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New Hermite–Hadamard Type Inequalities in Connection with Interval-Valued Generalized Harmonically (h_1, h_2)-Godunova–Levin Functions

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Abstract: As is known, integral inequalities related to convexity have a close relationship with symmetry. In this paper, we introduce a new notion of interval-valued harmonically (m, h_1, h_2)-Godunova–Levin functions, and we establish some new Hermite–Hadamard inequalities. Moreover, we show how this new notion of interval-valued convexity has a close relationship with many existing definitions in the literature. As a result, our theory generalizes many published results. Several interesting examples are provided to illustrate our results.

Keywords: interval-valued analysis; Hermite–Hadamard type inequalities; Godunova–Levin functions; harmonically convex functions

MSC: 26A51; 26A33; 26D10; 26D15

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1. Introduction

Interval analysis is a subset of set-valued analysis, which is the study of sets in the context of mathematics and general topology. The Archimedean approach, which includes determining a circle's circumference, is a classic illustration of interval enclosure. This theory addresses the interval uncertainty that exists in many computational and mathematical models of deterministic real-world systems. With this approach, errors that result in incorrect conclusions are avoided by studying interval variables instead of point variables and expressing computation results as intervals. Consideration of the error estimates of the numerical solutions for finite state machines was one of the initial goals of the interval-valued analysis. Interval analysis, which Moore first described in his renowned book [1], is one of the fundamental techniques in numerical analysis. As a result, it has found applications in many fields, including differential equations for intervals [2], neural network output optimization [3], automatic error analysis [4], computer graphics [5], and many more. For results and applications, we refer interested readers to [6–10].

Inequalities have a significant impact on mathematics, particularly those connected to the Jensen, Ostrowski, Hermite–Hadamard, Bullen, Simpson, and Opial inequalities. Many of these inequalities have recently been extended to interval-valued functions by some well-known researchers (see, for example, [11–14]), and many have also researched the Hermite–Hadamard inequality for convex functions. The traditional H–H inequality is given as:

$$\mathcal{G}\left(\frac{\mathtt{c}+\mathtt{d}}{2}\right) \leq \frac{1}{\mathtt{d}-\mathtt{c}} \int_{\mathtt{c}}^{\mathtt{d}} \mathcal{G}(\alpha) d\alpha \leq \frac{\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})}{2},$$

where $\mathcal{G}$ is a convex function.

On the other hand, the generalized convexity of mappings is a potent tool for addressing a broad range of issues in applied analysis and nonlinear analysis, including various problems in mathematical physics. Recently, a number of generalizations of convex functions have been thoroughly researched. Mathematical analytic study on the idea of integral inequalities is interesting. The study of differential and integral equations has also been considered to be relevant for inequalities and various extended convex mappings. Electrical networks, symmetry analysis, operations research, finance, decision making, numerical analysis, and equilibrium are just a few areas where they have had a substantial impact. We investigate how the subjective properties of convexity might be encouraged by using a number of fundamental integral inequalities.

The Hermite–Hadamard inequality is related to various classes of convexity; for some examples, see [15–19]. Iscan [20], in 2014, presented the idea of harmonic convexity and established a few related Hermite–Hadamard type inequalities. Harmonic h-convex functions and some associated Hermite–Hadamard inequalities were first described by the authors of [21] in 2015. Numerous researchers have linked integral inequalities with interval-valued functions in recent years, producing many significant findings. The Opial-type inequalities were introduced by Costa [22], the Ostrowski-type inequalities were investigated by Chalco-Cano [23] by using the generalized Hukuhara derivative, the Minkowski-type inequalities and the Beckenbach-type inequalities were established by Roman-Flores [24]. By introducing interval-valued coordinated convex functions and creating related $\mathrm{H}-\mathrm{H}$-type inequalities, Zhao et al. [25] recently improved on this idea. It was also utilized to support the $\mathrm{H}-\mathrm{H}$- and Fejér-type inequalities for the n-polynomial convex interval-valued function [26] and preinvex function [27,28]. Interval-valued coordinated preinvex functions are a recent extension of the interval-valued preinvex function notion introduced by Lai et al. [29]. Combined with interval analysis, the $\mathrm{H}-\mathrm{H}$ inequality was extended to interval h-convex functions in [30], to interval harmonic h-convex functions in [31], to interval (h_1, h_2)-convex functions in [32] and to interval harmonically (h_1, h_2)-convex functions in [33]. The definition of the h-Godunova–Levin function was utilized by the authors in [34] to take into account this inequality. Additionally, the author in [35] published a fuzzy Jensen-type integral inequality for fuzzy interval-valued functions, while the authors in [36] created a Jensen-type inequality for (h_1, h_2) interval-nonconvex functions.

