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Spatial Decay Bounds for the Brinkman Fluid Equations in Double-Diffusive Convection

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Abstract: In this paper, we consider the Brinkman equations pipe flow, which includes the salinity and the temperature. Assuming that the fluid satisfies nonlinear boundary conditions at the finite end of the cylinder, using the symmetry of differential inequalities and the energy analysis methods, we establish the exponential decay estimates for homogeneous Brinkman equations. That is to prove that the solutions of the equation decay exponentially with the distance from the finite end of the cylinder. To make the estimate of decay explicit, the bound for the total energy is also derived.

Keywords: spatial decay estimates; Brinkman equations; Saint-Venant principle

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1. Introduction

The Brinkman equations are one of the most important models in fluid mechanics. This model are mainly used to describe flow in a porous medium. For more details, one can refer to Nield and Bejan [1] and Straughan [2]. In the present paper, we define the Brinkman flow depending on the salinity and the temperature in a semi-infinite cylindrical pipe and derive the spatial decay properties. When the homogeneous initial-boundary conditions are applied on the lateral surface of the cylinder, We prove that the solutions of Brinkman equations decays exponentially with spatial variable.

In fact, the Brinkman equations have been studied by many papers in the literature. For example, Straughan [2] considered the mathematical properties of Brinkman equations as well as Darcy and Forchheimer equations, and stated how these equations describe the flow of porous media. Ames and Payne [3] studied the structural stability for the solutions to the viscoelasticity in an ill-posed problem. Franchi and Straughan [4] proved the structural stability for the solutions to the Brinkman equations in porous media in a bounded region. More relevant results one can see [5–10]. Paper [11] studied the double diffusive convection in porous medium and obtained the structural stability for the solutions. The continuous dependence for a thermal convection model with temperature-dependent solubility can be found in [12]. For more recent work about continuous dependence, one may refer to [13–19].

In this paper, let R be a semi-infinite cylinder and ∂R represents the boundary of R. D denotes the cross section of the cylinder with the smooth boundary ∂D (see Figure 1).

In this paper, we also use the following notations

$$R_z = \left\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D, \quad x_3 > z \geq 0\right\},$$

$$D_z = \left\{(x_1, x_2, x_3)\Big|(x_1, x_2) \in D, \quad x_3 = z \geq 0\right\},$$

where z is a point along the x_3 axis. Clearly, $R_0 = R$ and $D_0 = D$. Letting u_i, T, C and p denote the fluid velocity, temperature, salt concentration and pressure, respectively.

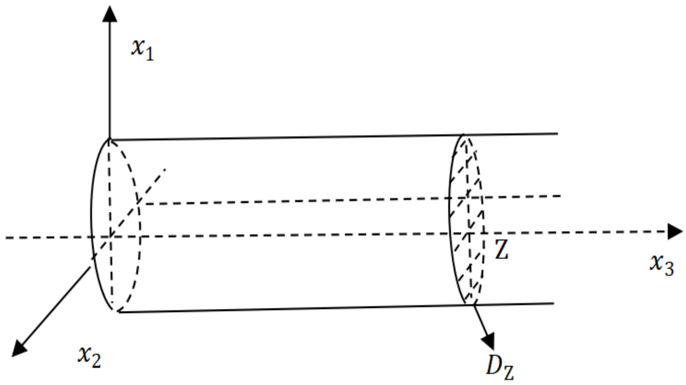

Figure 1. Cylindrical pipe.

The Brinkman equations we study can be written as [20]

$$\frac{\partial u_i}{\partial t} = \nu\Delta u_i - k_1 u_i - p_{,i} + g_i T + h_i C, \ \ in\ R \times \{t \geq 0\}, \tag{1}$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = k_2 \Delta T, \qquad in\ R \times \{t \geq 0\}, \tag{2}$$

$$\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = k_3 \Delta C + \sigma \Delta T, \qquad in\ R \times \{t \geq 0\}, \tag{3}$$

$$u_{i,i} = 0, \qquad in\ R \times \{t \geq 0\}, \tag{4}$$

where ν, $\sigma > 0$ denote the Brinkman coefficient, and the Soret coefficient, respectively. $k_1, k_2, k_3 > 0$. Without losing generality, we take them equal to 1. Δ is the Laplacian operator. $g_i(x)$ and $h_i(x)$ are gravity field, which are given functions. We suppose that (1)–(4) have the following initial-boundary conditions

$$u_i = 0, \quad T = C = 0, \qquad on\ \ \partial D \times \{t \geq 0\}, \tag{5}$$

$$u_i = 0, \quad T = C = 0, \qquad on\ \ R \times \{t = 0\}. \tag{6}$$

$$u_i = f_i(x_1, x_2, t),\ T = F(x_1, x_2, t),\ C = G(x_1, x_2, t),\ \ on\ D_0 \times \{t \geq 0\}, \tag{7}$$

$$u_i, u_{i,j}, u_{i,t}, T, T_{,i}, C, C_{,i}, p = o(x_3^{-1})\ \ uniformly\ in\ x_1, x_2, t, \quad as\ x_3 \to \infty. \tag{8}$$

In (1)–(8) and in the following, the usual summation convention is employed with repeated Latin subscripts summed from 1 to 3 and repeat Greek subscript summed from 1 to 2. The comma is used to indicate partial differentiation, i.e., $u_{i,j}u_{i,j} = \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j}\right)^2$, $\varphi_{\alpha,\beta}\varphi_{\alpha,\beta} = \sum_{\alpha,\beta=1}^{2} \left(\frac{\partial \varphi_\alpha}{\partial x_\beta}\right)^2$.

The purpose of this paper is to consider the spatial decay properties of the Equations (1)–(8) in a semi-infinite cylindrical pipe by using the symmetry of differential inequalities, that is, to prove that the solutions of the equations decay exponentially with the distance from the finite end of the cylindrical pipe.

In Section 2, some auxiliary inequalities are presented. We establish some useful lemmas in Section 3. The spatial exponential decay estimate for the solution is established in Section 4. Finally, in Section 5 we derive the bounds for the total energies.

2. Auxiliary Results

In this paper, we will use some inequalities in the following sections. Thus, we firstly list them as follows.

Lemma 1. *Let D be a plane domain D with the smooth boundary ∂D. If $w = 0$ on ∂D, then*

$$\int_D w_{,\alpha} w_{,\alpha} dA \geq \lambda_1 \int_{R_z} w^2 dx, \tag{9}$$

where λ_1 is the smallest eigenvalue of the problem

$$\Delta\phi + \lambda\phi = 0 \quad in \ \ D,$$

$$\phi = 0 \quad on \ \ \partial D.$$

Many papers have studied this inequality, e.g., one may see [21,22].

A representation theorem will be also used in next sections. We write this theorem as

Lemma 2. *Let D be a plane Lipschitz bound region and w be a differential function in D which satisfies $\int_D w dA = 0$, then there exists a vector function $\varphi_\alpha(x_1, x_2)$ such that*

$$\varphi_{\alpha,\alpha} = w \quad in \quad D,$$

$$\varphi_\alpha = 0 \quad on \quad \partial D,$$

and a positive constant Λ depending only on the geometry of D such that

$$\int_D \varphi_{\alpha,\beta}\varphi_{\alpha,\beta} dA \leq \Lambda \int_D \varphi^2_{\alpha,\alpha} dA. \tag{10}$$

The Lemma 2 was proofed by Babuška and Aziz [23] and Horgan and Wheeler [24] have used the Lemma 1 to viscous flow problems. The explicit upper bound of Λ can be found in Horgan and Payne [25]. In this paper, this Lemma 2 is used to eliminate the pressure function difference terms p, since we can prove that u_3 satisfy the hypothesis of this Lemma 2 later.

If $w \in C^1_0(D)$ and $w \in C^1_0(R)$, the following Sobolev inequalities hold

$$\int_D w^4 dA \leq \frac{1}{2}\Big[\int_D w^2 dA\Big]\Big[\int_D w_{,\alpha} w_{,\alpha} dA\Big], \tag{11}$$

$$\int_{R_z} w^6 dx \leq \Omega\Big[\int_{R_z} w_{,i} w_{,i} dx\Big]^3. \tag{12}$$

For (11), we assume that $w \to 0$ as $x_3 \to \infty$. Payne [26] has given the derivation of (12). For a special case of the results one can see [27,28]. They have obtained the optimal value of Ω

$$\Omega = \frac{1}{27}\Big(\frac{3}{4}\Big)^4.$$

In the following, we also use the following lemma.

Lemma 3. *If $w \in C^1(R_z)$, $w_i\Big|_{\partial D} = 0$ and $w_i \to 0$ as $x_3 \to \infty$, then*

$$\int_{D_z} \Big(w_i w_i\Big)^2 dA \leq 4\sqrt{\Omega}\Big[\int_{R_z} w_{i,j} w_{i,j} dx\Big]^2. \tag{13}$$

We will also use the following lemmas which were derived in [29].

