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Integral Representation and Explicit Formula at Rational Arguments for Apostol–Tangent Polynomials

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Abstract: The Fourier series expansion of Apostol–tangent polynomials is derived using the Cauchy residue theorem and a complex integral over a contour. This Fourier series and the Hurwitz–Lerch zeta function are utilized to obtain the explicit formula at rational arguments of these polynomials. Using the Lipschitz summation formula, an integral representation of Apostol–tangent polynomials is also obtained.

Keywords: tangent polynomials; bernoulli polynomials; euler polynomials; genocchi polynomials; generating functions; Fourier series; integral representation; Cauchy residue theorem

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1. Introduction

Generating function is one of the important properties for special functions and other mathematical objects such as that in [1]. Some studies construct generating functions aiming to connect some combinatorial numbers and polynomials to some well-known special polynomials and distributions (see [2]). Other studies use generating functions to derive Fourier series, integral representation and explicit formula of some special numbers and functions, which is also the main object of this study. It is important to note that the integral representation is necessary in finding explicit formula and asymptotic approximation of a function (see [3,4]).

Tangent polynomials together with Bernoulli, Euler and Genocchi polynomials have been the object of recent extensive investigation in the field of computational mathematics and physics (see [5–7]). Analogues, explicit identities and symmetric properties of tangent polynomials are derived in [8–10].

Some interesting analogues of the classical Bernoulli, Euler and Genocchi polynomials were investigated by Apostol [11], Luo and Srivastava (see [12–16]) These analogues are called the Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials of higher order defined by the following relations, respectively (see [17]). For $\lambda \in C\backslash\{0\}$, $\log\lambda$ and $\log(-\lambda)$ are taken to be their principal value,

$$\sum_{n=0}^{\infty} G_n^m(x,\lambda)\frac{z^n}{n!} = \left(\frac{2z}{\lambda e^z+1}\right)^m e^{xz}, \begin{cases} |z|<\pi \text{ when } \lambda=1 \\ |z|<|\log(-\lambda)| \text{ when } \lambda\neq 1 \end{cases} \tag{1}$$

$$\sum_{N=0}^{\infty} B_n^m(x,\lambda)\frac{z^n}{n!} = \left(\frac{z}{\lambda e^z-1}\right)^m e^{xz}, \begin{cases} |z|<2\pi \text{ when } \lambda=1 \\ |z|<|\log\lambda| \text{ when } \lambda\neq 1 \end{cases} \tag{2}$$

$$\sum_{n=0}^{\infty} E_n^m(x,\lambda)\frac{z^n}{n!} = \left(\frac{2}{\lambda e^z+1}\right)^m e^{xz}, \begin{cases} |z|<\pi \text{ when } \lambda=1 \\ |z|<|\log(-\lambda)| \text{ when } \lambda\neq 1 \end{cases} \tag{3}$$

$$\sum_{n=0}^{\infty} T_n\ (x;\lambda)\frac{z^n}{n!} = \left(\frac{2}{\lambda e^{2z}+1}\right)e^{xz}\ ,\ (|2z+\log\lambda| < \pi) \tag{4}$$

when $m = 1$, the above equations give the generating functions for the Apostol–Genocchi, Apostol–Bernoulli, and Apostol–Euler polynomials, respectively (see [18]). Parallel to these, we can also extend the tangent polynomials as follows:

For $\lambda \in \mathbb{C}\backslash\{0\}$ and $\log\lambda$ is taken to be the principal value, the Apostol–tangent polynomials $T_n(x,\lambda)$ are defined by means of the generating function:

which is valid within the circle $C : z = Re^{i\theta}, -\pi < \theta \le \pi$ with the radius

$$R < \min\left\{\frac{1}{2}|\pi i + \log\lambda|, \frac{1}{2}|\pi i - \log\lambda|\right\}$$

This validity can be obtained as follows: set the denominator of the generating function equal to 0 and solve for z.

$$\begin{gathered}\lambda e^{2z} + 1 = 0\\ 2z = \log(-1) - \log\lambda\\ z = \tfrac{1}{2}(\ln|-1| - \pi i + 2k\pi i - \log\lambda),\ \ k \in \mathbb{Z}\\ z = \tfrac{1}{2}((2k-1)\pi i - \log\lambda),\ \ k \in \mathbb{Z}\end{gathered}$$

These values of z, which we denote by z_k, are the singularities of the generating Function (4). We impose that R should be less than the modulus of the nearest singularity, which is $z_0 = \frac{1}{2}[\pi i - \log\lambda]$ or $z_{-1} = \frac{1}{2}[\pi i + \log\lambda]$. Thus, $R < \min\{|z_0|,\ |z_1|\}$ as prescribed above.

Note that when $\lambda = 1$, the Apostol–tangent polynomial $T_n(x,\lambda)$ reduces to the tangent polynomial $T_n\ (x;1) = T_n\ (x)$.

Fourier series is an expansion of a periodic function as an infinite sum of sines and cosines which can be easily differentiated and integrated. It is a useful tool in modeling and analyzing functions such as saw waves, which are common signals in experimentation [19]. Its applications are used in electronics, quantum mechanics, acoustics, and communications. For instance, Fourier series are utilized in audio compression [20].

In the study of Luo [21], the Lipschitz summation formula was used to obtain the Fourier series expansion of Genocchi polynomials. Araci and Acikgoz [22] used the Cauchy residue theorem and a complex integral over a contour to establish the Fourier expansion of Apostol–Euler polynomials. Motivated by the studies in [13,21,22], Corcino et al. [23] derived the integral representation and explicit formula at rational arguments for Genocchi polynomials of higher order. However, there was no available literature or related study that mentions about the Fourier series of Apostol–tangent polynomials [24].