Our research is inspired by the strong literature and the specific articles [33,34]. The idea of interval-valued harmonically $(m, h1, h2)$-Godunova–Levin functions is introduced first, and new $\mathrm{H}-\mathrm{H}$-type inequalities are then constructed for the aforementioned notion. The structure of the paper is as follows: In Section 2, the introduction and the mathematical background are given. The issue and our key findings are discussed in Section 3. Section 5 contains the conclusion and future scope.

2. Preliminaries

We begin by introducing some of the terms, characteristics, and notations that will be utilized in the article. Let I be represented as the intervals of the collection of real numbers R.

$[\mathtt{a}] = [\underline{\mathtt{a}}, \overline{\mathtt{a}}] = \{x \in R \mid \underline{\mathtt{a}} \leq x \leq \overline{\mathtt{a}}\}, \underline{\mathtt{a}}, \overline{\mathtt{a}} \in R$, where the real interval $[\mathtt{a}]$ is a closed and bounded subset of R. We call $[\mathtt{a}]$ positive when $\underline{\mathtt{a}} > 0$, and $[\mathtt{a}]$ is negative when $\overline{\mathtt{a}} < 0$. Let us denote all intervals of the set of real numbers by R_I of R, all positive intervals by R_I^+, and all negative intervals by R_I^-. The inclusion" $\subseteq$ " is defined as:

$$[\mathtt{a}] \subseteq [\mathtt{b}] \iff [\underline{\mathtt{a}}, \overline{\mathtt{a}}] \subseteq [\underline{\mathtt{b}}, \overline{\mathtt{b}}] \iff \underline{\mathtt{b}} \leq \underline{\mathtt{a}}, \overline{\mathtt{a}} \leq \overline{\mathtt{b}}.$$

Suppose λ is any real number, and $[\mathtt{a}]$ is an interval; then, the $v[\mathtt{a}]$ is given as:

$$\lambda \cdot [\underline{a}, \overline{a}] = \begin{cases} [\lambda \underline{a}, \lambda \overline{a}], & \text{if} \quad \lambda > 0 \\ \{0\}, & \text{if} \quad \lambda = 0 \\ [\lambda \overline{a}, \lambda \underline{a}], & \text{if} \quad \lambda < 0. \end{cases}$$

For $[a] = [\underline{a}, \overline{a}]$, and $[b] = [\underline{b}, \overline{b}]$, the following algebraic operations hold true:

$$[a] + [b] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$
$$[a] - [b] = [\underline{a} - \underline{b}, \overline{a} - \overline{b}],$$
$$[a].[b] = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}],$$
$$[a]/[b] = [\min\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}, \max\{\underline{a}/\underline{b}, \underline{a}/\overline{b}, \overline{a}/\underline{b}, \overline{a}/\overline{b}\}],$$

where $0 \notin [\underline{a}, \overline{a}]$.

For the intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ the Hausdorff–Pompeiu distance is defined as:

$$d([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}.$$

Definition 1 (see [37]). *Let $\mathcal{G} : [c, d] \to R_I$ be an interval valued function such that $\mathcal{G}(u) = [\underline{\mathcal{G}}(u), \overline{\mathcal{G}}(u)]$ for each $u \in [c, d]$. Then, the function $\mathcal{G}$ is Riemann integrable over the interval $[c, d]$, and*

$$(IR) \int_c^d \mathcal{G}(u) du = \left[(R) \int_c^d \underline{\mathcal{G}}(u) du, (R) \int_c^d \overline{\mathcal{G}}(u) du \right],$$

where $\underline{\mathcal{G}}, \overline{\mathcal{G}}$ are Riemann integrable over the interval $[c, d]$.