Lemma 4. *Let that the function φ is the solution of the problem*

$$\begin{aligned} &\Delta\varphi = 0 \quad in \;\; R_z, \\ &\frac{\partial\varphi}{\partial n} = 0 \quad on \;\; \partial D_z, \\ &\frac{\partial\varphi}{\partial n} = g \quad in \;\; D_z, \end{aligned} \tag{14}$$

where $\int_{D_z} g dA = 0$*. Then*

$$\int_{D_z} \varphi_{,\alpha}\varphi_{,\alpha} dA = \int_{D_z} g^2 dA, \tag{15}$$

$$\int_{R_z} \varphi_{,i}\varphi_{,i} dA = \frac{1}{\sqrt{\mu}} \int_{D_z} g^2 dA, \tag{16}$$

3. Some Useful Lemmas

In this section, we derive some useful lemmas which will be used in next section. First, we define a weighted energy expression

$$\begin{aligned} E(z,t) &= k\int_0^t \int_{R_z} (\xi - z) u_{i,\eta} u_{i,\eta} dx d\eta + \nu \int_0^t \int_{R_z} (\xi - z) u_{i,j} u_{i,j} dx d\eta \\ &+ \rho_1 \int_0^t \int_{R_z} (\xi - z) T_{,j} T_{,j} dx d\eta + \rho_2 \int_0^t \int_{R_z} (\xi - z) C_{,j} C_{,j} dx d\eta \\ &= E_1(z,t) + E_2(z,t) + E_3(z,t) + E_4(z,t), \end{aligned} \tag{17}$$

where k, ρ_1, ρ_2 are positive parameters and $\xi > z > 0$.

By using the divergence theorem and Equations (1) and (4), we obtain

$$\begin{aligned} E_1(z,t) &= k\int_0^t \int_{R_z} (\xi - z) u_{i,\eta} \Big[\nu\Delta u_i - u_i - p_{,i} + g_i T + h_i C\Big] dx d\eta \\ &= k\nu \int_0^t \int_{R_z} u_{i,\eta} u_{i,3} dx d\eta + k\int_0^t \int_{R_z} u_{3,\eta} p dx d\eta \\ &+ k\int_0^t \int_{R_z} (\xi - z) u_{i,\eta} g_i T dx d\eta + k\int_0^t \int_{R_z} (\xi - z) u_{i,\eta} h_i C dx d\eta \\ &- k\nu \int_{R_z} (\xi - z) u_{i,j} u_{i,j} dx\Big|_{\eta=t} - k\int_{R_z} (\xi - z) u_i u_i dx\Big|_{\eta=t} \\ &\doteq \sum_{i=1}^{4} A_i \\ &- \frac{1}{2}k\nu \int_{R_z} (\xi - z) u_{i,j} u_{i,j} dx\Big|_{\eta=t} - \frac{1}{2}k\int_{R_z} (\xi - z) u_i u_i dx\Big|_{\eta=t}. \end{aligned} \tag{18}$$

Using the Schwarz inequality, the arithmetic geometric mean inequality and (9), we can obtain

$$\begin{aligned} A_1 &\le k\nu \Big[\int_0^t \int_{R_z} u_{i,\eta} u_{i,\eta} dx d\eta \int_0^t \int_{R_z} u_{i,3} u_{i,3} dx d\eta\Big]^{\frac{1}{2}} \\ &\le \frac{\sqrt{k\nu}}{2}\Big[k\int_0^t \int_{R_z} u_{i,\eta} u_{i,\eta} dx d\eta + \nu\int_0^t \int_{R_z} u_{i,3} u_{i,3} dx d\eta\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} A_3 &\le \frac{k}{\sqrt{\lambda_1}}\Big[\int_0^t \int_{R_z} (\xi - z) u_{i,\eta} u_{i,\eta} dx d\eta \int_0^t \int_{R_z} (\xi - z) T_{,\alpha} T_{,\alpha} dx d\eta\Big]^{\frac{1}{2}} \\ &\le \frac{\varepsilon_1}{2} k\int_0^t \int_{R_z} (\xi - z) u_{i,\eta} u_{i,\eta} dx d\eta + \frac{k\delta_1^2}{2\lambda_1\varepsilon_1}\int_0^t \int_{R_z} (\xi - z) T_{,\alpha} T_{,\alpha} dx d\eta, \end{aligned} \tag{20}$$

and

$$A_4 \leq \frac{\varepsilon_2}{2} k \int_0^t \int_{R_z} (\xi - z) u_{i,\eta} u_{i,\eta} dx d\eta + \frac{k\delta_2^2}{2\lambda_1 \varepsilon_2} \int_0^t \int_{R_z} (\xi - z) C_{,\alpha} C_{,\alpha} dx d\eta, \tag{21}$$

where $\varepsilon_1, \varepsilon_2 > 0$ will be determined later and

$$\delta_1^2 = \max_D (g_i g_i), \quad \delta_2^2 = \max_D (h_i h_i), \tag{22}$$

We note that for any $z^* > 0$, using (4) and (5),

$$\begin{aligned}
\int_{D_z} u_{3,\eta} dA &= \int_{D_{z^*}} u_{3,\eta} dA - \int_z^{z^*} \int_{D_\xi} u_{3,3\eta} dA d\xi \\
&= \int_{D_{z^*}} u_{3,\eta} dA + \int_z^{z^*} \int_{D_\xi} u_{\alpha,\alpha\eta} dA d\xi \\
&= \int_{D_{z^*}} u_{3,\eta} dA.
\end{aligned}$$

Since

$$\int_{D_0} f_{3,\eta} dA = 0, \quad t \geq 0, \tag{23}$$

then,

$$\int_{D_z} u_{3,\eta} dA = 0.$$

Under this assumption, using Lemma 2, there exist vector functions (φ_1, φ_2) such that

$$\varphi_{\alpha,\alpha} = u_{3,\eta} \quad in \;\; D, \;\; \varphi_\alpha = 0 \quad on \;\; \partial D. \tag{24}$$

Hence we have

$$\begin{aligned}
A_2 &= k \int_0^t \int_{R_z} \varphi_{\alpha,\alpha} p dx d\eta = -k \int_0^t \int_{R_z} \varphi_\alpha p_{,\alpha} dx d\eta \\
&= k \int_0^t \int_{R_z} \varphi_\alpha \Big[u_{\alpha,\eta} - \nu \Delta u_\alpha + u_\alpha - g_\alpha T - h_\alpha C \Big] dx d\eta \\
&= k \int_0^t \int_{R_z} \varphi_\alpha u_{\alpha,\eta} dx d\eta + k\nu \int_0^t \int_{R_z} \varphi_{\alpha,\beta} u_{\alpha,\beta} dx d\eta \\
&+ k\nu \int_0^t \int_{D_z} \varphi_\alpha u_{\alpha,3} dx d\eta + k \int_0^t \int_{R_z} \varphi_\alpha u_\alpha dx d\eta \\
&- k \int_0^t \int_{R_z} \varphi_\alpha g_\alpha T dx d\eta - k \int_0^t \int_{R_z} \varphi_\alpha h_\alpha C dx d\eta \\
&= A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26}.
\end{aligned} \tag{25}$$

Using the Schwarz, Poincaré and the AG mean inequalities, (9) and (10), we can obtain

$$
\begin{aligned}
A_{21} &\leq \Big(\int_0^t\int_{R_z}\varphi_\alpha\varphi_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\eta}u_{\alpha,\eta}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{1}{\sqrt{\lambda_1}}\Big(\int_0^t\int_{R_z}\varphi_{\alpha,\beta}\varphi_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\eta}u_{\alpha,\eta}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{\Lambda^{\frac{1}{2}}}{\sqrt{\lambda_1}}\Big(\int_0^t\int_{R_z}u_3^2d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\eta}u_{\alpha,\eta}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}\int_0^t\int_{R_z}u_{i,\eta}u_{i,\eta}dxd\eta, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
A_{22} &\leq k\nu\Big(\int_0^t\int_{R_z}\varphi_{\alpha,\beta}\varphi_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq k\nu\Lambda^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\nu\Lambda^{\frac{1}{2}}}{2}\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta+\frac{k\nu\Lambda^{\frac{1}{2}}}{2}\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
A_{23} &\leq k\nu\Big(\int_0^t\int_{D_z}\varphi_\alpha\varphi_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{D_z}u_{\alpha,3}u_{\alpha,3}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\nu\Lambda^{\frac{1}{2}}}{\sqrt{\lambda_1}}\Big(\int_0^t\int_{D_z}u_{3,\eta}^2dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{D_z}u_{\alpha,3}u_{\alpha,3}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}\int_0^t\int_{D_z}u_{3,\eta}^2dxd\eta+\frac{k\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}\int_0^t\int_{D_z}u_{\alpha,3}u_{\alpha,3}dxd\eta, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
A_{24} &\leq k\Big(\int_0^t\int_{R_z}\varphi_\alpha\varphi_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_\alpha u_\alpha dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\Lambda^{\frac{1}{2}}}{\lambda_1}\Big(\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta+\frac{k\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
A_{25} &\leq k\Big(\int_0^t\int_{R_z}\varphi_\alpha\varphi_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}g_\alpha g_\alpha T^2dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\delta_1\Lambda^{\frac{1}{2}}}{\lambda_1}\Big(\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}T_{,i}T_{,i}dxd\eta\Big)^{\frac{1}{2}}\\
&\leq \frac{k\delta_1\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta+\frac{k\delta_1\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}T_{,i}T_{,i}dxd\eta, \qquad (30)
\end{aligned}
$$

$$
A_{26} \leq \frac{k\delta_2\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}u_{3,\eta}^2dxd\eta+\frac{k\delta_2\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta. \qquad (31)
$$

Inserting (26)–(31) into (25), then (19)–(21) and (25) into (18), and choosing $\varepsilon_1=\varepsilon_2=\frac{1}{2}$, we obtain the following lemma.