In this paper, the Fourier series expansion of Apostol–tangent polynomials will be derived using the method of [22,24]. Moreover, using the method of Luo (see [13,21]), the integral representation and explicit formula at rational arguments of these polynomials will be established.

2. Fourier Expansion for Apostol–Tangent Polynomials

The first step of the method in [22] and [24] in deriving the Fourier series expansion of Apostol–tangent polynomials is to show the convergence of certain integral to 0. The following lemma contains such convergence. In proving the lemma, we used analytic method as performed in [18].

Lemma 1. *Let C_N be a circle about the origin of radius $\left(\frac{1}{2}(2N-1+\epsilon)\pi\right)$, $N \in \mathbb{Z}^+$ with ϵ being a fixed real number such that $\epsilon\pi i \pm \log\lambda\ \neq 0\ (mod\ \pi i)$. Then, as $N \to \infty$, $n > 0$ and $0 \le x \le 1$,*

$$\int_{C_N} \frac{2e^{xz}}{(\lambda e^{2z}+1)z^{n+1}}\,dz \to 0$$

Proof. Using the basic property of integration,

$$\left|\int_{C_N} \frac{2e^{xz}\ dz}{(\lambda e^{2z}+1)z^{n+1}}\right| \leq \int_{C_N} \frac{|2e^{xz}||dz|}{|\lambda e^{2z}+1||z^{n+1}|}$$

For $0 \leq x \leq 1$, $\left|\lambda e^{2z}+1\right| > \left|\lambda e^{2z}\right|$. Let $z = a+bi$,

$$\frac{|e^{xz}|}{|\lambda e^{2z}+1|} = \frac{\left|e^{x(a+bi)}\right|}{|\lambda e^{2z}+1|} = \frac{e^{ax}}{|\lambda e^{2z}+1|} \leq \frac{e^{x\Re(z)}}{|\lambda e^{2z}|} \leq \frac{1}{|\lambda|}$$

Thus,

$$\left|\int_{C_N} \frac{2e^{xz}dz}{(\lambda e^{2z}+1)z^{n+1}}\right| \leq \frac{2}{|\lambda|}\int_{C_N} \frac{|dz|}{|z^{n+1}|} = \frac{2^{n+1}}{|\lambda|(2N-1+\epsilon)\pi))^n}$$

As $N \to \infty$, the last expression goes to 0. Hence, as $N \to \infty$, $n \geq 1$,

$$\int_{C_N} \frac{2e^{xz}}{(\lambda e^{2z}+1)z^{n+1}}\ dz \to 0$$

□

The Fourier series expansion of Apostol–tangent polynomials is stated in the following theorem.

Theorem 1. *For* $n > 0$, $0 \leq x \leq 1$ *and* $\lambda \in \mathbb{C}\backslash\{0\}$,

$$\begin{aligned} T_n(x;\lambda) &= \tfrac{2^{n+1}n!}{\lambda^{x/2}} \sum_{k\in z} \tfrac{e^{\frac{1}{2}(2k-1)\pi ix}}{[(2k-1)\pi i-\log(\lambda)]^{n+1}} \\ &= \tfrac{2^{n+1}n!i^{n+1}}{\lambda^{x/2}} \left[\sum_{k=0}^{\infty} \tfrac{e^{\frac{1}{2}[-(n+1)+(2k+1)x]\pi i}}{[(2k+1)\pi i-\log(\lambda)]^{\ n+1}}\right. \end{aligned} \tag{5}$$

$$\left. + \sum_{k=0}^{\infty} \frac{e^{\frac{1}{2}[(n+1)-(2k+1)x]\pi i}}{[(2k+1)\pi i+\log(\lambda)]^{n+1}}\right] \tag{6}$$

Proof. Consider the integral $\int_{C_N} f_n(z)dz$ where

$$f_n(z) = \frac{2e^{xz}}{(\lambda e^{2z}+1)z^{n+1}}$$

and the circle C_N is as described in the Lemma 1.
The function $f_n\ (z)$ has poles at $z = 0$ of order $n+1$ and at $z_k = \frac{1}{2}[(2k-1)\pi i - \log \lambda]$, $k \in \mathbb{Z}$. The poles z_k are simple poles. Using the Cauchy Residue Theorem,

$$\int_{C_N} f_n(z)\ dz = 2\pi i\ Res\ (f_n(z),\ z=0)\ +\ 2\pi i \sum_{k\epsilon Z,\ k<N} Res\ (f_n\ (z), z=z_k)$$

Taking $N \to\ \infty$, by Lemma 1,

$$0 = Res\ (f_n(z),\ z=0) + \sum_{k\epsilon Z,} Res\ (f_n\ (z), z=z_k).$$

Now, the first residue $Res\ (f_n(z), z = 0)$ is given as

$$
\begin{aligned}
es(f_n(z), z = 0) &= \lim_{z\to 0} \frac{1}{n!} \frac{d^n}{dz^n} (z-0)^{n+1} \frac{1}{z^{n+1}} \left(\frac{2e^{xz}}{\lambda e^{2z}+1} \right) \\
&= \lim_{z\to 0} \frac{1}{n!} \frac{d^n}{dz^n} \left(\frac{2e^{xz}}{\lambda e^{2z}+1} \right) \\
&= \lim_{z\to 0} \frac{1}{n!} \frac{d^n}{dz^n} \left(\sum_{l=0}^{\infty} T_l(x;\lambda) \frac{z^l}{l!} \right) \\
&= \lim_{z\to 0} \frac{1}{n!} \sum_{l=n}^{\infty} T_l(x;\lambda) \frac{z^{l-n}}{(l-n)!}
\end{aligned}
$$

