symmetry

Article

Best Dominants and Subordinants for Certain Sandwich-Type Theorems

Adriana Cătaş [*,†], Emilia Borşa [†] and Loredana Iambor [†]

Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania; eborsa@uoradea.ro (E.B.); lgalea@uoradea.ro (L.I.)
* Correspondence: acatas@uoradea.ro
† These authors contributed equally to this work.

Abstract: In this paper, we aim to present a survey on subordination and superordination theorems related to the class of analytic functions defined in a symmetric domain, which is the open unit disc. The results were deduced by making use of a new differential operator. We present two properties of this operator from which we constructed the final results. Moreover, based on the obtained outcomes, we give two sandwich-type theorems. Some interesting further consequences are also taken into consideration.

Keywords: best dominant; best subordinant; differential subordination; superordination; differential operator

check for updates

Citation: Cătaş, A.; Borşa, E.; Iambor, L. Best Dominants and Subordinants for Certain Sandwich-Type Theorems. *Symmetry* **2022**, *14*, 62. https://doi.org/10.3390/sym14010062

Academic Editor: Ioan Raşa

Received: 2 December 2021
Accepted: 14 December 2021
Published: 3 January 2022

1. Introduction and Definitions

Let us denote by $\mathcal{H}$ the set of analytic functions defined in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Consider also $\mathcal{H}[a,n]$ a subset of $\mathcal{H}$ with the following form of functions

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots .$$

Let $\mathcal{A}(p,n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

which are analytic in the open unit disc. In particular, we set $\mathcal{A}(p,1) := \mathcal{A}_p$ and $\mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1$. Let $\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots\}$ with $\mathcal{A}_1 := \mathcal{A}$.

Let f and g be two analytic functions in U. We recall here the well-known principle of subordination. We say that the analytic function f is subordinate to g, if there exists a Schwarz function w in U such that $f(z) = g(w(z))$, $z \in U$. We will denote this subordination relation by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad (z \in U).$$

The subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U),$$

if g is univalent in U. If f is subordinate to F, then F is called to be superordinate to f.

In the well-known paper [1], Miller and Mocanu studied second order differential superordinations. Let $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If we consider two univalent functions p and $\phi(p(z), zp'(z), z^2p''(z); z)$ and if the function p verifies the second order differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1}$$

then we say that p is a solution of the second order superordination from (1).

For all functions q verifying the subordination (1), we say that q is a subordinant if $q \prec p$. If $\widetilde{q}$, a univalent function, verifies $q \prec \widetilde{q}$ for all differential subordinants q of the relation (1), then we say that $\widetilde{q}$ is the best subordinant. The above mentioned paper [1] obtained special conditions on the functions h, q, and ϕ that satisfy the following implication:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \;\Rightarrow\; q(z) \prec p(z).$$

Based on the results from [1], in the papers [2] and [3], Bulboacă obtained several classes of first-order differential superordinations and superordination-preserving integral operators, respectively. Using the results derived by Bulboacă in [2], Ali et al. in paper [4], considered sufficient conditions for certain normalized analytic functions f that verify

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z).$$

In the above double subordination the functions q_1 and q_2 are univalent in the symmetric domain U such that $q_1(0) = 1$ and $q_2(0) = 1$.

Referring to the paper [5], we notice that Shanmugam et al. deduced sufficient conditions for f, which is a normalized analytic function to satisfy the following double subordinations

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = 1$ and $q_2(0) = 1$.

For a certain form of functions, namely, $\left(\frac{f(z)}{z}\right)^{\mu}$, in the work [6], Obradović and Owa derived several subordination results. Regarding the convex functions of complex order and starlike functions of complex order, and they were recently studied by Srivastava and Lashin [7] using Briot–Bouquet differential subordination techniques. There are many results concerning the theory of differential subordination and superordination techniques involving differential operators and integral operators as we can mention here [8]. For special function see [9].

Definition 1. *[1] Let Q represent the class of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$, where*

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$, where ∂U is the boundary of the unit disc.