Lemma 5. *Let u, T, C, p be solutions of Equations (1)–(8) with $g, h \in L_\infty(R \times \{t > 0\})$ and $\int_D f_3 dA = 0$. Then*

$$
\begin{aligned}
E_1(z,t) &+ k\nu \int_{R_z} (\xi - z) u_{i,j} u_{i,j} dx\Big|_{\eta=t} + k \int_{R_z} (\xi - z) u_i u_i dx\Big|_{\eta=t} \\
&\leq a_1 k \int_0^t \int_{R_z} u_{i,\eta} u_{i,\eta} dx d\eta + a_2 \nu \int_0^t \int_{R_z} u_{i,j} u_{i,j} dx d\eta \\
&+ \frac{k\delta_1 \Lambda^{\frac{1}{2}}}{\lambda_1} \int_0^t \int_{R_z} T_{,i} T_{,i} dx d\eta + \frac{k\delta_2 \Lambda^{\frac{1}{2}}}{\lambda_1} \int_0^t \int_{R_z} C_{,i} C_{,i} dx d\eta \\
&+ \frac{k\nu \Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}} \int_0^t \int_{D_z} u_{3,\eta}^2 dx d\eta + \frac{k\nu \Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}} \int_0^t \int_{D_z} u_{\alpha,3} u_{\alpha,3} dx d\eta \\
&+ \frac{2k\delta_1^2}{\lambda_1} \int_0^t \int_{R_z} (\xi - z) T_{,\alpha} T_{,\alpha} dx d\eta + \frac{2k\delta_2^2}{\lambda_1} \int_0^t \int_{R_z} (\xi - z) C_{,\alpha} C_{,\alpha} dx d\eta,
\end{aligned}
$$

where

$$
\begin{aligned}
a_1 &= \sqrt{k\nu} + \frac{\Lambda^{\frac{1}{2}}}{\sqrt{\lambda_1}} + \nu \Lambda^{\frac{1}{2}} + \frac{\Lambda^{\frac{1}{2}}}{\lambda_1} + \frac{2\delta_1 \Lambda^{\frac{1}{2}}}{\lambda_1} + \frac{2\delta_2 \Lambda^{\frac{1}{2}}}{\lambda_1}, \\
a_2 &= \frac{\sqrt{k\nu}}{2} + \frac{k\Lambda^{\frac{1}{2}}}{2} + \frac{k\Lambda^{\frac{1}{2}}}{2\nu\sqrt{\lambda_1}}.
\end{aligned}
$$

Similar to Lemma 5, for $E_2(z,t)$ we can obtain the following lemma.

Lemma 6. *Let u, T, C, p be solutions of Equations (1)–(8) with $g, h \in L_\infty(R \times \{t > 0\})$ and $\int_D f_3 dA = 0$. Then*

$$
\begin{aligned}
E_2(z,t) &+ \frac{1}{2} \int_0^t \int_{R_z} (\xi - z) u_i u_i dx d\eta + \frac{1}{2} \int_{R_z} (\xi - z) u_i u_i dx\Big|_{\eta=t} \\
&\leq a_3 \int_0^t \int_{R_z} u_{i,j} u_{i,j} dx d\eta + \frac{\nu}{2} \int_0^t \int_{R_z} u_i u_i dx d\eta \\
&+ \frac{\Lambda^{\frac{1}{2}}}{2\lambda_1} \int_0^t \int_{R_z} u_{i,\eta} u_{i,\eta} dx d\eta + \frac{\delta_1 \Lambda^{\frac{1}{2}}}{2\lambda_1} \int_0^t \int_{R_z} T_{,\alpha} T_{,\alpha} dx d\eta \\
&+ \frac{\delta_2 \Lambda^{\frac{1}{2}}}{2\lambda_1} \int_0^t \int_{R_z} C_{,\alpha} C_{,\alpha} dx d\eta \\
&+ \frac{\delta_1^2}{2\lambda_1 \varepsilon_3} \int_0^t \int_{R_z} (\xi - z) T_{,\alpha} T_{,\alpha} dx d\eta + \frac{\delta_2^2}{2\lambda_1 \varepsilon_2} \int_0^t \int_{R_z} (\xi - z) C_{,\alpha} C_{,\alpha} dx d\eta,
\end{aligned}
$$

where

$$
a_3 = \frac{\nu}{2} + \frac{\Lambda^{\frac{1}{2}}}{2\lambda_1} + \frac{\nu \Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}} + \frac{\nu \Lambda^{\frac{1}{2}}}{2\lambda_1} + \frac{k\Lambda^{\frac{1}{2}}}{2\lambda_1} + \frac{\delta_1 \Lambda^{\frac{1}{2}}}{2\lambda_1} + \frac{\delta_2 \Lambda^{\frac{1}{2}}}{2\lambda_1}.
$$

Proof. By the divergence theorem and Equations (1)–(8), we have

$$\begin{aligned}
E_2(z,t) = &-\int_0^t\int_{R_z}(\xi-z)u_iu_idxd\eta-\frac{1}{2}\int_{R_z}(\xi-z)u_iu_idx\Big|_{\eta=t}\\
&-\nu\int_0^t\int_{R_z}u_iu_{i,3}dxd\eta+\int_0^t\int_{R_z}(\xi-z)u_ig_iTdxd\eta\\
&+\int_0^t\int_{R_z}(\xi-z)u_ih_iCdxd\eta+\int_0^t\int_{R_z}u_3pdxd\eta\\
&\doteq-\int_0^t\int_{R_z}(\xi-z)u_iu_idxd\eta-\frac{1}{2}\int_{R_z}(\xi-z)u_iu_idx\Big|_{\eta=t}\\
&+\sum_{i=1}^4B_i.
\end{aligned}\tag{32}$$

Using the Schwarz inequality, the Poincaré inequality and the AG mean inequality, we can obtain

$$\begin{aligned}
B_1&\le\nu\Big[\int_0^t\int_{R_z}u_{i,3}u_{i,3}dxd\eta\int_0^t\int_{R_z}u_iu_idxd\eta\Big]^{\frac{1}{2}}\\
&\le\frac{\nu}{2}\Big[\int_0^t\int_{R_z}u_{i,3}u_{i,3}dxd\eta+\int_0^t\int_{R_z}u_iu_idxd\eta\Big].
\end{aligned}\tag{33}$$

Similar to (20) and (21), we have for B_2 and B_3

$$B_2\le\frac{\varepsilon_3}{2}\int_0^t\int_{R_z}(\xi-z)u_iu_idxd\eta+\frac{\delta_1^2}{2\lambda_1\varepsilon_3}\int_0^t\int_{R_z}(\xi-z)T_{,\alpha}T_{,\alpha}dxd\eta,\tag{34}$$

and

$$B_3\le\frac{\varepsilon_4}{2}\int_0^t\int_{R_z}(\xi-z)u_iu_idxd\eta+\frac{\delta_2^2}{2\lambda_1\varepsilon_2}\int_0^t\int_{R_z}(\xi-z)C_{,\alpha}C_{,\alpha}dxd\eta,\tag{35}$$

where $\varepsilon_3,\varepsilon_4$ are positive constants.

To bound B_4 in (32), we also require that

$$\int_D f_3dA=0.$$

Then using to the Lemma 2 in Section 2, there exist vector functions $(\widehat{\varphi}_1,\widehat{\varphi}_2)$ such that

$$\widehat{\varphi}_{\alpha,\alpha}=u_3,\quad in\ \ D,\ \ \widehat{\varphi}_\alpha=0,\quad on\ \partial D.\tag{36}$$

Therefore, we have

$$\begin{aligned}
B_4&=\int_0^t\int_{R_z}\widehat{\varphi}_{\alpha,\alpha}pdxd\eta\\
&=-\int_0^t\int_{R_z}\widehat{\varphi}_\alpha p_{,\alpha}dxd\eta\\
&=\int_0^t\int_{R_z}\widehat{\varphi}_\alpha\Big[u_{\alpha,\eta}-\nu\Delta u_\alpha+u_\alpha-g_\alpha T-h_\alpha C\Big]dxd\eta\\
&=\int_0^t\int_{R_z}\widehat{\varphi}_\alpha u_{\alpha,\eta}dxd\eta+\nu\int_0^t\int_{R_z}\widehat{\varphi}_{\alpha,\beta}u_{\alpha,\beta}dxd\eta+\nu\int_0^t\int_{D_z}\widehat{\varphi}_\alpha u_{\alpha,3}dAd\eta\\
&+\int_0^t\int_{R_z}\widehat{\varphi}_\alpha u_\alpha dxd\eta-\int_0^t\int_{R_z}g_\alpha T\widehat{\varphi}_\alpha dxd\eta-\int_0^t\int_{R_z}h_\alpha C\widehat{\varphi}_\alpha dxd\eta\\
&\doteq\sum_{i=1}^6B_{4i}.
\end{aligned}\tag{37}$$