Note that the limit of each term of the expansion is 0 as $z \to 0$ except the term when $l = n$. This gives

$$
Res\ (f_n(z), z = 0) = \frac{T_n(x;\lambda)}{n!}
$$

On the other hand, the residue $Res\ (f_n(z),\ z = z_k)$ is given by

$$
\begin{aligned}
Res\ (f_n(z), z = z_k) &= \lim_{z\to z_k} (z - z_k) \frac{1}{z^{n+1}} \left(\frac{2e^{xz}}{\lambda e^{2z}+1} \right) \\
&= \frac{2e^{xz_k}}{z_k^{n+1}} \lim_{z\to z_k} \left(\frac{z-z_k}{\lambda e^{2z}+1} \right) = \frac{e^{(x-2)z_k}}{\lambda z_k^{n+1}}
\end{aligned}
$$

Since $z_k = \frac{1}{2}[(2k-1)\pi i - \log\lambda]$,

$$
\begin{aligned}
\text{Res}\ (f_n(z), z = z_k) &= \frac{e^{(x-2)\{\frac{1}{2}[(2k-1)\pi i - \log\lambda]\}}}{\lambda\left\{\frac{1}{2}[(2k-1)\pi i - \log\lambda]\right\}^{\,n+1}} \\
&= \frac{2^{\,n+1}\ e^{\frac{1}{2}(x-2)(2k-1)\pi i}\ e^{\,\log\lambda}}{\lambda\ [(2k-1)\pi i - \log\lambda]^{\,n+1}\ e^{\,\frac{x}{2}(\log\lambda)}} \\
&= \frac{2^{n+1}\ e^{\frac{1}{2}(2k-1)x\pi i}\ e^{\,(-2k+1)\pi i}}{\lambda^{\,\frac{x}{2}}\ [(2k-1)\pi i - \log\lambda]^{\,n+1}} \\
&= \frac{-2^{n+1}\ e^{\frac{1}{2}(2k-1)x\pi i}}{\lambda^{\,\frac{x}{2}}\ [(2k-1)\pi i - \log\lambda]^{\,n+1}}
\end{aligned}
$$

Combining these residues gives,

$$
0 = \frac{T_n(x\ ;\ \lambda)}{n!} + \sum_{k\in\mathbb{Z}} \frac{-2^{n+1}\ e^{\frac{1}{2}(2k-1)x\pi i}}{\lambda^{\,\frac{x}{2}}\ [(2k-1)\pi i - \log\lambda]^{\,n+1}}
$$

Hence,

$$
T_n(x;\lambda) = \frac{2^{n+1}\ n!}{\lambda^{\frac{x}{2}}} \sum_{k\in\mathbb{Z}} \frac{e^{\frac{1}{2}(2k-1)x\pi i}}{[(2k-1)\pi i - \log\lambda]^{\,n+1}} \tag{7}
$$

Note that $i^{-(n+1)} = e^{\frac{-(n+1)\pi i}{2}}$ and $(-1)^{\,n+1} = e^{(n+1)\pi i}$. Thus, replacing k with $k+1$ in (7) yields

$$\begin{aligned}
T_n(x;\lambda) &= \frac{2^{n+1}\,n!}{\lambda^{\frac{x}{2}}}\sum_{k\in\mathbb{Z}}\frac{e^{\frac{1}{2}(2k+2-1)\pi ix}}{[(2k+2-1)\pi i-\log\lambda]^{\,n+1}}\\
&= \frac{2^{n+1}n!\;i^{\,n+1}}{\lambda^{\frac{x}{2}}}\sum_{k\in\mathbb{Z}}\frac{i^{-(n+1)}\;e^{\frac{1}{2}(2k+1)\pi ix}}{[(2k+1)\pi i-\log\lambda]^{\,n+1}}\\
&= \frac{2^{n+1}n!\;i^{\,n+1}}{\lambda^{\frac{x}{2}}}\left[\sum_{k=0}^{\infty}\frac{e^{-\frac{(n+1)\pi i}{2}}\;e^{\frac{1}{2}(2k+1)\pi ix}}{[(2k+1)\pi i-\log\lambda]^{\,n+1}}\right.\\
&\qquad\left.+\sum_{k=-\infty}^{-1}\frac{e^{-\frac{(n+1)\pi i}{2}}\;e^{\frac{1}{2}(2k+1)\pi ix}}{[(2k+1)\pi i-\log\lambda]^{\,n+1}}\right]\\
&= \frac{2^{n+1}n!\;i^{\,n+1}}{\lambda^{\frac{x}{2}}}\left[\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}(-(n+1)\pi i+(2k+1)\pi ix)}}{[(2k+1)\pi i-\log\lambda]^{\,n+1}}\right.\\
&\qquad\left.+\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}(-(n+1)\pi i-(2k+1)\pi ix)}}{(-1)^{\,n+1}\,[(2k+1)\pi i+\log\lambda]^{\,n+1}}\right]\\
&= \frac{2^{n+1}n!\;i^{\,n+1}}{\lambda^{\frac{x}{2}}}\left[\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}(-(n+1)+(2k+1)x)\pi i}}{[(2k+1)\pi i-\log\lambda]^{\,n+1}}\right.\\
&\qquad\left.+\sum_{k=0}^{\infty}\frac{e^{\,(n+1)\;\pi i}\;\;e^{\frac{1}{2}(-(n+1)-(2k+1)x)\pi i}}{[(2k+1)\pi i+\log\lambda]^{\,n+1}}\right]\\
&= \frac{2^{n+1}n!\;i^{\,n+1}}{\lambda^{\frac{x}{2}}}\left[\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}(-(n+1)+(2k+1)x)\pi i}}{[(2k+1)\pi i-\log\lambda]^{\,n+1}}\right.\\
&\qquad\left.+\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}((n+1)-(2k+1)x)\pi i}}{[(2k+1)\pi i+\log\lambda]^{\,n+1}}\right].
\end{aligned}$$