Theorem 1. *[10] Consider q as an univalent function in the open unit disc U and θ and ϕ as analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)), \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

(1) *$Q(z)$ is starlike univalent in Δ and*

(2) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for $z \in U$.*

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then

$$p(z) \prec q(z),$$

and q is the best dominant.

Theorem 2. *[2] Let the function q be univalent in the unit disc U and v and φ be analytic in a domain D containing $q(U)$. Suppose that*
1. $\operatorname{Re}\left\{\frac{v'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$ and*
2. $Q(z) = zq'(z)\varphi(q(z))$ *is starlike univalent in U.*

If the function $p \in \mathcal{H}[q(0), 1] \cap Q$ such that $p(U) \subseteq D$, and $v(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U and

$$v(q(z)) + zq'(z)\varphi(q(z)) \prec v(p(z)) + zp'(z)\varphi(p(z)) \tag{2}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

The theory of subordinations and superordinations has recently become a broad area of study. It is related to the theory of inequalities, operators, and other important branches of mathematics and their applications. In this direction, many inequalities and differential operators have been studied to obtain specific symmetry properties. These operators appears in various problems related to differential subordinations. We survey certain outcomes concerning the best dominants and best subordinants for certain sandwich-type theorems. Through the obtained consequences, we settled several type of functions q that have symmetry properties and are convex univalent functions.

2. Main Results

Definition 2. *Let $f \in \mathcal{A}$. For $m, \beta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $l \geq 0$, we propose a differential operator $I^{m,\beta}(\lambda, l)$ on $\mathcal{A}$ with the following form*

$$I^{m,\beta}(\lambda, l)f(z) := z + \sum_{k=2}^{\infty}\left[\frac{1 + \lambda(k-1) + kl}{1+l}\right]^m C(\beta, k)a_k z^k$$

where

$$C(\beta, k) := \binom{k+\beta-1}{\beta} = \frac{(\beta+1)_{k-1}}{(k-1)!}$$

and

$$(a)_n := \begin{cases} 1, & n = 0 \\ a(a+1)\ldots(a+n-1), & n \in \mathbb{N} = \mathbb{N}_0 - \{0\} \end{cases}$$

is a Pochhamer symbol.

Remark 1. *We reobtain several operators obtained earlier by various researchers. Recall here the Ruscheweyh derivative operator $I^{0,\beta}(\lambda, 0) \equiv D_\beta$ defined in [11], the Sălăgean derivative operator $I^{m,0}(1, 0) \equiv D^m$, studied in [12], the generalized Sălăgean operator $I^{m,0}(\lambda, 0) \equiv D^m_\lambda$ defined by Al-Oboudi in [13], the generalized Ruscheweyh operator $I^{1,\beta}(\lambda, 0) \equiv D_{\lambda,\beta}$ introduced in [14], the operator $I^{m,\beta}(\lambda, 0) \equiv D^m_{\lambda,\beta}$ defined by K. Al-Shaqsi and M. Darus in [15], and, for $\beta = 0$, a similar operator introduced in [16]. The operator $I^{m,0}(\lambda, 1-\lambda) \equiv I^m_\lambda$ (for $p = 1$) was studied by Cho and Srivastava [17] and Cho and Kim [18].*

By making use of a simple computation technique, one obtains the following result.

Proposition 1. *Consider $m, \beta \in \mathbb{N}_0$, $\lambda \geq 0$, $l \geq 0$*

$$(1+l)I^{m+1,\beta}(\lambda, l)f(z) = (1-\lambda)I^{m,\beta}(\lambda, l)f(z) + (\lambda + l)z(I^{m,\beta}(\lambda, l)f(z))'. \tag{3}$$

and regarding parameter β

$$z\left(I^{m,\beta}(\lambda, l)f(z)\right)' = (1+\beta)I^{m,\beta+1}(\lambda, l)f(z) - \beta I^{m,\beta}(\lambda, l)f(z). \tag{4}$$

In the present paper, we deduce sufficient conditions for normalized analytic functions f, which satisfy the next double differential subordination

$$q_1(z) \prec \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec q_2(z),$$

where $m, \beta \in \mathbb{N}_0$, $\lambda \geq 0$, $\mu, \eta \in \mathbb{C}$, $\eta \neq 0$ and q_1, q_2 are given univalent functions in U.