The set of all Riemann integrable interval-valued functions and real-valued functions are represented by the symbols $IR_{[c,d]}$ and $R_{[c,d]}$, respectively.

Definition 2 (see [20]). *A set, $S \subset R - \{0\}$, is said to be a harmonic convex set if*

$$\frac{cd}{uc + (1-u)d} \in S,$$

where $\forall c, d \in S$ and $u \in [0, 1]$.

Definition 3 (see [38]). *A nonnegative function $\mathcal{G} : S \to R$ is said to be a Godunova–Levin function, if*

$$\mathcal{G}(uc + (1-u)d) \leq \frac{\mathcal{G}(c)}{u} + \frac{\mathcal{G}(d)}{(1-u)},$$

where $\forall c, d \in S$ and $u \in (0, 1)$.

Definition 4 (see [20]). *A function $\mathcal{G} : S \to R$ is said to be a harmonically convex function, if*

$$\mathcal{G}\left(\frac{cd}{uc + (1-u)d}\right) \leq u\mathcal{G}(d) + (1-u)\mathcal{G}(c),$$

where $\forall c, d \in S$ and $u \in [0, 1]$.

Definition 5 (see [21]). *The function $\mathcal{G} : S \to R$ is said to be a harmonically h-convex function, if $\forall c, d \in S$ and $u \in [0, 1]$, we have*

$$\mathcal{G}\left(\frac{cd}{uc + (1-u)d}\right) \leq h(u)\mathcal{G}(d) + h(1-u)\mathcal{G}(c),$$

where $h : [0, 1] \subseteq S \to R$ is a nonnegative function with $h \neq 0$.

Definition 6 (see [39]). *The function $\mathcal{G} : S \to R$ is said to be a harmonically (m,h)-convex function, if $\forall \mathtt{c}, \mathtt{d} \in S$, $m \in (0,1]$ and $\mathtt{u} \in [0,1]$, we have*

$$\mathcal{G}\left(\frac{m\mathtt{cd}}{m\mathtt{u}\mathtt{c} + (1-\mathtt{u})\mathtt{d}}\right) \leq h(\mathtt{u})\mathcal{G}(\mathtt{d}) + mh(1-\mathtt{u})\mathcal{G}(\mathtt{c}),$$

where $h : [0,1] \subseteq S \to R$ is a nonnegative function with $h \neq 0$.

Definition 7 (see [33]). *The function $\mathcal{G} : S \to R$ is said to be a harmonically (h_1, h_2)-convex function, if $\forall \mathtt{c}, \mathtt{d} \in S$ and $\mathtt{u} \in [0,1]$, we have*

$$\mathcal{G}\left(\frac{\mathtt{cd}}{\mathtt{u}\mathtt{c} + (1-\mathtt{u})\mathtt{d}}\right) \leq h_1(\mathtt{u})h_2(1-\mathtt{u})\mathcal{G}(\mathtt{c}) + h_1(1-\mathtt{u})h_2(\mathtt{u})\mathcal{G}(\mathtt{d}),$$

where $h : [0,1] \subseteq S \to R$ is a nonnegative function with $h \neq 0$.

Definition 8 (see [34]). *The function $\mathcal{G} : S \to R$ is said to be a h-Godunova–Levin function, if for all $\mathtt{c}, \mathtt{d} \in S$ and $\mathtt{u} \in (0,1)$, we have*

$$\mathcal{G}(\mathtt{u}\mathtt{c} + (1-\mathtt{u})\mathtt{d}) \leq \frac{\mathcal{G}(\mathtt{c})}{h(\mathtt{u})} + \frac{\mathcal{G}(\mathtt{d})}{h(1-\mathtt{u}),}$$

where $h : (0,1) \subseteq S \to R$ is a nonnegative function.

3. Main Results

We will now introduce harmonically interval-valued (m, h_1, h_2) Godunova–Levin functions (this idea was motivated by [32]).

Definition 9. *A function $\mathcal{G} : S \to R$ is said to be a harmonically (m, h)-Godunova–Levin function, if for all $\mathtt{c}, \mathtt{d} \in S$, $\mathtt{u} \in (0,1)$ and $m \in (0,1]$, we have*

$$\mathcal{G}\left(\frac{m\mathtt{cd}}{m\mathtt{u}\mathtt{c} + (1-\mathtt{u})\mathtt{d}}\right) \leq \frac{\mathcal{G}(\mathtt{d})}{h(\mathtt{u})} + \frac{m\mathcal{G}(\mathtt{c})}{h(1-\mathtt{u})},$$

where $h : (0,1) \subseteq S \to R$ is a nonnegative function.