□

Remark 1. *When*$\lambda = 1$, *the Fourier series expansion in Theorem 1 gives*

$$\begin{aligned}
T_n(x) = T_n(x;1) &= 2^{n+1}n!\sum_{k\in z}\frac{e^{\frac{1}{2}(2k-1)\pi ix}}{[(2k-1)\pi i]^{n+1}}\\
&= 2^{n+1}n!i^{n+1}\left[\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}(-(n+1))+(2k+1)x)\pi i}}{[(2k+1)\pi i]^{\,n+1}}+\sum_{k=0}^{\infty}\frac{e^{\frac{1}{2}((n+1)-(2k+1)x)\pi i}}{[(2k+1)\pi i]^{n+1}},\right]
\end{aligned}$$

which is the Fourier series expansion of tangent polynomials.

3. Integral Representation for the Apostol–Tangent Polynomials

In this section, an integral representation for the Apostol–tangent polynomials will be obtained. For convenience, we take $\lambda = e^{2\pi i\xi}\left(\xi\in\mathbb{R},\ |\xi| < \frac{1}{2}\right)$.

Theorem 2. *For*$n \geq 0,\ 0 \leq x \leq 1,\ \xi \in \mathbb{R}$, *we have*

$$T_n\left(x;e^{2\pi i\xi}\right) = 2^{n+1}e^{-\pi i\xi x}\int_0^{\infty}\frac{M(n;x,t)\cosh(2\xi\pi t)+iN(n;x,t)\sinh(2\xi\pi t)}{\cosh(2\pi t)-\cos(\pi x)}\,t^n\,dt \qquad (8)$$

where

$$M(n;x,t) = \sin\left(\frac{\pi x}{2}+\frac{n\pi}{2}\right)e^{-\pi t}+\sin\left(\frac{\pi x}{2}-\frac{n\pi}{2}\right)e^{\pi t}$$

and

$$N(n;x,t) = \cos\left(\frac{\pi x}{2}+\frac{n\pi}{2}\right)e^{-\pi t}-\cos\left(\frac{\pi x}{2}-\frac{n\pi}{2}\right)e^{\pi t}$$

Proof. Setting $\lambda = e^{2\pi i \xi}$, the Fourier series (6) yields

$$\begin{aligned} T_n\left(x; e^{2\pi i\xi}\right) &= \frac{2^{n+1}\; n! i^{n+1}}{\left(e^{2\pi i\xi}\right)^{\frac{x}{2}}}\left[\sum_{k=0}^{\infty} \frac{e^{\frac{1}{2}(-(n+1)+(2k+1)x)\pi i}}{\left[(2k+1)\pi i - \log\left(e^{2\pi i\xi}\right)\right]^{n+1}} \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \frac{e^{\frac{1}{2}((n+1)-(2k+1)x)\pi i}}{\left[(2k+1)\pi i + \log\left(e^{2\pi i\xi}\right)\right]^{n+1}}\right] \\ &= \frac{2^{n+1}\; n! i^{n+1}}{e^{\pi i \xi x}}\left[\sum_{k=0}^{\infty} \frac{e^{\frac{1}{2}(-(n+1)+(2k+1)x)\pi i}}{\left[(2k-2\xi+1)\pi i\right]^{n+1}} + \sum_{k=0}^{\infty} \frac{e^{\frac{1}{2}((n+1)-(2k+1)x)\pi i}}{\left[(2k+2\xi+1)\pi i\right]^{n+1}}\right] \\ &= \frac{2^{n+1}}{e^{\pi i \xi x}\; \pi^{n+1}}\left[\sum_{k=0}^{\infty}\left(e^{\frac{1}{2}(-(n+1)+(2k+1)x)\pi i}\right)\left(\frac{n!}{(2k-2\xi+1)^{n+1}}\right) \right. \\ &\quad \left. + \sum_{k=0}^{\infty}\left(e^{\frac{1}{2}((n+1)-(2k+1)x)\pi i}\right)\left(\frac{n!}{(2k+2\xi+1)^{n+1}}\right)\right]. \end{aligned} \tag{9}$$