For a certain operator $I^{m,\beta}(\lambda,l)$, various interesting outcomes concerning differential subordination and differential superordination relations were obtained. For the first step, we will prove the following subordination result involving the operator $I^{m,\beta}(\lambda,l)$.

Theorem 3. *Consider the number $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$ and q be a univalent function in the open unit disc U with $q(z) \neq 0$.*

Assume that $\dfrac{zq'(z)}{q(z)}$ is a starlike univalent function in U. Let

$$\operatorname{Re}\left\{\frac{b}{\xi}q(z) + \frac{2c}{\xi}(q(z))^2\right\} > 0 \tag{5}$$

and

$$\begin{aligned}\psi^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f) := a + b\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{\mu} \cdot \left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{\eta} + \\ +c\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{2\mu} \cdot \left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{2\eta} + \\ +\frac{\xi(l+1)}{\lambda+l} \cdot \left[\mu\left(\frac{I^{m+2,\beta}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)} - 1\right) + \eta\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)} - 1\right)\right].\end{aligned} \tag{6}$$

If q satisfies the following subordination

$$\psi^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f) \prec a + bq(z) + c(q(z))^2 + \xi\frac{zq'(z)}{q(z)} \tag{7}$$

then,

$$\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec q(z),\ z \in U,\ z \neq 0, \eta \in \mathbb{C},\ \eta \neq 0, \tag{8}$$

and q is the best dominant.

Proof. Let us define the function $p(z)$ by

$$p(z) := \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}, \quad z \in \mathbb{C},\ z \neq 0,\ f \in \mathcal{A}. \tag{9}$$

By a straightforward computation, one obtains

$$\frac{zp'(z)}{p(z)} = \mu\left(\frac{z[I^{m+1,\beta}(\lambda,l)f(z)]'}{I^{m+1,\beta}(\lambda,l)f(z)} - 1\right) + \eta\left(\frac{z[I^{m,\beta}(\lambda,l)f(z)]'}{I^{m,\beta}(\lambda,l)f(z)} - 1\right).$$

Using the identity

$$(l+1)I^{m+2,\beta}(\lambda,l)f(z) = (1-\lambda)I^{m+1,\beta}(\lambda,l)f(z) + (l+\lambda)z(I^{m+1,\beta}(\lambda,l)f(z))',$$

we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\mu(l+1)}{l+\lambda}\left(\frac{I^{m+2,\beta}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)}-1\right)+ \frac{\eta(l+1)}{l+\lambda}\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)}-1\right). \tag{10}$$

By substituting the above equality into (7), we deduce

$$a+bp(z)+c(p(z))^2+\xi\frac{zp'(z)}{p(z)} \prec a+bq(z)+c(q(z))^2+\xi\frac{zq'(z)}{q(z)}.$$

Letting

$$\theta(w) := a+bw+cw^2 \text{ and } \phi(w) := \frac{\xi}{w}$$

it can be easily observed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\setminus\{0\}$, and $\phi(w)\neq 0$, $w\in\mathbb{C}\setminus\{0\}$. By setting

$$Q(z) = zq'(z)\phi(q(z)) = \xi\frac{zq'(z)}{q(z)}$$

and

$$h(z) := \theta(q(z))+Q(z) = a+bq(z)+c(q(z))^2+\xi\frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}+\frac{b}{\xi}q(z)+\frac{2c}{\xi}(q(z))^2\right\},$$

$$(a,b,c,\xi\in\mathbb{C},\xi\neq 0).$$

Knowing by hypothesis that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U, which is

$$\operatorname{Re}\left\{1-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right\}>0,$$

we deduce that $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}>0$.

The assertion (8) of Theorem 3 follows by an application of Theorem 1. □

For the choices $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ and $q(z)=\left(\frac{1+z}{1-z}\right)^{\delta}$, $0\le\delta\le 1$ in Theorem 3, one obtains the following two corollaries.