Definition 10. *A function $\mathcal{G} : S \to R$ is said to be a harmonically (m, h_1, h_2)-Godunova–Levin function, if for all $\mathtt{c}, \mathtt{d} \in S$, $\mathtt{u} \in (0,1)$ and $m \in (0,1]$, we have*

$$\mathcal{G}\left(\frac{m\mathtt{cd}}{m\mathtt{u}\mathtt{c} + (1-\mathtt{u})\mathtt{d}}\right) \leq \frac{m\mathcal{G}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\mathcal{G}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})},$$

where $h_1, h_2 : (0,1) \subseteq S \to R$ is a nonnegative functions.

Remark 1. • *If wechoose $m = 1$ and $h_1(\mathtt{u}) = h_2(\mathtt{u}) = 1$ in Definition 10, then the notion of a harmonically P-convex function [21] is recovered.*

• *If we choose $m = 1$, $h_1(\mathtt{u}) = 1$, and $h_2(\mathtt{u}) = \frac{1}{\mathtt{u}}$ in Definition 10, then the notion of harmonically convex function [20] is recovered.*

• *If we choose $m = 1$, $h_1(\mathtt{u}) = 1$, and $h_2(\mathtt{u}) = \frac{1}{h(\mathtt{u})}$ in Definition 10, then the notion of harmonic h-convex function [21] is recovered.*

• *If we choose $m = 1$, $h_1(\mathtt{u}) = 1$, and $h_2(\mathtt{u}) = (\mathtt{u})^s$ in Definition 10, then the notion of harmonically s-Godunova–Levin function [40] is recovered.*

Definition 11. *The function $\mathcal{G} : S \to R_I^+$ is said to be a harmonically (m, h_1, h_2)-Godunova–Levin interval-valued convex function, if $\forall \mathtt{c}, \mathtt{d} \in S$, $\mathtt{u} \in (0,1)$ and $m \in (0,1]$, we have*

$$\frac{m\mathcal{G}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\mathcal{G}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})} \subseteq \mathcal{G}\left(\frac{m\mathtt{cd}}{m\mathtt{uc} + (1-\mathtt{u})\mathtt{d}}\right), \tag{1}$$

where $h_1, h_2 : (0,1) \subseteq S \to R$ are non-negative functions such that $h_1, h_2 \neq 0$.

If the above inequality is in reverse order, then the function $\mathcal{G}$ is known as a harmonically (m, h_1, h_2)-Godunova–Levin concave interval-valued function. The space of all harmonically (m, h_1, h_2)-Godunova–Levin convex and (m, h_1, h_2)–Godunova-Levin concave interval-valued functions are denoted by $SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), S, {R_I}^+\right)$and $SGHV\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), S, R_I^+\right)$, respectively.

Proposition 1. *Suppose $\mathcal{G} : [x, y] \to R_I^+$ is a harmonically (m, h_1, h_2)-Godunova–Levin interval-valued function such that $\mathcal{G}(\mathtt{u}) = [\underline{\mathcal{G}}(\mathtt{u}), \overline{\mathcal{G}}(\mathtt{u})]$, $m \in (0,1]$. Then, $\mathcal{G} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right)$ if and only if*

$$\underline{\mathcal{G}} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right) \quad \textit{and} \quad \overline{\mathcal{G}} \in SGHV\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right).$$

Proof. Let $\mathcal{G}$ be a harmonically (m, h_1, h_2)-Godunova–Levin interval valued convex function. Assume $\mathtt{c}, \mathtt{d} \in [x, y], \mathtt{u} \in (0,1)$ and $m \in (0,1]$. Then, we have

$$\frac{m\mathcal{G}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\mathcal{G}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})} \subseteq \mathcal{G}\left(\frac{m\mathtt{cd}}{m\mathtt{uc} + (1-\mathtt{u})\mathtt{d}}\right).$$