Applying the integral formula

$$\int_0^{\infty} t^n\, e^{-at}\, dt = \frac{n!}{a^{n+1}},\ for\ a > 0$$

(9) becomes

$$\begin{aligned} T_n\left(x; e^{2\pi i\xi}\right) &= \frac{2^{n+1}}{e^{\pi i \xi x}\; \pi^{n+1}}\left[\sum_{k=0}^{\infty} e^{\frac{1}{2}(-(n+1)+(2k+1)x)\pi i}\int_0^{\infty} t^n\, e^{-(2k-2\xi+1)t}\, dt \right. \\ &\quad \left. + \sum_{k=0}^{\infty} e^{\frac{1}{2}((n+1)-(2k+1)x)\pi i}\int_0^{\infty} t^n\, e^{-(2k+2\xi+1)t}\, dt\right] \\ &= \frac{2^{n+1}}{e^{\pi i \xi x}\; \pi^{n+1}}\left[\int_0^{\infty} t^n\, e^{-t}\, e^{2\xi t}\, e^{-\frac{(n+1)\pi i}{2}}\, e^{\frac{x\pi i}{2}} \sum_{k=0}^{\infty} e^{(\pi i x - 2t)k}\, dt \right. \\ &\quad \left. + \int_0^{\infty} t^n\, e^{-t}\, e^{-2\xi t}\, e^{\frac{(n+1)\pi i}{2}}\, e^{-\frac{x\pi i}{2}} \sum_{k=0}^{\infty} e^{(-\pi i x - 2t)k}\, dt\right] \end{aligned} \tag{10}$$

Note that

$$\sum_{k=0}^{\infty} e^{(\pi i x - 2t)k} = \frac{1}{1 - e^{\pi i x - 2t}} = \frac{1}{1 - \frac{e^{\pi i x}}{e^{2t}}} = \frac{1}{\frac{e^{2t} - e^{\pi i x}}{e^{2t}}} = \frac{e^{2t}}{e^{2t} - e^{\pi i x}} \tag{11}$$

and

$$\sum_{k=0}^{\infty} e^{(-\pi i x - 2t)k} = \frac{1}{1 - e^{-\pi i x - 2t}} = \frac{1}{1 - \frac{e^{-\pi i x}}{e^{2t}}} = \frac{1}{\frac{e^{2t} - e^{-\pi i x}}{e^{2t}}} = \frac{e^{2t}}{e^{2t} - e^{-\pi i x}} \tag{12}$$

Applying (11) and (12) to (10) yields

$$\begin{aligned} T_n\left(x; e^{2\pi i\xi}\right) &= \frac{2^{n+1}}{e^{\pi i \xi x}\; \pi^{n+1}}\left[\int_0^{\infty} t^n\, e^{-t}\, e^{2\xi t}\, e^{-\frac{(n+1)\pi i}{2}}\, e^{\frac{x\pi i}{2}}\left(\frac{e^{2t}}{e^{2t} - e^{\pi i x}}\right) dt \right. \\ &\quad \left. + \int_0^{\infty} t^n\, e^{-t}\, e^{-2\xi t}\, e^{\frac{(n+1)\pi i}{2}}\, e^{-\frac{x\pi i}{2}}\left(\frac{e^{2t}}{e^{2t} - e^{-\pi i x}}\right) dt\right] \\ &= \frac{2^{n+1}}{e^{\pi i \xi x}\; \pi^{n+1}}\left[\int_0^{\infty} \frac{e^{-\frac{(n+1)\pi i}{2}}\, e^{\frac{x\pi i}{2}}}{e^{2t} - e^{\pi i x}}\, e^{(2\xi+1)t}\, t^n\, dt \right. \\ &\quad \left. + \int_0^{\infty} \frac{e^{\frac{(n+1)\pi i}{2}}\, e^{-\frac{x\pi i}{2}}}{e^{2t} - e^{-\pi i x}}\, e^{(1-2\xi)t}\, t^n\, dt\right] \end{aligned} \tag{13}$$

Now,

$$\frac{e^{\frac{\pi i x}{2}}}{e^{2t} - e^{\pi i x}} = \frac{\frac{1}{2}\, e^{\frac{-\pi i x}{2}}\, \left(e^{\pi i x} - e^{-2t}\right)}{cosh(2t) - cos(\pi x)} \tag{14}$$

Similarly,

$$\frac{e^{\frac{-\pi i x}{2}}}{e^{2t} - e^{-\pi i x}} = \frac{\frac{1}{2}\, e^{\frac{\pi i x}{2}}\, \left(e^{-\pi i x} - e^{-2t}\right)}{cosh(2t) - cos(\pi x)} \tag{15}$$

Applying (14) and (15) to (13) yields

$$T_n\left(x;e^{2\pi i\xi}\right)=\frac{2^n}{e^{\pi i\xi x}\pi^{n+1}}\left[\int_0^\infty \frac{e^{\frac{-(n+1)\pi i}{2}}e^{\frac{-\pi i x}{2}}\left(e^{\pi i x}-e^{-2t}\right)e^{(2\xi+1)t}}{cosh(2t)-cos(\pi x)}t^n dt\right.$$
$$\left.+\int_0^\infty \frac{e^{\frac{(n+1)\pi i}{2}}e^{\frac{\pi i x}{2}}\left(e^{-\pi i x}-e^{-2t}\right)e^{(1-2\xi)t}}{cosh(2t)-cos(\pi x)}t^n dt\right] \tag{16}$$

Using the transformation $t=\pi t$, (16) becomes

$$T_n\left(x;e^{2\pi i\xi}\right)=\frac{2^n}{e^{\pi i\xi x}}\left[\int_0^\infty \frac{e^{\frac{-(n+1)\pi i}{2}}e^{\frac{-\pi i x}{2}}\left(e^{\pi i x}-e^{-2\pi t}\right)e^{(2\xi+1)\pi t}}{cosh(2\pi t)-cos(\pi x)}t^n dt\right.$$
$$\left.+\int_0^\infty \frac{e^{\frac{(n+1)\pi i}{2}}e^{\frac{\pi i x}{2}}\left(e^{-\pi i x}-e^{-2\pi t}\right)e^{(1-2\xi)\pi t}}{cosh(2\pi t)-cos(\pi x)}t^n dt\right] \tag{17}$$