Corollary 1. *Consider the numbers $a,b,c,\xi,\mu,\eta\in\mathbb{C}$, $\eta\neq 0$, $\xi\neq 0$, $-1\le B<A\le 1$ and*

$$\operatorname{Re}\left\{\frac{b}{\xi}\frac{1+Az}{1+Bz}+\frac{2c}{\xi}\left(\frac{1+Az}{1+Bz}\right)^2\right\}>0. \tag{11}$$

If $f\in\mathcal{A}$, then the following differential subordination

$$\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \prec a+b\frac{1+Az}{1+Bz}+c\left(\frac{1+Az}{1+Bz}\right)^2+\xi\frac{(A-B)z}{(1+Az)(1+Bz)} \tag{12}$$

implies

$$\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu}\cdot\left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec \frac{1+Az}{1+Bz}, \tag{13}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ *is defined in (6), and* $\dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2. *Consider* $a,b,c,\xi,\mu,\eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $0 \leq \delta \leq 1$ *and*

$$\operatorname{Re}\left\{\frac{b}{\xi}\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2c}{\xi}\left(\frac{1+z}{1-z}\right)^{2\delta}\right\} > 0. \tag{14}$$

If $f \in \mathcal{A}$, *then the following differential subordination*

$$\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \prec a + b\left(\frac{1+z}{1-z}\right)^{\delta} + c\left(\frac{1+z}{1-z}\right)^{2\delta} + \frac{2\xi\delta z}{1-z^2} \tag{15}$$

implies

$$\left(\frac{I^{m+1}(\lambda,\beta,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m}(\lambda,\beta,l)f(z)}{z}\right)^{\eta} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \tag{16}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ *is defined in (6), and* $\left(\dfrac{1+z}{1-z}\right)^{\delta}$ *is the best dominant.*

We consider the special case $q(z) = e^{\upsilon Az}$, such that $|\upsilon A| < \pi$, and then Theorem 3 easily produces the following corollary.

Corollary 3. *Consider the numbers* $a,b,c,\xi,\mu,\eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $|\upsilon A| < \pi$ *and*

$$\operatorname{Re}\left\{\frac{b}{\xi}e^{\upsilon Az} + \frac{2c}{\xi}e^{2\upsilon Az}\right\} > 0. \tag{17}$$

If $f \in \mathcal{A}$, *then differential subordination*

$$\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \prec a + be^{\upsilon Az} + ce^{2\upsilon Az} + \xi\upsilon Az \tag{18}$$

implies

$$\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec e^{\upsilon Az}, \tag{19}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ *is defined in (6) and* $e^{\upsilon Az}$ *is the best dominant.*

For $q(z) = (1+Bz)^{\frac{\delta(A-B)}{B}}$, $-1 \leq B < A < 1$, $B \neq 0$, we deduce the next known result.

Corollary 4. *Let* $a,b,c,\xi,\mu,\eta,\delta \in \mathbb{C}$, $\eta \neq 0$, $\delta \neq 0$, $\xi \neq 0$, $-1 \leq B < A < 1$, $B \neq 0$ *and*

$$\operatorname{Re}\left\{\frac{b}{\xi}(1+Bz)^{\frac{\delta(A-B)}{B}} + \frac{2c}{\xi}(1+Bz)^{\frac{2\delta(A-B)}{B}}\right\} > 0. \tag{20}$$

If $f \in \mathcal{A}$, *then differential subordination*

$$\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \prec a + b(1+Bz)^{\frac{\delta(A-B)}{B}} + c(1+Bz)^{\frac{2\delta(A-B)}{B}} + \xi\frac{z\delta(A-B)}{1+Bz} \tag{21}$$

implies

$$\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec (1+Bz)^{\frac{\delta(A-B)}{B}}, \tag{22}$$

where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ *is defined in (6), and* $(1+Bz)^{\frac{\delta(A-B)}{B}}$ *is the best dominant.*

We remark that $q(z) = (1+Bz)^{\frac{\delta(A-B)}{B}}$ is univalent if and only if either

$$\left|\frac{\delta(A-B)}{B} - 1\right| \leq 1 \text{ or } \left|\frac{\delta(A-B)}{B} + 1\right| \leq 1.$$

Regarding parameter β, we derive the next result.