That is,

$$\left[\frac{\underline{m\mathcal{G}}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\underline{\mathcal{G}}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})}, \frac{\overline{m\mathcal{G}}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\overline{\mathcal{G}}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})}\right]$$
$$\subseteq \mathcal{G}\left(\frac{m\mathtt{cd}}{m\mathtt{uc} + (1-\mathtt{u})\mathtt{d}}\right).$$

It follows that

$$\frac{\underline{m\mathcal{G}}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\underline{\mathcal{G}}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})} \geq \underline{\mathcal{G}}\left(\frac{m\mathtt{cd}}{m\mathtt{uc} + (1-\mathtt{u})\mathtt{d}}\right),$$

and

$$\frac{\overline{m\mathcal{G}}(\mathtt{c})}{h_1(\mathtt{u})h_2(1-\mathtt{u})} + \frac{\overline{\mathcal{G}}(\mathtt{d})}{h_1(1-\mathtt{u})h_2(\mathtt{u})} \leq \overline{m\mathcal{G}}\left(\frac{m\mathtt{cd}}{m\mathtt{uc} + (1-\mathtt{u})\mathtt{d}}\right).$$

This shows that $\underline{\mathcal{G}} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right)$, and $\overline{\mathcal{G}} \in SGHV\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right)$. Conversely, let $\underline{\mathcal{G}} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right)$, and $\overline{\mathcal{G}} \in SGHV\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right)$. Based on the above definition and set inclusion, we obtain $\mathcal{G} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [x, y], R_I^+\right)$. This completes the proof. □

Proposition 2. *Suppose $\mathcal{G} : [\mathtt{c}, \mathtt{d}] \to R_I^+$ is a harmonically interval-valued (m, h_1, h_2)-Godunova–Levin function such that $\mathcal{G}(\mathtt{u}) = [\underline{\mathcal{G}}(\mathtt{u}), \overline{\mathcal{G}}(\mathtt{u})]$, $m \in (0,1]$.*

Then, $\mathcal{G} \in SGHV\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right)$ if and only if

$$\underline{\mathcal{G}} \in SGHV\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right) \quad \textit{and} \quad \overline{\mathcal{G}} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right).$$

The proof can be conducted in a similar manner as in Proposition 1; hence, it is omitted for the readers.

Hermite–Hadamard Inequalities

Throughout, $H(x,y) = h_1(x)h_2(y) \forall x, y \in (0,1)$.

Theorem 1. *Let $h_1, h_2 : (0,1) \to R^+$ and $\mathcal{G} : [\mathtt{c}, \mathtt{d}] \to R_I^+$ be an interval-valued function defined with $\underline{\mathcal{G}}, \overline{\mathcal{G}}$. If $\mathcal{G} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right)$, $m \in (0,1]$, and $\mathcal{G} \in IR_{[\mathtt{c},\mathtt{d}]}$, then*

$$\frac{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}{2} \mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c} + \mathtt{d}}\right) \supseteq \frac{m\mathtt{cd}}{\mathtt{d} - m\mathtt{c}} \int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2} d\mathtt{u} \supseteq [m\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})] \int_0^1 \frac{d\sigma}{H(\sigma, 1-\sigma)}. \quad (2)$$

Proof. Assume that $\mathcal{G} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right)$. Then,

$$\frac{\mathcal{G}(p)}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]} + \frac{\mathcal{G}(q)}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]} \subseteq \mathcal{G}\left(\frac{2pq}{p+q}\right),$$

where

$$p = \frac{m\mathtt{cd}}{m\sigma\mathtt{c} + (1-\sigma)\mathtt{d}},$$

and

$$q = \frac{m\mathtt{cd}}{m(1-\sigma)\mathtt{c} + \sigma\mathtt{d}}; \text{ here } \sigma \in [0,1].$$

Then,

$$\frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]} \left[\mathcal{G}\left(\frac{m\mathtt{cd}}{m\sigma\mathtt{c} + (1-\sigma)\mathtt{d}}\right) + \mathcal{G}\left(\frac{m\mathtt{cd}}{m(1-\sigma)\mathtt{c} + \sigma\mathtt{d}}\right)\right] \subseteq \mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c} + \mathtt{d}}\right). \quad (3)$$

