in [a,b] \subset (0,1)$.

Proof. The first part follows by Theorem 1, while for the second part one adapts the proof of Theorem 2, by noting that in this case, the boundedness of $\{x_n\}$ and $\{y_n\}$ is ensured by the hypothesis. □

The next example shows that Theorem 2 is an effective generalization of Theorem 1 in [11].

Example 1 (Example 2.1, [13]). *Let* $X = \left[\frac{1}{2}, 2\right]$ *be endowed with the usual norm and let* $T : X \to X$ *be defined by* $Tx = \frac{1}{x}$, *for all* $x \in \left[\frac{1}{2}, 2\right]$. *Then*

(i) *T is not nonexpansive.*
(ii) *T is a 3/2-enriched nonexpansive mapping.*
(iii) $Fix\,(T) = \{1\}$.

Proof. (i) Assume T is nonexpansive. Then

$$|Tx - Ty| \leq |x - y|, \forall x, y \in X,$$

which, for $x = 1$ and $y = 1/2$, leads to the contradiction $1 \leq 1/2$.

(ii) The enriched nonexpansive condition (4) reduces in this case to

$$\left|b(x-y) + \frac{1}{x} - \frac{1}{y}\right| \leq (b+1)|x-y| \Leftrightarrow \left|b - \frac{1}{xy}\right| \cdot |x-y| \leq (b+1) \cdot |x-y|.$$

It easy to check that, for any $b \geq 3/2$, we have

$$\left|b - \frac{1}{xy}\right| \leq b + 1, \forall x, y \in \left[\frac{1}{2}, 2\right],$$

which proves that T is a 3/2-enriched nonexpansive mapping. □

Therefore, all assumptions of Theorem 2 are satisfied and, for any $x_0 \in \left[\frac{1}{2}, 2\right]$, the sequence

$$x_{n+1} = \frac{(5 - 2\beta_n)(1 - \alpha_n)}{5} \cdot x_n + \frac{2\beta_n}{5(1 - \alpha_n)x_n}, n \geq 0,$$

generated by the modified Krasnosel'skiǐ–Mann algorithm (6), converges to 1, the unique fixed point of T, provided that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions (C_1)–(C_3).

3. Numerical Experiments and Conclusions

Our aim in this section is to present a comparative study of the modified Krasnosel'skiǐ–Mann algorithm (6), involved in Theorems 2 and 3, and of the simpler Krasnosel'skiǐ algorithm (5), involved in Theorem 1, for the case of the enriched nonexpansive function T in Example 1. For the numerical experiments, which are given in Tables 1–4, we consider different values of the parameters $\{\alpha_n\}$, $\{\beta_n\}$, and λ and of the starting points x_0. By N, we denote the number of iterations needed to obtain the fixed point with six exact digits.

Table 1. Numerical experiments for Krasnosel'skiǐ iteration and starting points $x_0 = 1.95$ and $x_0 = 2$.

n	$\lambda = 1/2$	$\lambda = 1/3$	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 4/5$	$\lambda = 8/9$
0	1.95	1.95	1.95	1.95	2	2
1	1.23141	1.47094	0.99188	0.872115	0.8	0.666667
2	1.02174	1.20724	1.00275	1.07801	1.16	1.40741
3	1.00023	1.08094	0.999088	0.96523	0.921655	0.787958
4	1	1.029	1.0003	1.01832	1.05233	1.21564
5	1	1.00994	0.999899	0.991085	0.970681	0.86628
6	1	1.00335	1.00003	1.00452	1.0183	1.12235
7	1	1.00112	0.999989	0.997756	0.989283	0.916693

Table 1. *Cont.*

n	$\lambda = 1/2$	$\lambda = 1/3$	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 4/5$	$\lambda = 8/9$
8	1	1.00037	1	1.00113	1.00652	1.07152
9	1	1.00012	0.999999	0.999438	0.99612	0.948614
10	1	1.00004	1	1.00028	1.00234	1.04244
11	1	1.00001	1	0.999859	0.9986	0.968526
12	1	1	1	1.00007	1.00084	1.02539
13	1	1	1	0.999965	0.999496	0.980812
14	1	1	1	1.00002	1.0003	1.01526
15	1	1	1	0.999991	0.999818	0.988337
N	3	11	9	20	27	57

Table 2. Numerical experiments for Krasnosel'skiǐ iteration and starting point $x_0 = 0.5$.

n	$\lambda = 1/2$	$\lambda = 1/3$	$\lambda = 2/3$	$\lambda = 3/4$	$\lambda = 4/5$	$\lambda = 8/9$
0	0.5	0.5	0.5	0.5	0.5	0.5
1	1.25	1	1.5	1.625	1.7	1.83333
2	1.025	1	0.944444	0.867788	0.810588	0.688552
3	1.003	1	1.0207	1.08121	1.14906	1.36746
4	1	1	0.993381	0.963969	0.926035	0.801969
5	1	1	1.00224	1.01903	1.04911	1.19749
6	1	1	0.999258	0.990754	0.972376	0.875348
7	1	1	1.00025	1.00469	1.0172	1.11273
8	1	1	0.999917	0.997672	0.989911	0.922473
9	1	1	1.00003	1.00117	1.00614	1.06609
10	1	1	0.999991	0.999417	0.996349	0.952238
11	1	1	1	0.999863	1.0022	1.03928
12	1	1	0.999999	0.999854	0.998683	0.97077
13	1	1	1	1.00007	1.00079	1.02352
14	1	1	1	0.999964	0.999526	0.98219
15	1	1	1	1.00002	1.00028	1.01414
N	3	1	13	22	28	56