Theorem 4. *Consider q as a univalent function in the unit disc U such that $q(z) \neq 0$ and $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$.*

Suppose that $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in U, and the inequality (5) holds. Let the function

$$\begin{aligned} F_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) = \xi(\beta+1)\left[\mu\left(\frac{I^{m+1,\beta+1}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)} - 1\right) + \eta\left(\frac{I^{m,\beta+1}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)} - 1\right)\right] \\ +a + b\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{\mu} \cdot \left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{\eta} + \\ +c\left[\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right]^{2\mu} \cdot \left[\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right]^{2\eta}. \end{aligned} \tag{23}$$

If q verifies the following subordination

$$F_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;z) \prec a + bq(z) + c(q(z))^2 + \xi\frac{zq'(z)}{q(z)}, \tag{24}$$

then

$$\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec q(z),\ z \in U,\ z \neq 0, \eta \in \mathbb{C},\ \eta \neq 0, \tag{25}$$

and q is the best dominant.

Proof. Consider the function p as defined as in (9). Using the identity

$$z\Big(I^{m+1,\beta}(\lambda,l)f(z)\Big)' = (1+\beta)I^{m+1,\beta+1}(\lambda,l)f(z) - \beta I^{m+1,\beta}(\lambda,l)f(z),$$

we obtain

$$\begin{aligned} \frac{zp'(z)}{p(z)} = \mu(\beta+1)\left[\frac{I^{m+1,\beta+1}(\lambda,l)f(z)}{I^{m+1,\beta}(\lambda,l)f(z)} - 1\right] + \\ + \eta(\beta+1)\left[\frac{I^{m,\beta+1}(\lambda,l)f(z)}{I^{m,\beta}(\lambda,l)f(z)} - 1\right]. \end{aligned} \tag{26}$$

By substituting the last equality into (24), we find

$$a + bp(z) + c(p(z))^2 + \xi\frac{zp'(z)}{p(z)} \prec a + bq(z) + c(q(z))^2 + \xi\frac{zq'(z)}{q(z)}.$$

Letting

$$\theta(w) := a + bw + cw^2 \text{ and } \phi(w) := \frac{\xi}{w}$$

one can be observed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C} \setminus \{0\}$, and $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Considering

$$Q(z) = zq'(z)\phi(q(z)) = \xi\frac{zq'(z)}{q(z)}$$

and
$$h(z) := \theta(q(z)) + Q(z) = a + bq(z) + c(q(z))^2 + \xi \frac{zq'(z)}{q(z)},$$
we deduce that $Q(z)$ is starlike univalent in U and
$$\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{\frac{b}{\xi}q(z) + \frac{2c}{\xi}(q(z))^2\right\} > 0,$$
$$(a, b, c, \xi \in \mathbb{C}, \xi \neq 0).$$

Applying Theorem 1, the assertion (25) of Theorem 4 is obtained. □

Remark 2. *We remark here that Theorem 4 can be easily reformulated for various choices of the functions q (as in Corollaries 1–4).*

We shall prove Theorem 5 below by appealing to Theorem 2 of the previous section.

Theorem 5. *Consider $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$ and let q be a convex univalent function in U with $q(z) \neq 0, q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Moreover, let us presume that*
$$\text{Re}\left\{\frac{b}{\xi}q(z) + \frac{2c}{\xi}(q(z))^2\right\} > 0, \quad z \in U. \tag{27}$$

If $f \in \mathcal{A}$,
$$0 \neq \left(\frac{I^{m+1,\beta}(\lambda, l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda, l)f(z)}{z}\right)^{\eta} \in \mathcal{H}[q(0), 1] \cap Q$$
and $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a, b, c, \xi; f)$ is univalent in U, then
$$a + bq(z) + c(q(z))^2 + \xi \frac{zq'(z)}{q(z)} \prec \psi_{\mu,\eta}^{m,\lambda,\beta,l}(a, b, c, \xi; f)$$
implies
$$q(z) \prec \left(\frac{I^{m+1,\beta}(\lambda, l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda, l)f(z)}{z}\right)^{\eta}, \quad z \in U,\ z \neq 0 \tag{28}$$
and q is the best subordinant where $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a, b, c, \xi; f)$ is given in (6).