Multiplying both sides by

$$H\left(\frac{1}{2}, \frac{1}{2}\right),$$

we obtain

$$\left[\mathcal{G}\left(\frac{m\mathtt{cd}}{m\sigma\mathtt{c} + (1-\sigma)\mathtt{d}}\right) + \mathcal{G}\left(\frac{m\mathtt{cd}}{m(1-\sigma)\mathtt{c} + \sigma\mathtt{d}}\right)\right] \subseteq H\left(\frac{1}{2}, \frac{1}{2}\right) \mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c} + \mathtt{d}}\right). \quad (4)$$

If we integrate the above inequality over $(0,1)$ with respect to "œ", we have

$$\int_0^1 \left[\mathcal{G}\left(\frac{m\mathtt{cd}}{m\sigma\mathtt{c} + (1-\sigma)\mathtt{d}}\right) + \mathcal{G}\left(\frac{m\mathtt{cd}}{m(1-\sigma)\mathtt{c} + \sigma\mathtt{d}}\right)\right] d\sigma \subseteq H\left(\frac{1}{2}, \frac{1}{2}\right) \int_0^1 \mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c} + \mathtt{d}}\right) d\sigma.$$

Thus,

$$\int_0^1 \underline{\mathcal{G}}\left(\frac{m\mathtt{cd}}{m\sigma\mathtt{c} + (1-\sigma)\mathtt{d}}\right) d\sigma + \int_0^1 \underline{\mathcal{G}}\left(\frac{m\mathtt{cd}}{m(1-\sigma)\mathtt{c} + \sigma\mathtt{d}}\right) d\sigma \geq H\left(\frac{1}{2}, \frac{1}{2}\right) \int_0^1 \underline{\mathcal{G}}\left(\frac{2m\mathtt{cd}}{m\mathtt{c} + \mathtt{d}}\right) d\sigma,$$

and

$$\int_0^1 \overline{\mathcal{G}}\left(\frac{m\mathtt{cd}}{m\sigma\mathtt{c} + (1-\sigma)\mathtt{d}}\right) d\sigma + \int_0^1 \overline{\mathcal{G}}\left(\frac{m\mathtt{cd}}{m(1-\sigma)\mathtt{c} + \sigma\mathtt{d}}\right) d\sigma \leq H\left(\frac{1}{2}, \frac{1}{2}\right) \int_0^1 \overline{\mathcal{G}}\left(\frac{2m\mathtt{cd}}{m\mathtt{c} + \mathtt{d}}\right) d\sigma.$$

It follows (change of variables) that

$$\frac{2m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}}\frac{\underline{\mathcal{G}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\geq H\left(\frac{1}{2},\frac{1}{2}\right)\int_0^1\underline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)d\sigma=H\left(\frac{1}{2},\frac{1}{2}\right)\underline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)$$

and

$$\frac{2m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}}\frac{\overline{\mathcal{G}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\leq H\left(\frac{1}{2},\frac{1}{2}\right)\int_0^1\overline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)d\sigma=H\left(\frac{1}{2},\frac{1}{2}\right)\overline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right).$$

As a result, we have

$$\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]\left[\underline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right),\overline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\right]\supseteq\frac{2m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}.$$

Upon dividing both sides by $\frac{1}{2}$, we obtain the desired first inclusion of Theorem 1.

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\left[\underline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right),\overline{\mathcal{G}}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\right]\supseteq\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}. \tag{5}$$

From our hypothesis, we have

$$\frac{m\mathcal{G}(\mathtt{c})}{h_1(\sigma)h_2(1-\sigma)}+\frac{\mathcal{G}(\mathtt{d})}{h_1(1-\sigma)h_2(\sigma)}\subseteq\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right).$$

$$\frac{m\mathcal{G}(\mathtt{c})}{h_1(1-\sigma)h_2(\sigma)}+\frac{\mathcal{G}(\mathtt{d})}{h_1(\sigma)h_2(1-\sigma)}\subseteq\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m(1-\sigma)\mathtt{c}+\sigma\mathtt{d}}\right).$$