As the derivation of (26)–(32), we conclude that

$$
\begin{aligned}
B_{41} &\le \Big(\int_0^t\int_{R_z}\widehat{\varphi}_\alpha\widehat{\varphi}_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\eta}u_{\alpha,\eta}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{1}{\sqrt{\lambda_1}}\Big(\int_0^t\int_{R_z}\widehat{\varphi}_{\alpha,\beta}\widehat{\varphi}_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\eta}u_{\alpha,\eta}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\Lambda^{\frac{1}{2}}}{\sqrt{\lambda_1}}\Big(\int_0^t\int_{R_z}u_3^2xd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\eta}u_{\alpha,\eta}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\Lambda^{\frac{1}{2}}}{2\lambda_1}\Big[\int_0^t\int_{R_z}u_{3,\alpha}u_{3,\alpha}xd\eta+\int_0^t\int_{R_z}u_{i,\eta}u_{i,\eta}dxd\eta\Big],
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
B_{42} &\le \nu\Big(\int_0^t\int_{R_z}\widehat{\varphi}_{\alpha,\beta}\widehat{\varphi}_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \nu\Lambda^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_3^2dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_{\alpha,\beta}u_{\alpha,\beta}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
B_{43} &\le \nu\Big(\int_0^t\int_{D_z}\widehat{\varphi}_\alpha\widehat{\varphi}_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{D_z}u_{\alpha,3}u_{\alpha,3}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\nu\Lambda^{\frac{1}{2}}}{\sqrt{\lambda_1}}\Big(\int_0^t\int_{D_z}u_3^2dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{D_z}u_{\alpha,3}u_{\alpha,3}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\nu\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{D_z}u_{i,j}u_{i,j}dxd\eta,
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
B_{44} &\le \Big(\int_0^t\int_{R_z}\widehat{\varphi}_\alpha\widehat{\varphi}_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}u_\alpha u_\alpha dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\Lambda^{\frac{1}{2}}}{\lambda_1}\int_0^t\int_{R_z}u_iu_idxd\eta\\
&\le \frac{k\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}u_{i,\alpha}u_{i,\alpha}dxd\eta,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
B_{45} &\le \Big(\int_0^t\int_{R_z}\widehat{\varphi}_\alpha\widehat{\varphi}_\alpha dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}g_\alpha g_\alpha T^2dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\delta_1\Lambda^{\frac{1}{2}}}{\lambda_1}\Big(\int_0^t\int_{R_z}u_{3,\alpha}u_{3,\alpha}dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}dxd\eta\Big)^{\frac{1}{2}}\\
&\le \frac{\delta_1\Lambda^{\frac{1}{2}}}{2\lambda_1}\Big[\int_0^t\int_{R_z}u_{3,\alpha}u_{3,\alpha}dxd\eta+\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}dxd\eta\Big],
\end{aligned}
\tag{42}
$$

$$
B_{46} \le \frac{\delta_2\Lambda^{\frac{1}{2}}}{2\lambda_1}\Big[\int_0^t\int_{R_z}u_{3,\alpha}u_{3,\alpha}dxd\eta+\int_0^t\int_{R_z}C_{,\alpha}C_{,\alpha}dxd\eta\Big].
\tag{43}
$$

Inserting (38)–(43) into (37), we obtain

$$
\begin{aligned}
B_4 \le{}& \Big[\frac{\Lambda^{\frac{1}{2}}}{2\lambda_1}+\frac{\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}+\frac{\nu\Lambda^{\frac{1}{2}}}{2\lambda_1}+\frac{k\Lambda^{\frac{1}{2}}}{2\lambda_1}+\frac{\delta_1\Lambda^{\frac{1}{2}}}{2\lambda_1}+\frac{\delta_2\Lambda^{\frac{1}{2}}}{2\lambda_1}\Big]\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta\\
&+\frac{\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}u_{i,\eta}u_{i,\eta}dxd\eta+\frac{\delta_1\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}T_{,\alpha}T_{,\alpha}dxd\eta\\
&+\frac{\delta_2\Lambda^{\frac{1}{2}}}{2\lambda_1}\int_0^t\int_{R_z}C_{,\alpha}C_{,\alpha}dxd\eta.
\end{aligned}
\tag{44}
$$

Inserting (33), (34), (35) and (44) into (32) and choosing $\varepsilon_3 = \varepsilon_4 = \frac{1}{2}$, we can obtain Lemma 5.

Next we may bound $E_3(z,t)$. First we let T_M denotes that the maximum of T by using the maximum principle in R, i.e.,

$$T_M = \max_{D\times\{t>0\}} F(x_1, x_2, t). \tag{45}$$

Integrating by parts, using (3), (5), (6), (7) together with (9) and the AG mean inequality, we have

$$\begin{aligned}
E_3(z,t) &= -\rho_1 \int_0^t \int_{R_z} TT_{,3}dxd\eta - \rho_1 \int_0^t \int_{R_z} T(T_{,\eta} + u_i T_{,i})dxd\eta \\
&= -\frac{\rho_1}{2} \int_{R_z} (\xi - z)T^2 dx\Big|_{\eta=t} + \frac{\rho_1}{2} \int_0^t \int_{D_z} T^2 dAd\eta + \frac{\rho_1}{2} \int_0^t \int_{R_z} u_3 T^2 dxd\eta \\
&\leq -\frac{\rho_1}{2} \int_{R_z} (\xi - z)T^2 dx\Big|_{\eta=t} + \frac{\rho_1}{2\lambda_1} \int_0^t \int_{D_z} T_{,\alpha} T_{,\alpha} dAd\eta \\
&+ \frac{\rho_1 T_M}{2} \Big(\int_0^t \int_{R_z} u_3^2 dxd\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_{R_z} T^2 dxd\eta \Big)^{\frac{1}{2}} \\
&\leq -\frac{\rho_1}{2} \int_{R_z} (\xi - z)T^2 dx\Big|_{\eta=t} + \frac{\rho_1}{2\lambda_1} \int_0^t \int_{D_z} T_{,\alpha} T_{,\alpha} dAd\eta \\
&+ \frac{\rho_1 T_M}{4\lambda_1} \Big(\int_0^t \int_{R_z} u_{3,\alpha} u_{3,\alpha} dxd\eta + \int_0^t \int_{R_z} T_{,i} T_{,i} dxd\eta \Big).
\end{aligned} \tag{46}$$

Using Equations (3)–(7) and integrating by parts, we obtain

$$\begin{aligned}
E_4(z,t) &= -\rho_2 \int_0^t \int_{R_z} CC_{,3}dxd\eta - \rho_2 \int_0^t \int_{R_z} (\xi - z)C\Big[C_{,\eta} + u_i C_{,i} - \sigma\Delta T\Big]dxd\eta \\
&\leq \frac{\rho_2}{2} \int_0^t \int_{D_z} C^2 dAd\eta - \frac{\rho_2}{2} \int_{R_z} (\xi - z)C^2 dx\Big|_{\eta=t} + \frac{\rho_2}{2} \int_0^t \int_{R_z} u_3 C^2 dxd\eta \\
&- \sigma\rho_2 \int_0^t \int_{R_z} CT_{,3}dxd\eta - \sigma\rho_2 \int_0^t \int_{R_z} (\xi - z)C_{,i}T_{,i}dxd\eta.
\end{aligned} \tag{47}$$

By the Schwarz and the AG mean inequalities, it follows that from (47)

$$\begin{aligned}
&E_4(z,t) + \frac{\rho_2}{2} \int_{R_z} (\xi - z)C^2 dx\Big|_{\eta=t} \\
&\leq \frac{\rho_2}{2\lambda_1} \int_0^t \int_{D_z} C^2 dAd\eta + \frac{\sigma\rho_2}{2\sqrt{\lambda_1}} \int_0^t \int_{R_z} C_{,i}C_{,i}dxd\eta \\
&+ \frac{\sigma\rho_2}{2\sqrt{\lambda_1}} \int_0^t \int_{R_z} T_{,3}T_{,3}dxd\eta + \frac{\sigma\rho_2\varepsilon_5}{2} \int_0^t \int_{R_z} (\xi - z)C_{,i}C_{,i}dxd\eta \\
&+ \frac{\sigma\rho_2}{2\varepsilon_5} \int_0^t \int_{R_z} (\xi - z)T_{,i}T_{,i}dxd\eta + \frac{\rho_2}{2} \int_0^t \int_{R_z} u_3 C^2 dxd\eta,
\end{aligned} \tag{48}$$

for an arbitrary constant $\varepsilon_5 > 0$.

In order to bound the last term on the right of (48), using the Equations (9), (11) and (13), the Schwarz inequality and the AG mean inequality to obtain

$$\begin{aligned}
\frac{\rho_2}{2}\int_0^t\int_{R_z} u_3C^2dxd\eta &\le \frac{\rho_2}{2}\int_0^t\Big(\int_{R_z}(u_3C)^2dx\Big)^{\frac{1}{2}}\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}d\eta \\
&\le \frac{\rho_2}{2}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}\int_0^t\int_z^\infty\Big(\int_{D_\xi}u_3^4dA\Big)^{\frac{1}{4}}\Big(\int_{D_\xi}C^4dA\Big)^{\frac{1}{4}}d\xi d\eta \\
&\le \frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{3}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}\Big(\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta\Big)^{\frac{1}{2}} \\
&\le \frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{7}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}\Big(\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta+\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta\Big),
\end{aligned} \tag{49}$$

where the bound for $\max_t\Big\{(\int_{R_z}C^2dx)^{\frac{1}{2}}\Big\}$ will be derived later.

Inserting (49) back into (48), we have

$$\begin{aligned}
&E_4(z,t)+\frac{\rho_2}{2}\int_{R_z}(\xi-z)C^2dx\Big|_{\eta=t} \\
&\le \frac{\rho_2}{2\lambda_1}\int_0^t\int_{D_z}C_{,\alpha}C_{,\alpha}dAd\eta+\frac{\sigma\rho_2}{2\sqrt{\lambda_1}}\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta+\frac{\sigma\rho_2}{2\sqrt{\lambda_1}}\int_0^t\int_{R_z}T_{,3}T_{,3}dxd\eta \\
&+\frac{\sigma\rho_2\varepsilon_6}{2}\int_0^t\int_{R_z}(\xi-z)C_{,i}C_{,i}dxd\eta+\frac{\sigma\rho_2}{2\varepsilon_6}\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}dxd\eta \\
&+\frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{7}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}\Big(\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta+\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta\Big).
\end{aligned} \tag{50}$$

Combining (46) and (50), we obtain the following Lemma.