Now,

$$\begin{aligned}&e^{\frac{-(n+1)\pi i}{2}}e^{\frac{-\pi i x}{2}}\left(-e^{-2\pi t}\right)e^{(2\xi+1)\pi t}\\&=-i\left[\cos\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)+i\sin\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)\right]e^{2\xi\pi t}e^{\pi t}\\&+i\left[\cos\left(\tfrac{x\pi}{2}+\tfrac{n\pi}{2}\right)-i\sin\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)\right]e^{2\xi\pi t}e^{-\pi t}\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}&e^{\frac{(n+1)\pi i}{2}}e^{\frac{\pi i x}{2}}\left(e^{-\pi i x}-e^{-2\pi t}\right)e^{(1-2\xi)\pi t}\\&=i\left[\cos\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)-i\sin\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)\right]e^{-2\xi\pi t}e^{\pi t}\\&-i\left[\cos\left(\tfrac{x\pi}{2}+\tfrac{n\pi}{2}\right)+i\sin\left(\tfrac{x\pi}{2}+\tfrac{n\pi}{2}\right)\right]e^{-2\xi\pi t}e^{-\pi t}\end{aligned} \tag{19}$$

Combining (18) and (19) yields

$$\begin{aligned}&e^{\frac{-(n+1)\pi i}{2}}e^{\frac{-\pi i x}{2}}\left(e^{\pi i x}-e^{-2\pi t}\right)e^{(2\xi+1)\pi t}+e^{\frac{(n+1)\pi i}{2}}e^{\frac{\pi i x}{2}}\left(e^{-\pi i x}-e^{-2\pi t}\right)e^{(1-2\xi)\pi t}\\&=2\sinh(2\xi\pi t)\left[\cos\left(\tfrac{x\pi}{2}+\tfrac{n\pi}{2}\right)e^{-\pi t}-\cos\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)e^{\pi t}\right]i\\&+2\cosh(2\xi\pi t)\left[\sin\left(\tfrac{x\pi}{2}+\tfrac{n\pi}{2}\right)e^{-\pi t}+\sin\left(\tfrac{x\pi}{2}-\tfrac{n\pi}{2}\right)e^{\pi t}\right]\end{aligned} \tag{20}$$

Applying (20) to (17) gives

$$T_n\left(x;e^{2\pi i\xi}\right)=\frac{2^n}{e^{\pi i\xi x}}\left[\int_0^\infty \frac{2\sinh(2\xi\pi t)\left[\cos\left(\frac{x\pi}{2}+\frac{n\pi}{2}\right)e^{-\pi t}-\cos\left(\frac{x\pi}{2}-\frac{n\pi}{2}\right)e^{\pi t}\right]i}{\cosh(2\pi t)-\cos(\pi x)}\right.$$
$$\left.+\frac{2\cosh(2\xi\pi t)\left[\sin\left(\frac{x\pi}{2}+\frac{n\pi}{2}\right)e^{-\pi t}+\sin\left(\frac{x\pi}{2}-\frac{n\pi}{2}\right)e^{\pi t}\right]}{\cosh(2\pi t)-\cos(\pi x)}t^n dt\right]$$

which is exactly the integral representation in (8). □

4. Explicit Formula for the Apostol–Tangent Polynomials at Rational Arguments

To obtain the explicit formula for the Apostol–tangent polynomials at rational arguments, the Fourier expansion derived above will be used.

Recall the Hurwitz–Lerch zeta function [25], which is defined by

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{z^k}{(k+a)^s}\ , \tag{21}$$

for $\left(a\in\mathbb{C}\backslash\mathbb{Z}_0^-\ ;s\in\mathbb{C}\right)$ when $|z|\ 1<;\ \Re(s)>1$ when $|z|=1$, which contains as its special case:

$$\zeta(s,a);=\Phi(1,\ s,a)=\sum_{k=0}^{\infty}\frac{1}{(k+a)^s}$$

Theorem 3. *For n, q, p ∈ N; $0 < \frac{2p}{q} \leq 1$; ξ∈ R, the following formula of Apostol–tangent polynomials at rational arguments holds,*

$$T_n\left(\frac{2p}{q}; e^{2\pi i\xi}\right) = \frac{n!}{(\pi q)^{n+1}}\left[\sum_{j=1}^{q} \zeta\left(n+1, \frac{2j-2\xi-1}{2q}\right) e^{\left[-\frac{(n+1)}{2} + \frac{(2j-2\xi-1)p}{q}\right]\pi i} + \sum_{j=1}^{q} \zeta\left(n+1, \frac{2j+2\xi-1}{2q}\right) e^{\left[\frac{(n+1)}{2} - \frac{(2j+2\xi-1)p}{q}\right]\pi i}\right]$$

Proof. By replacing k with $k-1$ in (6), we have

$$T_n(x;\lambda) = \frac{2^{n+1}\, n!\, i^{n+1}}{\lambda^{\frac{x}{2}}}\left[\sum_{k=1}^{\infty} \frac{e^{\frac{1}{2}(-(n+1)+(2k-1)x)\pi i}}{[(2k-1)\pi i - \log\lambda]^{n+1}} + \sum_{k=1}^{\infty} \frac{e^{\frac{1}{2}((n+1)-(2k-1)x)\pi i}}{[(2k-1)\pi i + \log\lambda]^{n+1}}\right].$$