Proof. Consider the function p be in the form
$$p(z) := \left(\frac{I^{m+1,\beta}(\lambda, l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda, l)f(z)}{z}\right)^{\eta}, \quad z \in U,\ z \neq 0,\ f \in \mathcal{A}.$$

A straightforward computation yields
$$a + bq(z) + c(q(z))^2 + \xi \frac{zq'(z)}{q(z)} \prec a + bp(z) + c(p(z))^2 + \xi \frac{zp'(z)}{p(z)}.$$

Setting
$$v(w) := a + bw + cw^2 \text{ and } \varphi(w) := \frac{\xi}{w}$$
it is easily to observe that $v(w)$ is analytic in $\mathbb{C}$. In addition, φ is analytic in $\mathbb{C} \setminus \{0\}$ and $\varphi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$.

As q is a convex (univalent) function, we deduce that

$$\operatorname{Re}\left\{\frac{\upsilon'(q(z))}{\varphi(q(z))}\right\} = \operatorname{Re}\left\{\frac{\beta}{\xi}q(z) + \frac{2c}{\xi}(q(z))^2\right\} > 0.$$

By an application of Theorem 2, the assertion (28) of Theorem 5 is obtained. □

Remark 3. *We remark that Theorem 5 can be easily reformulated for various choices of the function $q(z)$ (as in Corollaries 1–4).*

Appealing to a similar method used in the proof of Theorem 5, we find the proof of the next superordination result regarding parameter β.

Theorem 6. *Consider $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$ and let q be convex univalent in U with $q(z) \neq 0, q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Presume that $\operatorname{Re}\left\{\frac{\beta}{\xi}q(z) + \frac{2c}{\xi}(q(z))^2\right\} > 0, \quad z \in U$.*

If $f \in \mathcal{A}$, $0 \neq \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \in \mathcal{H}[q(0),1] \cap Q$ and $F^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f)$ is univalent in U, then

$$a + bq(z) + c(q(z))^2 + \xi\frac{zq'(z)}{q(z)} \prec F^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f)$$

implies

$$q(z) \prec \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta}, \quad z \in U,\ z \neq 0 \tag{29}$$

and q is the best subordinant, where $F^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f)$ is given in (23).

Combining Theorem 3 and Theorem 5, we deduce the next sandwich-type theorem.

Theorem 7. *Consider $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$ and let q_i be convex univalent functions in U such that $q_i(z) \neq 0, q_i(0) = 1$ for $i \in \{1,2\}$. Suppose that $\frac{zq_i'(z)}{q_i(z)}$ is starlike univalent in U for $i = 1,2$ and q_1, q_2 satisfy (5). If $f \in \mathcal{A}$, $\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \in \mathcal{H}[q_i(0),1] \cap Q$ and $\psi^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f)$ is univalent in U, then*

$$\alpha + \beta q_1(z) + \gamma(q_1(z))^2 + \xi\frac{zq_1'(z)}{q_1(z)} \prec \psi^{m,\lambda,\beta,l}_{\mu,\eta}(a,b,c,\xi;f) \tag{30}$$

$$\prec \alpha + \beta q_2(z) + \gamma(q_2(z))^2 + \xi\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec q_2(z), \quad z \in U,\ z \neq 0 \tag{31}$$

and q_1, q_2 are the best subordinant and the best dominant, respectively.

We deduce a similar result from Theorems 4 and 6.