Lemma 7. *Let u,T,C,p be solutions of Equations (1)–(8) with $g,h\in L_\infty(R\times\{t>0\})$ and $\int_D f_3dA=0$. Then*

$$\begin{aligned}
&E_3(z,t)+E_4(z,t)\frac{1}{2}\int_{R_z}(\xi-z)\Big[\rho_1T^2+\rho_2C^2\Big]dx\Big|_{\eta=t} \\
&\le \frac{\rho_1}{2\lambda_1}\int_0^t\int_{D_z}T_{,\alpha}T_{,\alpha}dAd\eta+\frac{\rho_2}{2\lambda_1}\int_0^t\int_{D_z}C_{,\alpha}C_{,\alpha}dAd\eta \\
&+\Big[\frac{\sigma\rho_2}{2\sqrt{\lambda_1}}+\frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{7}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}\Big]\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta \\
&+\Big[\frac{\sigma\rho_2}{2\sqrt{\lambda_1}}+\frac{\rho_1T_M}{4\lambda_1}\Big]\int_0^t\int_{R_z}T_{,i}T_{,i}dxd\eta \\
&+\frac{\sigma\rho_2\varepsilon_6}{2}\int_0^t\int_{R_z}(\xi-z)C_{,i}C_{,i}dxd\eta+\frac{\sigma\rho_2}{2\varepsilon_6}\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}dxd\eta \\
&+\Big[\frac{\rho_1T_M}{4\lambda_1}+\frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{7}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}\Big]\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta,
\end{aligned}$$

where ε_6 is a positive constant. Next, we use Lemmas 5–7 to prove our main result.

□

4. Main Result

First, we introduce a new function

$$
\begin{aligned}
\psi(z,t) &= k\int_0^t\int_{R_z}(\xi-z)u_{i,\eta}u_{i,\eta}dxd\eta+\nu\int_0^t\int_{R_z}(\xi-z)u_{i,j}u_{i,j}dxd\eta\\
&+\rho_1\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}dxd\eta+\rho_2\int_0^t\int_{R_z}(\xi-z)C_{,i}C_{,i}dxd\eta\\
&+k\nu\int_{R_z}(\xi-z)u_{i,j}u_{i,j}dx\Big|_{\eta=t}+(k+\frac{1}{2})\int_{R_z}(\xi-z)u_iu_idx\Big|_{\eta=t}\\
&+\frac{1}{2}\int_0^t\int_{R_z}(\xi-z)u_iu_idxd\eta+\frac{1}{2}\int_{R_z}(\xi-z)\Big[\rho_1T^2+\rho_2C^2\Big]dx\Big|_{\eta=t}.
\end{aligned}
\tag{51}
$$

Using Lemmas 4–6 and in view of (51), we have

$$
\begin{aligned}
\psi(z,t) &\le a_4\int_0^t\int_{R_z}u_{i,\eta}u_{i,\eta}dxd\eta+a_5\int_0^t\int_{R_z}u_{i,j}u_{i,j}dxd\eta\\
&+a_6\int_0^t\int_{R_z}T_{,i}T_{,i}dxd\eta+\frac{\nu}{2}\int_0^t\int_{R_z}u_iu_idxd\eta+a_7\int_0^t\int_{R_z}C_{,i}C_{,i}dxd\eta\\
&+\frac{k\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}\int_0^t\int_{D_z}u_{3,\eta}^2dAd\eta+\frac{k\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}\int_0^t\int_{D_z}u_{\alpha,3}u_{\alpha,3}dAd\eta\\
&+\frac{\rho_1}{2\lambda_1}\int_0^t\int_{D_z}T_{,\alpha}T_{,\alpha}dAd\eta+\frac{\rho_2}{2\lambda_1}\int_0^t\int_{D_z}C_{,\alpha}C_{,\alpha}dAd\eta\\
&+\frac{2k\delta_1^2}{\lambda_1}\int_0^t\int_{R_z}(\xi-z)T_{,\alpha}T_{,\alpha}dxd\eta+\frac{2k\delta_2^2}{\lambda_1}\int_0^t\int_{R_z}(\xi-z)C_{,\alpha}C_{,\alpha}dxd\eta\\
&+\frac{\delta_1^2}{2\lambda_1\varepsilon_3}\int_0^t\int_{R_z}(\xi-z)T_{,\alpha}T_{,\alpha}dxd\eta+\frac{\delta_2^2}{2\lambda_1\varepsilon_2}\int_0^t\int_{R_z}(\xi-z)C_{,\alpha}C_{,\alpha}dxd\eta\\
&+\frac{\sigma\rho_2\varepsilon_6}{2}\int_0^t\int_{R_z}(\xi-z)C_{,i}C_{,i}dxd\eta+\frac{\sigma\rho_2}{2\varepsilon_6}\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}dxd\eta,
\end{aligned}
\tag{52}
$$

where

$$
a_4=a_1k+\frac{\Lambda^{\frac{1}{2}}}{2\lambda_1},a_5=a_2\nu+a_3+\frac{\rho_1T_M}{4\lambda_1}+\frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{7}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\},
$$

$$
a_6=\frac{k\delta_1\Lambda^{\frac{1}{2}}}{\lambda_1}+\frac{\delta_1\Lambda^{\frac{1}{2}}}{2\lambda_1},a_7=\frac{k\delta_2\Lambda^{\frac{1}{2}}}{\lambda_1}+\frac{\delta_2\Lambda^{\frac{1}{2}}}{2\lambda_1}+\frac{\sigma\rho_2}{2\sqrt{\lambda_1}}+\frac{\rho_2\Omega^{\frac{1}{8}}}{2^{\frac{7}{4}}\lambda_1^{\frac{1}{4}}}\max_t\Big\{\Big(\int_{R_z}C^2dx\Big)^{\frac{1}{2}}\Big\}.
$$

Choosing $\varepsilon_6=\frac{1}{2\sigma},\rho_2=\frac{8k\delta_2^2}{\lambda_1}+\frac{2\delta_2^2}{\lambda_1\varepsilon_2},\rho_1=\frac{4k\delta_1^2}{\lambda_1}+\frac{\delta_1^2}{\lambda_1\varepsilon_3}+\frac{\sigma\rho_2}{\varepsilon_6}$ and define

$$
\begin{aligned}
\Psi(z,t) &= k\int_0^t\int_{R_z}(\xi-z)u_{i,\eta}u_{i,\eta}dxd\eta+\nu\int_0^t\int_{R_z}(\xi-z)u_{i,j}u_{i,j}dxd\eta\\
&+\frac{1}{2}\rho_1\int_0^t\int_{R_z}(\xi-z)T_{,i}T_{,i}dxd\eta+\frac{1}{2}\rho_2\int_0^t\int_{R_z}(\xi-z)C_{,i}C_{,i}dxd\eta\\
&+k\nu\int_{R_z}(\xi-z)u_{i,j}u_{i,j}dx\Big|_{\eta=t}+(k+\frac{1}{2})\int_{R_z}(\xi-z)u_iu_idx\Big|_{\eta=t}\\
&+\frac{1}{2}\int_0^t\int_{R_z}(\xi-z)u_iu_idxd\eta+\frac{1}{2}\int_{R_z}(\xi-z)\Big[\rho_1T^2+\rho_2C^2\Big]dx\Big|_{\eta=t},
\end{aligned}
\tag{53}
$$

we can have from (52)

$$
\begin{aligned}
\Psi(z,t) \le{}& a_4 \int_0^t \int_{R_z} u_{i,\eta} u_{i,\eta} dx d\eta + a_5 \int_0^t \int_{R_z} u_{i,j} u_{i,j} dx d\eta \\
&+ a_6 \int_0^t \int_{R_z} T_{,i} T_{,i} dx d\eta + \frac{\nu}{2} \int_0^t \int_{R_z} u_i u_i dx d\eta + a_7 \int_0^t \int_{R_z} C_{,i} C_{,i} dx d\eta \\
&+ \frac{k\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}} \int_0^t \int_{D_z} u_{3,\eta}^2 dA d\eta + \frac{k\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}} \int_0^t \int_{D_z} u_{\alpha,3} u_{\alpha,3} dA d\eta \\
&+ \frac{\rho_1}{2\lambda_1} \int_0^t \int_{D_z} T_{,\alpha} T_{,\alpha} dA d\eta + \frac{\rho_2}{2\lambda_1} \int_0^t \int_{D_z} C_{,\alpha} C_{,\alpha} dA d\eta.
\end{aligned} \tag{54}
$$