By applying the elementary series identity

$$\sum_{k=1}^{\infty} f(k) = \sum_{j=1}^{q}\sum_{k=0}^{\infty} f(qk+j),\ q \in \mathrm{N},$$

used by Luo in his papers ([13,21]), where $f : N \to C$ is a sequence of complex numbers, we obtain

$$\begin{aligned}
T_n(x;\lambda) &= \frac{(2i)^{n+1}\ n!}{\lambda^{\frac{x}{2}}}\left[\sum_{j=1}^{q}\sum_{k=0}^{\infty} \frac{e^{\frac{1}{2}(-(n+1)+(2qk+2j-1)x)\pi i}}{[(2qk+2j-1)\pi i-\log\lambda]^{n+1}} + \sum_{j=1}^{q}\sum_{K=0}^{\infty} \frac{e^{\frac{1}{2}((n+1)-(2qk+2j-1)x)\pi i}}{[(2qk+2j-1)\pi i+\log\lambda]^{n+1}}\right] \\
&= \frac{(2i)^{n+1} n!}{\lambda^{\frac{x}{2}}}\left[\sum_{j=1}^{q}\sum_{k=0}^{\infty} \frac{e^{qk\pi ix} e^{\frac{1}{2}(-(n+1)+(2j-1)x)\pi i}}{[(2qk+2j-1)\pi i-\log\lambda]^{n+1}} + \sum_{j=1}^{q}\sum_{k=0}^{\infty} \frac{e^{-qk\pi ix} e^{\frac{1}{2}((n+1)-(2j-1)x)\pi i}}{[(2qk+2j-1)\pi i+\log\lambda]^{n+1}}\right] \cdot \frac{\frac{1}{(2q\pi i)^{n+1}}}{\frac{1}{(2q\pi i)^{n+1}}} \\
&= \frac{n!}{\lambda^{\frac{x}{2}}(\pi q)^{n+1}}\left\{\sum_{j=1}^{q}\sum_{k=0}^{\infty} e^{qk\pi ix} \frac{e^{\frac{1}{2}\ (-(n+1)+(2j-1)x)\pi i}}{\left[k+\frac{(2j-1)\pi i-\log\lambda}{2q\pi i}\right]^{n+1}} + \sum_{j=1}^{q}\sum_{k=0}^{\infty} e^{-qk\pi ix} \frac{e^{\frac{1}{2}\ ((n+1)-(2j-1)x)\pi i}}{\left[k+\frac{(2j-1)\pi i+\log\lambda}{2q\pi i}\right]^{n+1}}\right\}
\end{aligned} \quad (22)$$

Using the Hurwitz–Lerch zeta function (21) becomes

$$T_n(x,\ \lambda) = \frac{n!}{\lambda^{\frac{x}{2}}(\pi q)^{n+1}}\left[\sum_{j=1}^{q} \Phi\left(e^{q\pi ix}, n+1,\ \frac{(2j-1)\pi i-\log\lambda}{2q\pi i}\right) \times e^{\frac{1}{2}(-(n+1)+(2j-1)x)\pi i} + \sum_{j=1}^{q} \Phi\left(e^{-q\pi ix}, n+1,\ \frac{(2j-1)\pi i+\log\lambda}{2q\pi i}\right) \times e^{\frac{1}{2}((n+1)-(2j-1)x)\pi i}\right] \quad (23)$$

Setting $\lambda = e^{2\pi i\xi}$ and $= \frac{2p}{q}$, (23) becomes

$$\begin{aligned}
&T_n\left(\frac{2p}{q};\ e^{2\pi i\xi}\right) \\
&= \frac{n!}{e^{\frac{2\pi i\xi p}{q}}(\pi q)^{n+1}}\left[\sum_{j=1}^{q} \Phi\left(e^{2\pi pi},\ n+1, \frac{2j-2\xi-1}{2q}\right) \times e^{\left[-\frac{(n+1)}{2} + \frac{(2j-1)p}{q}\right]}\pi i + \sum_{j=1}^{q} \Phi\left(e^{-2\pi pi},\ n+1, \frac{2j+2\xi-1}{2q}\right) \times e^{\left[\frac{(n+1)}{2} - \frac{(2j-1)p}{q}\right]}\pi i\right]
\end{aligned} \quad (24)$$

Since $e^{-2\pi pi} = e^{2\pi pi} = 1$, then by Hurwitz zeta function when $z = 1$, (24) becomes,

$$T_n\left(\frac{2p}{q}, e^{2\pi i\xi}\right) = \frac{n!}{(\pi q)^{n+1}}\left[\sum_{j=1}^{q}\zeta\left(n+1, \frac{2j-2\xi-1}{2q}\right) \times e^{\frac{1}{2}(-(n+1)+(2j-1)(\frac{2p}{q}))\pi i} e^{-\frac{2\pi i\xi p}{q}} + \sum_{j=1}^{q}\zeta\left(n+1, \frac{2j+2\xi-1}{2q}\right) \times e^{\frac{1}{2}((n+1)-(2j-1)\frac{2p}{q})\pi i} e^{-\frac{2\pi i\xi p}{q}}\right]$$

$$= \frac{n!}{(\pi q)^{n+1}}\left[\sum_{j=1}^{q}\zeta\left(n+1, \frac{2j-2\xi-1}{2q}\right) \times e^{-\frac{(n+1)}{2}\pi i+(2j-2\xi-1)\frac{p}{q}\pi i} + \sum_{j=1}^{q}\zeta\left(n+1, \frac{2j+2\xi-1}{2q}\right) \times e^{(\frac{n+1}{2})\pi i-(2j+2\xi-1)\frac{p}{q}\pi}\right].$$

□

5. Conclusions

The researchers obtained three formulas for the Apostol–tangent polynomials: the Fourier series, an integral representation, and an explicit formula at rational arguments. Taking into account all the residues of the generating function combined with the use of the Cauchy Residue Theorem proved to be a good technique to obtain the Fourier series while the method by Luo proved to be applicable with no major difficulty to obtain the latter two formulas. For future study, it will be interesting to obtain corresponding formulas for the generalized Apostol type Frobenius–Euler polynomials.