Theorem 8. *Consider $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$ and let q_i be convex univalent functions in U such that $q_i(z) \neq 0, q_i(0) = 1$ for $i \in \{1,2\}$. Suppose that $\frac{zq_i'(z)}{q_i(z)}$ is starlike univalent in U for $i = 1,2$ and q_1, q_2 satisfy (5). If $f \in \mathcal{A}$, $\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \in \mathcal{H}[q_i(0),1] \cap Q$ and $F_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is univalent in U, then*

$$\alpha + \beta q_1(z) + \gamma(q_1(z))^2 + \xi\frac{zq_1'(z)}{q_1(z)} \prec F_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \tag{32}$$

$$\prec \alpha + \beta q_2(z) + \gamma(q_2(z))^2 + \xi\frac{zq_2'(z)}{q_2(z)}$$

implies

$$q_1(z) \prec \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec q_2(z), \quad z \in U,\ z \neq 0 \tag{33}$$

and q_1, q_2 are the best subordinant and the best dominant, respectively.

Corollary 5. *Consider $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$. Suppose that $-1 \leq B_i < A_i < 1$. If $\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \in \mathcal{H}[1,1] \cap Q$ and $\psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is univalent in U, then*

$$a + b\frac{1+A_1z}{1+B_1z} + c\left(\frac{1+A_1z}{1+B_1z}\right)^2 + \xi\frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)} \prec \psi_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \tag{34}$$

$$\prec a + b\frac{1+A_2z}{1+B_2z} + c\left(\frac{1+A_2z}{1+B_2z}\right)^2 + \xi\frac{(A_2-B_2)z}{(1+A_2z)(1+B2z)}$$

implies

$$\frac{1+A_1z}{1+B_1z} \prec \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec \frac{1+A_2z}{1+B_2z}, \quad z \in U,\ z \neq 0, \tag{35}$$

and $\frac{1+A_1z}{1+B_1z}, \frac{1+A_2z}{1+B_2z}$ are the best subordinant and the best dominant, respectively.

Corollary 6. *Let $a, b, c, \xi, \mu, \eta \in \mathbb{C}$, $\eta \neq 0$, $\xi \neq 0$, $\lambda > 0$. Suppose that $-1 \leq B_i < A_i < 1, B_i \neq 0$, $\delta_i \in \mathbb{C}$, $\delta_i \neq 0$, for $i = 1,2$. If $\left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \in \mathcal{H}[1,1] \cap Q$ and $F_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f)$ is univalent in U, then*

$$a + b(1+B_1z)^{\frac{\delta_1(A_1-B_1)}{B_1}} + c(1+B_1z)^{\frac{2\delta_1(A_1-B_1)}{B_1}} + \xi\frac{z\delta_1(A_1-B_1)}{1+B_1z} \prec F_{\mu,\eta}^{m,\lambda,\beta,l}(a,b,c,\xi;f) \tag{36}$$

$$\prec a + b(1+B_2z)^{\frac{\delta_2(A_2-B_2)}{B_2}} + c(1+B_2z)^{\frac{2\delta_2(A_2-B_2)}{B_2}} + \xi\frac{z\delta_2(A_2-B_2)}{1+B_2z}$$

implies

$$(1+B_1z)^{\frac{\delta_1(A_1-B_1)}{B_1}} \prec \left(\frac{I^{m+1,\beta}(\lambda,l)f(z)}{z}\right)^{\mu} \cdot \left(\frac{I^{m,\beta}(\lambda,l)f(z)}{z}\right)^{\eta} \prec (1+B_2z)^{\frac{\delta_2(A_2-B_2)}{B_2}} \tag{37}$$

and $(1+B_1z)^{\frac{\delta_1(A_1-B_1)}{B_1}}$,$(1+B_2z)^{\frac{\delta_2(A_2-B_2)}{B_2}}$ are the best subordinant and the best dominant, respectively.

3. Discussion

In the present paper, we proposed a new form of a differential operator $I^{m,\beta}(\lambda,l)$, which generalizes several operators introduced earlier by many other researchers. Relevant connections of the proposed operator with other differential operators are considered. By making use of this operator, we derived the new results. First, we review some of the basic results on the theory of subordination and supeordination results.

Using the method of admissible function, we deduced certain differential subordination results associated with two properties of the newly introduced operator. After that, using the dual notion of subordination, namely that of superordination, we established the corresponding results in terms of superordination. In addition, using specific well-known univalent functions, we derived interesting corollaries that provide the best dominants and the best superordinats.