From (53), we have

$$
\begin{aligned}
-\frac{\partial \Psi(z,t)}{\partial z} ={}& k \int_0^t \int_{R_z} u_{i,\eta} u_{i,\eta} dx d\eta + \nu \int_0^t \int_{R_z} u_{i,j} u_{i,j} dx d\eta \\
&+ \frac{1}{2}\rho_1 \int_0^t \int_{R_z} T_{,i} T_{,i} dx d\eta + \frac{1}{2}\rho_2 \int_0^t \int_{R_z} C_{,i} C_{,i} dx d\eta \\
&+ k\nu \int_{R_z} u_{i,j} u_{i,j} dx \Big|_{\eta=t} + (k + \frac{1}{2}) \int_{R_z} u_i u_i dx \Big|_{\eta=t} \\
&+ \frac{1}{2} \int_0^t \int_{R_z} u_i u_i dx d\eta + \frac{1}{2} \int_{R_z} \Big[\rho_1 T^2 + \rho_2 C^2\Big] dx \Big|_{\eta=t}
\end{aligned} \tag{55}
$$

and

$$
\begin{aligned}
\frac{\partial^2 \Psi(z,t)}{\partial z^2} ={}& k \int_0^t \int_{D_z} u_{i,\eta} u_{i,\eta} dA d\eta + \nu \int_0^t \int_{D_z} u_{i,j} u_{i,j} dA d\eta \\
&+ \frac{1}{2}\rho_1 \int_0^t \int_{D_z} T_{,i} T_{,i} dA d\eta + \frac{1}{2}\rho_2 \int_0^t \int_{D_z} C_{,i} C_{,i} dA d\eta \\
&+ k\nu \int_{D_z} u_{i,j} u_{i,j} dA \Big|_{\eta=t} + (k + \frac{1}{2}) \int_{D_z} u_i u_i dA \Big|_{\eta=t} \\
&+ \frac{1}{2} \int_0^t \int_{D_z} u_i u_i dA d\eta + \frac{1}{2} \int_{D_z} \Big[\rho_1 T^2 + \rho_2 C^2\Big] dA \Big|_{\eta=t}.
\end{aligned} \tag{56}
$$

Combining (54), (55) and (56), we have

Thus

$$
\Psi(z,t) \le K_1 \Big[-\frac{\partial \Psi(z,t)}{\partial z} \Big] + K_2 \frac{\partial^2 \Psi(z,t)}{\partial z^2}, \tag{57}
$$

where

$$
K_1 = \max\{\frac{a_4}{k}, \frac{a_5}{\nu}, \nu, \frac{a_6}{\rho_1}, \frac{a_7}{\rho_2}\},
$$

$$
K_2 = \max\{\frac{\nu\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}, \frac{k\Lambda^{\frac{1}{2}}}{2\sqrt{\lambda_1}}, \frac{1}{2\lambda_1}\}.
$$

Inequality (57) can be rewritten as

$$
\frac{\partial}{\partial z}\Big\{ e^{-\ell_1 z} \Big(\frac{\partial \Psi}{\partial z} + \ell_2 \Psi \Big) \Big\} \ge 0, \tag{58}
$$

where

$$
\ell_1 = \frac{K_1}{2K_2} + \frac{1}{2}\sqrt{\frac{K_1^2}{K_2^2} + \frac{4}{K_2}}, \quad \ell_2 = -\frac{K_1}{2K_2} + \frac{1}{2}\sqrt{\frac{K_1^2}{K_2^2} + \frac{4}{K_2}}.
$$

Integrating (58) from z to ∞ leads to

$$\frac{\partial \Psi}{\partial z} + \ell_2 \Psi \leq 0,$$

and hence

$$\Psi(z,t) \leq \Psi(0,t) e^{-\ell_2 z}. \tag{59}$$

Combining (53) and (59), we can obtain the following theorem.

Theorem 1. *Let u, T, C, p be solutions of Equations (1)–(8) with $g, h \in L_\infty(R \times \{t > 0\})$ and $\int_D f_3 dA = 0$. Then*

$$\begin{aligned} & k \int_0^t \int_{R_z} (\xi - z) u_{i,\eta} u_{i,\eta} dx d\eta + \nu \int_0^t \int_{R_z} (\xi - z) u_{i,j} u_{i,j} dx d\eta \\ & + \frac{1}{2} \rho_1 \int_0^t \int_{R_z} (\xi - z) T_{,i} T_{,i} dx d\eta + \frac{1}{2} \rho_2 \int_0^t \int_{R_z} (\xi - z) C_{,i} C_{,i} dx d\eta \\ & + k\nu \int_{R_z} (\xi - z) u_{i,j} u_{i,j} dx \Big|_{\eta = t} + (k + \frac{1}{2}) \int_{R_z} (\xi - z) u_i u_i dx \Big|_{\eta = t} \\ & + \frac{1}{2} \int_0^t \int_{R_z} (\xi - z) u_i u_i dx d\eta + \frac{1}{2} \int_{R_z} (\xi - z) \Big[\rho_1 T^2 + \rho_2 C^2 \Big] dx \Big|_{\eta = t} \\ & \leq \Psi(0,t) e^{-\ell_2 z}. \end{aligned} \tag{60}$$

Remark 1. *The result of Theorem 1 belongs to the study of Saint-Venant principle, which shows that the fluid decays exponentially with spatial variables on the cylinder.*

Remark 2. *Theorem 1 shows that the solutions of Equations (1)–(8) decays exponentially as $z \to \infty$. To make the decay bound explicit, we have to derive the bounds for $\Psi(0,t)$ and $\max_t \int_R C^2 dx$ in next section.*

5. Bounds of $\Psi(0,t)$ and $\max_t \int_R C^2 dx$

From the previous section, we can see that a_3 involves the quantities $\max_t \int_R C^2 dx$. To make our main result explicit, we have to derive bounds of $\Psi(0,t)$ and $\max_t \int_R C^2 dx$ in term of the physical parameters σ, ν, g_i, h_i, the boundary data and so on. To do this, we begin with

$$\int_0^t \int_R T_{,i} T_{,i} dx d\eta = -\int_0^t \int_D F T_{,3} dA d\eta - \int_0^t \int_R T \Delta T dx d\eta. \tag{61}$$

Now we assume that S is a sufficiently smooth function satisfying the same initial and boundary conditions as T. Thus,

$$\begin{aligned} \int_0^t \int_R T_{,i} T_{,i} dx d\eta &= -\int_0^t \int_D S T_{,3} dA d\eta - \int_0^t \int_R T \Delta T dx d\eta \\ &= \int_0^t \int_R S_{,i} T_{,i} dx d\eta - \int_0^t \int_R (T - S) \Delta T dx d\eta \\ &= \int_0^t \int_R S_{,i} T_{,i} dx d\eta - \int_0^t \int_R (T - S)(T_{,\eta} + u_i T_{,i}) dx d\eta \\ &= \int_0^t \int_R S_{,i} T_{,i} dx d\eta - \int_R T^2 dx \Big|_{\eta = t} + \int_R T S dx \Big|_{\eta = t} \\ &- \int_0^t \int_R S_{,\eta} T dx d\eta - \int_0^t \int_R S_{,i} T u_i dx d\eta - \frac{1}{2} \int_0^t \int_D f F^2 dA d\eta. \end{aligned} \tag{62}$$

Using the Schwarz and the arithmetic-geometric mean inequalities, we can obtain

$$\begin{aligned}\int_R T^2dx\Big|_{\eta=t}+\int_0^t\int_R T_{,i}T_{,i}dxd\eta\le{}&\frac{1}{2}\int_R S^2dx\Big|_{\eta=t}-\frac{1}{2}\int_0^t\int_D fF^2dAd\eta\\&+\Big(\frac{\epsilon_1}{2}\int_0^t\int_R T_{,i}T_{,i}dxd\eta+\frac{1}{2\epsilon_1}\int_0^t\int_R S_{,i}S_{,i}dxd\eta\Big)\\&+\Big(\frac{\epsilon_2}{2\lambda_1}\int_0^t\int_R T_{,i}T_{,i}dxd\eta+\frac{1}{2\epsilon_2\lambda_1}\int_0^t\int_R S_{,\eta}S_{,\eta}dxd\eta\Big)\\&+\Big(\frac{\epsilon_3T_M}{2}\int_0^t\int_R u_iu_idxd\eta+\frac{T_M}{2\epsilon_3}\int_0^t\int_R S_{,i}S_{,i}dxd\eta\Big),\end{aligned}\tag{63}$$

where $\epsilon_1,\epsilon_2,\epsilon_3$ are positive constants. Choosing

$$\epsilon_1=\frac{1}{2},\quad \epsilon_2=\frac{\lambda_1}{2},\tag{64}$$

we can obtain

$$\int_R T^2dx\Big|_{\eta=t}+\int_0^t\int_R T_{,i}T_{,i}dxd\eta\le\frac{\epsilon_3T_M}{2}\int_0^t\int_R u_iu_idxd\eta+data.\tag{65}$$

Obviously, the data terms in (65) involve $\frac{1}{2}\int_R S^2dx\Big|_{\eta=t}$, $\int_0^t\int_R S_{,i}S_{,i}dxd\eta$, $\int_0^t\int_R S_{,\eta}S_{,\eta}dxd\eta$ and $-\frac{1}{2}\int_0^t\int_D fF^2dAd\eta$. Similarly, we can bound $\int_0^t\int_R C_{,i}C_{,i}dxd\eta$ as well as $\max_t\int_R C^2dx$. Firstly, we introduce a function H:

$$\begin{aligned}\frac{\partial H}{\partial t}+u_iH_{,i}=\Delta H,&\quad in\ R\times\{t>0\},\\ H=0,&\quad in\ R\times\{t=0\},\\ H=0,&\quad on\ \partial D\times\{x_3>0\}\times\{t\ge0\},\\ H=G(x_1,x_2,t),&\quad on\ D\times\{t>0\},\end{aligned}\tag{66}$$

Then we have

$$\begin{aligned}(C-H)_{,t}+u_i(C-H)_{,i}=\Delta(C-H)+\sigma\Delta T,&\quad in\ R\times\{t>0\},\\ C-H=0&\quad in\ R\times\{t=0\},\\ C-H=0&\quad on\ \partial D\times\{x_3>0\}\times\{t\ge0\},\\ C-H=0&\quad on\ D\times\{t>0\}.\end{aligned}\tag{67}$$