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References

1. Caratelli, D.; Ricci, P.E. Inversion of Tridiagonal Matrices Using the Dunford-Taylor's Integral. *Symmetry* **2021**, *13*, 870. [CrossRef]
2. Kucukoglu, I.; Simsek, B.; Simsek, Y. Generating Function for New Families of Combinatorial Numbers and Polynomials: Approach to Poisson-Charlier Polynomials and Probability Distribution Function. *Axioms* **2019**, *8*, 112. [CrossRef]
3. Juraev, D.A.; Noeiaghdam, S. Regularizationof the III-Posed Cauchy Problem for Matrix Factorization of the Helmholtz Equation on the Plane. *Axioms* **2021**, *10*, 82. [CrossRef]
4. Karachik, V. Dirichlet and Neumann Boundary Value Problems for the Polyharmonic Equation in the Unit Ball. *Mathematics* **2021**, *9*, 1907. [CrossRef]
5. Ryoo, C.S. A numerical investigation on the zeros of the Tangent polynomials. *J. Appl. Math. Inform.* **2014**, *32*, 315–322. [CrossRef]
6. Ryoo, C.S. Differential equations associated with Tangent numbers. *J. Appl. Math. Inform.* **2016**, *34*, 487–494. [CrossRef]
7. Ryoo, C.S. On the Twisted q-Tangent Numbers and Polynomials. *Appl. Math. Sci.* **2013**, *7*, 4935–4941. [CrossRef]
8. Ryoo, C.S. Explicit Identities for the Generalized Tangent Polynomials. *Nonlinear Anal. Differ. Equ.* **2018**, *6*, 43–51. [CrossRef]
9. Ryoo, C.S. On the analogues of Tangent numbers and polynomials associated with p-adic integral on Z_p. *Appl. Math. Sci.* **2013**, *7*, 3177–3183. [CrossRef]
10. Ryoo, C.S. A note on the symmetric properties for the Tangent polynomials. *Int. J. Math. Anal.* **2013**, *7*, 2575–2581. [CrossRef]

11. Apostol, T.M. On the Lerch zeta function. *Pac. J. Math.* **1951**, *1*, 161–167. [CrossRef]
12. Luo, Q.M. Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions. *Taiwan. J. Math.* **2006**, *10*, 917–925. [CrossRef]
13. Luo, Q.M. Extensions of the Genocchi Polynomials and their Fourier expansions and integral representations. *Osaka J. Math.* **2011**, *48*, 291–309.
14. Luo, Q.M.; Srivastava, H.M. Some relationships between the Apostol–Bernoulli and Apostol–Euler polynomials. *Comput. Math. Appl.* **2006**, *51*, 631–642. [CrossRef]
15. Luo, Q.M.; Srivastava, H.M. Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials. *J. Math. Anal. Appl.* **2005**, *308*, 290–302. [CrossRef]
16. Srivastava, H.M. Some formulas for the Bernoulli and Euler polynomials at rational arguments. *Math. Proc. Camb. Philos. Soc.* **2000**, *129*, 77–84. [CrossRef]
17. He, Y.; Araci, S.; Srivastava, H.M.; Abdel-Aty, M. Higher-order convolutions for Apostol–Bernoulli, Apostol–Euler and Apostol-Genocchi polynomials. *Mathematics* **2019**, *6*, 329. [CrossRef]
18. Bayad, A. Fourier expansions for Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials. *Math. Comp.* **2011**, *80*, 2219–2221. [CrossRef]
19. Hollingsworth, M. Applications of the Fourier Series; Semantic Scholar 2008. Available online: https://www.semanticscholar.org/paper/Applications-of-the-Fourier-Series-Hollingsworth/37f51fb07cd215c6db8c673971f2698ac5cff0fa (accessed on 25 November 2021).
20. Walker, J. Fourier Series. In *Encyclopedia of Physical Science and Technology*, 3rd ed.; Meyers, R., Ed.; Academic Press: Cambridge, UK, 2003; pp. 167–183.
21. Luo, Q.M. Fourier expansions and integral representations for Genocchi polynomials. *J. Integer Seq.* **2009**, *12*, 1–9.
22. Araci, S.; Acikgoz, M. Construction of Fourier expansion of Apostol Frobenius—Euler polynomials and its application. *Adv. Differ. Equ.* **2018**, *2018*, 1–14. [CrossRef]
23. Corcino, C.B.; Corcino, R.B. Asymptotics of Genocchi polynomials and higher order Genocchi polynomials using residues. *Afr. Math.* **2020**, *31*, 781–792. [CrossRef]
24. Corcino, C.B.; Damgo, B.; Corcino, R.B. Fourier expansions for Genocchi polynomials of higher order. *J. Math. Comput. Sci.* **2020**, *22*, 59–72. [CrossRef]
25. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company: New York, NY, USA; Toronto, ON, Canada; London, UK, 1953; Volume I.