Finally, future research could address the results of this study. The proposed methodology constructed in the obtained corollaries could inspire other papers in finding several particular function as examples. These new results provide a theoretical basis for further studies. Therefore, many interesting outcomes can be derived using the differential subordination and superordination theory.

Author Contributions: Conceptualization, A.C., E.B. and L.I.; methodology, A.C.; software, A.C.; validation, A.C., E.B. and L.I.; formal analysis, A.C., E.B. and L.I.; investigation, A.C.; resources, A.C.; data curation, A.C.; writing—original draft preparation, A.C.; writing—review and editing, A.C., E.B. and L.I.; visualization, A.C., E.B. and L.I.; supervision, A.C.; project administration, A.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Miller, S.S.; Mocanu, P.T. Subordinants of differential superordinations. *Complex Var. Elliptic Equ.* **2003**, *48*, 815–826. [CrossRef]
2. Bulboacă, T. Classes of first-order superordinations. *Demonstr. Math.* **2002**, *35*, 287–292.
3. Bulboacă, T. A class of superordination–preserving integral operators. *Indag. Math. New Ser.* **2002**, *13*, 301–311. [CrossRef]
4. Ali, R.M.; Ravichandran, V.; Hussain, M.; Subramanian, K.G. Differential sandwich theorems for certain analytic functions. *Far East J. Math. Sci.* **2005**, *15*, 87–94.
5. Shanmugam, T.N.; Ravichandran, V.; Sivasubramanian, S. Differential Sandwich Theorems for some Subclasses of Analytic Functions. *Aust. J. Math. Anal. Appl.* **2006**, *3*, 11.
6. Obradovic, M.; Owa, S. On certain properties for some classes of starlike functions. *J. Math. Anal. Appl.* **1990**, *145*, 357–364. [CrossRef]
7. Srivastava, H.M.; Lashin, A.Y. Some applications of the Briot-Bouquet differential subordination. *J. Inequal. Pure Appl. Math.* **2005**, *6*, 41.
8. Cotîrlă, L.I.; Cătaş, A. Differential sandwich theorem for certain class of analytic functions associated with an integral operator. *Stud. Univ. Babes-Bolyai Math.* **2020**, *65*, 487–494. [CrossRef]
9. Cotîrlă, L.I. New classes of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 10642–10651. [CrossRef]
10. Miller, S.S.; Mocanu, P.T. Differential Subordinations: Theory and Applications. *Pure Appl. Math.* **2000**, *225*.
11. Ruscheweyh, S. New criteria for univalent functions. *Proc. Amer. Math. Soc.* **1975**, *49*, 109–115. [CrossRef]
12. Şt. Sălăgean, G. Subclasses of univalent functions. *Lecture Note Math.* **1983**, *1013*, 362–372.
13. Al-Oboudi, F.M. On univalent functions defined by a generalized Sălăgean operator. *Int. J. Math. Math. Sci.* **2004**, *27*, 1429–1436. [CrossRef]
14. Darus, M.; Al-Shaqsi, K. Differential sandwich theorems with generalized derivative operator. *Int. J. Comput. Math. Sci.* **2008**, *2*, 75–78.
15. Al-Shaqsi, K.; Darus, M. On univalent functions with respect to k-symmetric points defined by a generalized Ruscheweyh derivatives operator. *J. Anal. Appl. 2009*, **7**, 53–61.
16. Cătaş, A. On certain class of p-valent functions defined by a new multiplier transformations. In *Proceedings Book of the International Symposium G.F.T.A.*; Istanbul Kultur University: Istanbul, Turkey, 2007; pp. 241–250.

17. Cho, N.E.; Srivastava, H.M. Argument estimates of certain analytic functions defined by a class of multiplier transformations. *Math. Comput. Model.* **2003**, *37*, 39–49. [CrossRef]
18. Cho, N.E.; Kim, T.H. Multiplier transformations and strongly close-to-convex functions. *Bull. Korean Math. Soc.* **2003**, *40*, 399–410. [CrossRef]