By the triangle inequality, we obtain that

$$\Big(\int_0^t\int_R C_{,i}C_{,i}dxd\eta\Big)^{\frac{1}{2}}\le\Big[\int_0^t\int_R(C-H)_{,i}(C-H)_{,i}dxd\eta\Big]^{\frac{1}{2}}+\Big[\int_0^t\int_R H_{,i}H_{,i}dxd\eta\Big]^{\frac{1}{2}},\tag{68}$$

and

$$\Big[\max_t\int_R C^2dx\Big]^{\frac{1}{2}}\le\Big[\max_t\int_R(C-H)^2dx\Big]^{\frac{1}{2}}+\Big[\max_t\int_R H^2dx\Big]^{\frac{1}{2}}.\tag{69}$$

Then,

$$\frac{1}{2}\int_R(C-H)^2dx\Big|_{\eta=t}+\int_0^t\int_R(C-H)_{,i}(C-H)_{,i}dxd\eta=-\sigma\int_0^t\int_R(C-H)_{,i}T_{,i}dxd\eta,\tag{70}$$

which follows that

$$\begin{aligned}\frac{1}{2}\int_R (C-H)^2 dx\Big|_{\eta=t} + \int_0^t \int_R (C-H)_{,i}(C-H)_{,i} dx d\eta \\ \leq \sigma^2 \int_0^t \int_R T_{,i}T_{,i} dx d\eta \\ \leq \frac{\epsilon_3 T_M \sigma^2}{2} \int_0^t \int_R u_i u_i dx d\eta + data.\end{aligned} \tag{71}$$

Just as in the computation for T, we have the following inequality

$$\frac{1}{2}\int_R H^2 dx|_{\eta=t} + \int_0^t \int_R H_{,i}H_{,i} dx d\eta \leq \epsilon_4 \int_0^t \int_R u_i u_i dx d\eta + data. \tag{72}$$

Thus,

$$\frac{1}{2}\int_R C^2 dx|_{\eta=t} + \int_0^t \int_R C_{,i}C_{,i} dx d\eta \leq \epsilon_5 \int_0^t \int_R u_i u_i dx d\eta + data, \tag{73}$$

where $\epsilon_5 > 0$ depends on ϵ_3, ϵ_4 and σ. Next we have to derive a bound for $\int_0^t \int_R u_i u_i dx d\eta$ in term of data. To do this, we define a function

$$\varpi_i = f_i e^{-\varsigma_1 z}, \tag{74}$$

for some positive constant ϖ_i. Then,

$$\begin{aligned}\Big[\int_0^t \int_R u_{i,j}u_{i,j} dx d\eta\Big]^{\frac{1}{2}} \leq \Big[\int_0^t \int_R (u_i-\varpi_i)_{,j}(u_i-\varpi_i)_{,j} dx d\eta\Big]^{\frac{1}{2}} \\ + \Big[\int_0^t \int_R \varpi_{i,j}\varpi_{i,j} dx d\eta\Big]^{\frac{1}{2}}.\end{aligned}$$

Obviously, we find that the last term of (75) is a data term. Now

$$\begin{aligned}\nu \int_0^t \int_R (u_i-\varpi_i)_{,j}(u_i-\varpi_i)_{,j} dx d\eta \\ = -\int_0^t \int_R (u_i-\varpi_i)_{,j}\Big[(u_i-\varpi_i) + p_{,i} - g_i T - h_i C + \varpi_i - \nu\Delta\varpi_i\Big] dx d\eta\end{aligned} \tag{75}$$

or

$$\begin{aligned}\frac{\nu}{2}\int_0^t \int_R (u_i-\varpi_i)_{,j}(u_i-\varpi_i)_{,j} dx d\eta \\ \leq -\int_0^t \int_R p\varpi_{i,i} dx d\eta + \frac{\delta_1^2}{2}\int_0^t \int_R T^2 dx d\eta + \frac{\delta_2^2}{2}\int_0^t \int_R C^2 dx d\eta + data.\end{aligned} \tag{76}$$

Noting that

$$\varpi_{i,i} = (f_{\alpha,\alpha} - \varsigma_1 f_3)e^{-\varsigma_1 z} = 0, \tag{77}$$

in R for $\varsigma_1 = \frac{f_{\alpha,\alpha}}{f_3}$, we can rewrite (76) as

$$\begin{aligned}\frac{\nu}{2}\int_0^t \int_R (u_i-\varpi_i)_{,j}(u_i-\varpi_i)_{,j} dx d\eta \\ \leq \frac{\delta_1^2}{2\lambda_1}\int_0^t \int_R T_{,i}T_{,i} dx d\eta + \frac{\delta_2^2}{2\lambda_1}\int_0^t \int_R C_{,i}C_{,i} dx d\eta + data.\end{aligned} \tag{78}$$

Inserting (78) back into (75), we may have a bound of the form

$$\int_0^t \int_R u_{i,j}u_{i,j}dxd\eta \leq C_1 \int_0^t \int_R T_{,i}T_{,i}dxd\eta + C_2 \int_0^t \int_R C_{,i}C_{,i}dxd\eta + data, \tag{79}$$

for computable C_1 and C_2. Combining (65) and (73) and by inequality (17), we have

$$\int_0^t \int_R u_{i,j}u_{i,j}dxd\eta \leq \frac{C_1 T_M}{2\lambda_1}\epsilon_3 \int_0^t \int_R u_{i,j}u_{i,j}dxd\eta + \frac{C_2}{\lambda_1}\epsilon_5 \int_0^t \int_R u_{i,j}u_{i,j}dxd\eta + data. \tag{80}$$

It is clear to see that

$$\int_0^t \int_R u_{i,j}u_{i,j}dxd\eta \leq data, \tag{81}$$

for $\epsilon_3 = \frac{\lambda_1}{2C_1T_M}, \epsilon_5 = \frac{\lambda_1}{4C_2}$. From (65) and (73), we can obtain

$$\max_t \int_R T^2dx \leq data, \quad \max_t \int_R C^2dx \leq data, \tag{82}$$

and

$$\int_0^t \int_R T_{,i}T_{,i}dxd\eta \leq data, \quad \int_0^t \int_R C_{,i}C_{,i}dxd\eta \leq data. \tag{83}$$

Next we seek bound for the total energy $\Psi(0,t)$. From (54) we can obtain for $\Psi(0,t)$

$$\begin{aligned}\Psi(0,t) \leq a_4 \int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta + \frac{\nu}{2}\int_0^t \int_R u_iu_idxd\eta \\ + b_1 \int_0^t \int_D u_{\alpha,3}u_{\alpha,3}dAd\eta + data.\end{aligned} \tag{84}$$

We are left to derive bounds for $\int_0^t \int_{R_z} u_{i,\eta}u_{i,\eta}dxd\eta$ and $\int_0^t \int_D u_{\alpha,3}u_{\alpha,3}dAd\eta$. Multiplying (1) with $u_{i,\eta}$ and integrating in the region $R \times [0,t]$, we have

$$\int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta = \int_0^t \int_R u_{i,\eta}\Big[\nu\Delta u_i - u_i - p_{,i} + g_iT + h_iC\Big]dxd\eta, \tag{85}$$

which follows that

$$\begin{aligned}\int_0^t \int_R u_{i,\eta}u_{i,\eta}dxd\eta &\leq -2\nu \int_0^t \int_D u_{\alpha,3}u_{\alpha,\eta}dAd\eta + 2\int_0^t \int_D u_{3,\eta}pdAd\eta \\ &+ \frac{\delta_1^2}{2\lambda_1}\int_0^t \int_R T_{,i}T_{,i}dxd\eta + \frac{\delta_1^2}{2\lambda_1}\int_0^t \int_R C_{,i}C_{,i}dxd\eta + data \\ &\leq -2\nu \int_0^t \int_D u_{\alpha,3}f_{\alpha,\eta}dAd\eta + 2\int_0^t \int_D f_{3,\eta}pdAd\eta + data,\end{aligned} \tag{86}$$

where we have used the fact $u_{3,3} = -u_{\alpha,\alpha} = -f_{\alpha,\alpha}$ on D_0 and (83), and ε_6 is a positive constants. For the first term of (86), using the Schwarz and the AG mean inequalities we have

$$\begin{aligned}-2\nu \int_0^t \int_{D_0} f_{\alpha,\eta}u_{\alpha,3}dxd\eta &\leq 2\nu\Big(\int_0^t \int_{D_0} u_{\alpha,3}u_{\alpha,3}dAd\eta\Big)^{\frac{1}{2}}\Big(\int_0^t \int_{D_0} f_{\alpha,\eta}f_{\alpha,\eta}dAd\eta\Big)^{\frac{1}{2}} \\ &\leq \int_0^t \int_{D_0} u_{\alpha,3}u_{\alpha,3}dAd\eta + data.\end{aligned} \tag{87}$$

To bound the second term on the right of (86), we define $\overline{p}$ to be the mean value of p over D_0, i.e.,

$$\overline{p} = \frac{1}{|D_0|} \int_{D_0} p dA, \tag{88}$$

where $|D_0|$ is the measure of D_0. Since

$$\int_{D_0} f_{3,\eta} \overline{p} dA = \overline{p} \int_{D_0} f_{3,\eta} dA = 0, \tag{89}$$

we obtain

$$\int_{D_0} f_{3,\eta} p dA = \int_{D_0} f_{3,\eta} (p - \overline{p}) dA. \tag{90}$$

It follows by using Schwarz inequality that

$$\int_0^t \int_{D_0} f_{3,\eta} p dx d\eta = \int_0^t \int_{D_0} f_{3,\eta} (p - \overline{p}) dA d\eta \leq data + \epsilon_6 \int_0^t \int_{D_0} (p - \overline{p})^2 dA d\eta, \tag{91}$$

where ϵ_6 is a positive constant to be determined later.