Table 3. Numerical experiments for modified Krasnosel'skiǐ–Mann iteration and starting point $x_0 = 1.95$.

n	$\alpha_n = 1/(2n+1)$; $\beta_n = n/(3n+2)$	$\alpha_n = 1/(n+1)$; $\beta_n = 2n/(3n+2)$	$\alpha_n = 1/(n+1)$; $\beta_n = n/(2n+2)$
0	1.95	1.95	1.95
1	1.25754	0.983103	0.980064
2	1.00483	0.829478	0.770328
3	0.893985	0.837089	0.750714
4	0.847654	0.857922	0.770893
5	0.832607	0.875828	0.794782

Table 3. *Cont.*

n	$\alpha_n = 1/(2n+1)$; $\beta_n = n/(3n+2)$	$\alpha_n = 1/(n+1)$; $\beta_n = 2n/(3n+2)$	$\alpha_n = 1/(n+1)$; $\beta_n = n/(2n+2)$
6	0.832469	0.890237	0.816099
7	0.839068	0.901868	0.83419
8	0.848366	0.911383	0.849433
9	0.858399	0.919283	0.862333
10	0.868233	0.92593	0.873334
11	0.877455	0.931592	0.882795
12	0.885909	0.936466	0.891
13	0.893569	0.940704	0.898172
14	0.90047	0.94442	0.904487
15	0.906673	0.947704	0.910084
N	>1000	>1000	>1000

Table 4. Numerical experiments for modified Krasnosel'skiǐ–Mann iteration and starting point $x_0 = 0.5$.

n	$\alpha_n = 1/(2n+1)$; $\beta_n = n/(3n+2)$	$\alpha_n = 1/n$; $\beta_n = 2n/(3n+2)$	$\alpha_n = 1/(n+1)$; $\beta_n = n/(2n+2)$
0	0.5	0.5	0.5
1	1	1	1.25
2	0.777778	0.833333	1.1
3	0.752976	0.837727	1.05227
4	0.771613	0.858056	1.03188
5	0.79504	0.875862	1.02142
6	0.8162	0.890246	1.01536
7	0.834232	0.901871	1.01155
8	0.849452	0.911384	1.009
9	0.862341	0.919283	1.00721
10	0.873338	0.92593	1.0059
11	0.882797	0.931592	1.00492
12	0.891001	0.936466	1.00417
13	0.898172	0.940704	1.00357
14	0.904487	0.94442	1.0031
15	0.910085	0.947704	1.00271
N	>600	363	362

By analyzing the results in Tables 1 and 2, some conclusions could be drawn:

1. The speed of convergence of Krasnosel'skiǐ iteration for the considered enriched nonexpansive mapping depends on both parameter λ and starting point x_0.

2. When starting from $x_0 = 1.95$ (see Table 1), the most rapid Krasnosel'skiǐ iteration corresponds to the value $\frac{1}{2}$ of the parameter λ (after three iterations we obtain the exact value of fixed point). In this case, Krasnosel'skiǐ iteration also converges quickly for the value $\frac{2}{3}$ of the parameter λ (after nine iterations we obtain the fixed point).

3. For $x_0 = 1.95$, Krasnosel'skiǐ iteration converges as slowly as the value of the parameter λ approaches 1 (note that for $\lambda = 1$, Krasnosel'skiǐ iteration reduces to Picard iteration, which is not convergent).

4. When starting from $x_0 = 0.5$ (see Table 2), the most rapid Krasnosel'skiǐ iteration corresponds to the value $\frac{1}{3}$ of the parameter λ (we obtain the fixed point after one iteration only), while for $\lambda = \frac{1}{2}$, the fixed point is obtained after three iterations.

5. Similarly to the case $x_0 = 1.95$, for $x_0 = 0.5$, the Krasnosel'skiǐ iteration converges as slowly as the parameter λ approaches 1.

6. Now, by analyzing the results in Tables 3 and 4, note that the modified Krasnosel'skiǐ–Mann iteration converges very slowly in comparison with the simple Krasnosel'skiǐ iteration. In all the three cases considered for the parameters α_n and β_n, and for any starting value $x_0 \in \{1.95; 0.5\}$, the modified Krasnosel'skiǐ–Mann iteration converges extremely slowly.

7. When starting from $x_0 = 1.95$, the exact value of the fixed point is not yet reached after 1000 iterations, while, for $x_0 = 0.5$, the modified Krasnosel'skiǐ–Mann iteration reaches the fixed point after 362 iterations in the third case and 362 iterations in the second case, while in the first case, more than 600 iterations are needed to reach the exact value of the fixed point.

8. Based on the numerical experiments reported here, we can conclude that for approximating the fixed points of some enriched nonexpansive mappings, it would be more convenient to use the Krasnosel'skiǐ iteration than the modified Krasnosel'skiǐ–Mann iteration.

9. Therefore, it is an open problem to study if a similar situation holds in the case of all enriched nonexpansive mappings or nonexpansive mappings.

10. For other related developments we refer to [18–25] and references therein.

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