To deal with the integral $\int_0^t \int_{D_0} (p - \overline{p})^2 dA d\eta$, we let an auxiliary function χ satisfying:

$$\Delta \chi = 0, \quad \frac{\partial \chi}{\partial n} = 0 \quad on\ \partial D_0, \quad \frac{\partial \chi}{\partial n} = p - \overline{p},\ in\ D_0. \tag{92}$$

From the definition of $\overline{p}$ in (88), it is clear that $\int_{D_0} (p - \overline{p}) dA = 0$. Thus, the necessary condition for the existence of a solution is satisfied and we compute

$$\begin{aligned} \int_0^t \int_{D_0} (p - \overline{p})^2 dA d\eta &= \int_0^t \int_{\partial R} (p - \overline{p}) \frac{\partial \chi}{\partial n} dx d\eta = \int_0^t \int_R (p - \overline{p})_{,i} \chi_{,i} dx d\eta \\ &= \int_0^t \int_R \chi_{,i} \Big[- u_{i,\eta} + \nu u_{i,jj} - u_i + g_i T + h_i C \Big] dx d\eta. \end{aligned} \tag{93}$$

Since

$$\begin{aligned} \nu \int_0^t \int_R \chi_{,i} u_{i,jj} dx d\eta &= -\nu \int_0^t \int_D \chi_{,i} u_{i,3} dx d\eta - \nu \int_0^t \int_R \chi_{,ij} u_{i,j} dx d\eta \\ &= \nu \int_0^t \int_D \chi_{,3} f_{\alpha,\alpha} dx d\eta - \nu \int_0^t \int_D \chi_{,\alpha} u_{\alpha,3} dx d\eta \\ &+ \nu \int_0^t \int_D \chi_{,j} u_{3,j} dx d\eta + \nu \int_0^t \int_R \chi_{,j} u_{i,ij} dx d\eta \\ &= -\nu \int_0^t \int_D \chi_{,\alpha} u_{\alpha,3} dx d\eta + \nu \int_0^t \int_D \chi_{,\alpha} f_{3,\alpha} dx d\eta. \end{aligned} \tag{94}$$

From (93), we can obtain

$$
\begin{aligned}
\int_0^t \int_{D_0} (p-\overline{p})^2 dA d\eta =& \int_0^t \int_R \chi_{,i} \Big[-u_{i,\eta} + \nu u_{i,jj} - u_i + g_i T + h_i C \Big] dx d\eta \\
\leq& \Big(\int_0^t \int_R \chi_{,i}\chi_{,i} dx d\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta \Big)^{\frac{1}{2}} \\
&+ \nu \Big(\int_0^t \int_D \chi_{,\alpha}\chi_{,\alpha} dA d\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_D u_{3,\alpha} u_{3,\alpha} dA d\eta \Big)^{\frac{1}{2}} \\
&+ \nu \Big(\int_0^t \int_D \chi_{,\alpha}\chi_{,\alpha} dA d\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_D f_{3,\alpha} f_{3,\alpha} dA d\eta \Big)^{\frac{1}{2}} \\
&+ \frac{1}{\sqrt{\lambda_1}} \Big(\int_0^t \int_R \chi_{,i}\chi_{,i} dx d\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_R u_{i,j} u_{i,j} dx d\eta \Big)^{\frac{1}{2}} \\
&+ \frac{\delta_1}{\sqrt{\lambda_1}} \Big(\int_0^t \int_R \chi_{,i}\chi_{,i} dx d\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_R T_{,j} T_{,j} dx d\eta \Big)^{\frac{1}{2}} \\
&+ \frac{\delta_2}{\sqrt{\lambda_1}} \Big(\int_0^t \int_R \chi_{,i}\chi_{,i} dx d\eta \Big)^{\frac{1}{2}} \Big(\int_0^t \int_R C_{,j} C_{,j} dx d\eta \Big)^{\frac{1}{2}}.
\end{aligned} \tag{95}
$$

Making use of (15), (16), (81) and (83) with $g = p - \overline{p}$, we have

$$
\begin{aligned}
&\Big[\int_0^t \int_{D_0} (p-\overline{p})^2 dA d\eta \Big]^{\frac{1}{2}} \\
&\quad \leq \frac{1}{\sqrt{\mu}} \Big(\int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta \Big)^{\frac{1}{2}} + \nu \Big(\int_0^t \int_{D_0} u_{\alpha,3} u_{\alpha,3} dx d\eta \Big)^{\frac{1}{2}} \\
&\quad + \nu \Big(\int_0^t \int_{D_0} f_{3,\alpha} f_{3,\alpha} dx d\eta \Big)^{\frac{1}{2}} + \frac{1}{\sqrt{\mu\lambda_1}} \Big(\int_0^t \int_R u_{i,j} u_{i,j} dx d\eta \Big)^{\frac{1}{2}} \\
&\quad + \frac{\delta_1}{\sqrt{\mu\lambda_1}} \Big(\int_0^t \int_R T_{,j} T_{,j} dx d\eta \Big)^{\frac{1}{2}} + \frac{\delta_2}{\sqrt{\mu\lambda_1}} \Big(\int_0^t \int_R C_{,j} C_{,j} dx d\eta \Big)^{\frac{1}{2}},
\end{aligned} \tag{96}
$$

which follows that

$$
\int_0^t \int_{D_0} (p-\overline{p})^2 dA d\eta \leq data + c_3 \int_0^t \int_{D_0} u_{\alpha,3} u_{\alpha,3} dx d\eta + c_4 \int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta. \tag{97}
$$

Obviously, from (97) we must establish a bound for the term $\int_0^t \int_{D_0} u_{\alpha,3} u_{\alpha,3} dx d\eta$. To do this, we begin with the identity

$$
\int_0^t \int_R u_{i,3} \Big[\nu u_{i,jj} - u_i - p_{,i} - u_{i,\eta} + g_i T + h_i C \Big] dx d\eta = 0. \tag{98}
$$

Integrating (98) by parts, we can have

$$
\begin{aligned}
&-\nu \int_0^t \int_{D_0} u_{i,3} u_{i,3} dx d\eta + \nu \int_0^t \int_R u_{i,j3} u_{i,j} dx d\eta + \int_0^t \int_R u_{i,3} u_i dx d\eta \\
&+ \int_0^t \int_R u_{i,3} p_{,i} dx d\eta + \int_0^t \int_R u_{i,3} u_{i,\eta} dx d\eta + \int_0^t \int_R u_{i,3} g_i T dx d\eta \\
&+ \int_0^t \int_R u_{i,3} h_i C dx d\eta = 0,
\end{aligned} \tag{99}
$$

which follows that

$$
\int_0^t \int_{D_0} u_{\alpha,3} u_{\alpha,3} dx d\eta \leq data + \int_0^t \int_{D_0} u_{\alpha,3} p dA d\eta + \epsilon_7 \int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta. \tag{100}
$$

where ϵ_7 is a positive constant.

As the derivation of (91), for the term $\int_0^t \int_{D_0} u_{\alpha,3} p dA d\eta$ we can obtain

$$\int_0^t \int_{D_0} u_{\alpha,3} p dA d\eta \leq data + \epsilon_8 \int_0^t \int_{D_0} (p - \overline{p})^2 dA d\eta, \tag{101}$$

where ϵ_8 is a positive constant.

Combining (97), (100) and (101), we have

$$\left(1 - \epsilon_8 c_3\right) \int_0^t \int_{D_0} (p - \overline{p})^2 dx d\eta \leq data + c_3 \epsilon_7 \int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta. \tag{102}$$

Combing (86), (87), (91) and (100), we obtain

$$(1 - \epsilon_7) \int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta \leq data + (\epsilon_6 + \epsilon_7) \int_0^t \int_{D_0} (p - \overline{p})^2 dx d\eta. \tag{103}$$

Choosing ε_7 and ϵ_8 small enough such that $1 - \epsilon_8 c_3 > 0$ and $1 - \epsilon_7 > 0$, from (102) and (103) we can obtain

$$\int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta \leq data, \tag{104}$$

and

$$\int_0^t \int_{D_0} (p - \overline{p})^2 dA d\eta \leq data. \tag{105}$$

Inserting (101) back into (100), we obtain

$$\int_0^t \int_{D_0} u_{\alpha,3} u_{\alpha,3} dx d\eta \leq data + \epsilon_8 \int_0^t \int_{D_0} (p - \overline{p})^2 dA d\eta + \epsilon_7 \int_0^t \int_R u_{i,\eta} u_{i,\eta} dx d\eta. \tag{106}$$

In light of (104) and (105), we have

$$\int_0^t \int_{D_0} u_{\alpha,3} u_{\alpha,3} dx d\eta \leq data. \tag{107}$$

Recalling (84) and using (104) and (107), we obtain

$$\Psi(0, t) \leq data, \tag{108}$$

which is to say that we have bounded the total energy.

6. Conclusions

In this paper, we consider the spatial decay bounds for the Brinkman equations in double-diffusive convection in a semi-infinite pipe. Using the results of this paper, we can continue to study the continuous dependence of the solution on the parameters in the system of equations. In addition, Using the results of this paper, we can continue to study the continuous dependence of the solution on the parameters in the system of equations. This research can refer to the method of [30,31]. In addition, if Equation (1) is replaced by a nonlinear problem (e.g., Forchheimer equations), it will be a more interesting topic.

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