Adding the above two inclusions and integrating over $(0,1)$ with respect to "σ" gives

$$\begin{aligned}[m\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_0^1\frac{1}{h_1(\sigma)h_2(1-\sigma)}d\sigma+[m\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_0^1\frac{1}{h_1(1-\sigma)h_2(\sigma)}d\sigma\\ \subseteq\int_0^1\left[\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)+\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m(1-\sigma)\mathtt{c}+\sigma\mathtt{d}}\right)\right]d\sigma.\end{aligned}$$

It is easily seen that $\int_0^1\frac{1}{h_1(\sigma)h_2(1-\sigma)}d\sigma=\int_0^1\frac{1}{h_1(1-\sigma)h_2(\sigma)}d\sigma$. This implies

$$2[m\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_0^1\frac{1}{H(\sigma,1-\sigma)}d\sigma\subseteq\frac{2m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\kappa)}{\kappa^2}d\kappa.$$

From the above developments, it follows

$$[m\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_0^1\frac{1}{H(\sigma,1-\sigma)}d\sigma\subseteq\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\kappa)}{\kappa^2}d\kappa. \tag{6}$$

Now, combining (5) and (6), we obtain the required result

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathcal{G}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\supseteq\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\kappa)d\kappa}{\kappa^2}\supseteq[m\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}.$$

□

Remark 2. (1) *If we put $h_1(\sigma)=h_2(\sigma)=1$ into Theorem 1, then it gives results for interval-valued harmonically (m,P)-functions.*

$$\frac{1}{2}\mathcal{G}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\supseteq\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})d\mathtt{u}}{\mathtt{u}^2}\supseteq[m\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]$$

(2) *If we put $m = 1$ $h_1(\sigma) = h_2(\sigma) = 1$ into Theorem 1, then it gives results for interval-valued harmonically (m, P)-functions.*

$$\frac{1}{2}\mathcal{G}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})du}{\mathtt{u}^2} \supseteq [\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})].$$

(3) *Choosing $H(\sigma, y) = h(\sigma)$, Theorem 1 gives results for harmonically (m,h)-Godunova–Levin interval-valued functions.*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2}\mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq [m\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})]\int_0^1 \frac{d\sigma}{h(\sigma)}.$$

(4) *Choosing $m = 1$ and $H(\sigma, y) = h(\sigma)$, Theorem 1 gives results for harmonically h-Godunova–Levin interval-valued functions.*

$$\frac{\left[h\left(\frac{1}{2}\right)\right]}{2}\mathcal{G}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq [\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})]\int_0^1 \frac{d\sigma}{h(\sigma)}.$$

(5) *If we choose $H(\sigma, y) = \frac{1}{h(\sigma)}$, then Theorem 1 gives results for harmonically m-interval-valued h-convex functions.*

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq [m\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})]\int_0^1 h(\sigma)d\sigma.$$

(6) *If we choose $m = 1$ and $H(\sigma, y) = \frac{1}{h(\sigma)}$, then Theorem 1 gives results for harmonically interval-valued h-convex functions.*

$$\frac{1}{2\left[h\left(\frac{1}{2}\right)\right]}\mathcal{G}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq [\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})]\int_0^1 h(\sigma)d\sigma.$$

(7) *Again, if we choose $H(\sigma, y) = \frac{1}{H(\sigma,y)}$ in Theorem 1, it recovers a result for harmonic (h_1, h_2)-convex interval-valued functions.*

$$\frac{1}{2\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}\mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq [m\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})]\int_0^1 H(\sigma, 1-\sigma)d\sigma.$$

(8) *Again, if we choose $m = 1$ and $H(\sigma, y) = \frac{1}{H(\sigma,y)}$ in Theorem 1, it recovers a result for harmonic (h_1, h_2)-convex interval-valued functions.*

$$\frac{1}{2\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}\mathcal{G}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right) \supseteq \frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq [\mathcal{G}(\mathtt{c}) + \mathcal{G}(\mathtt{d})]\int_0^1 H(\sigma, 1-\sigma)d\sigma.$$

Theorem 2. *Let $\mathcal{G} : [\mathtt{c}, \mathtt{d}] \to R_I^+$ be an interval-valued function and $h_1, h_2 : (0,1) \to R^+$ such that $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. If $\mathcal{G} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right)$, $m \in (0,1]$ and $\mathcal{G} \in IR_{[\mathtt{c},\mathtt{d}]}$; then, the following inequality holds true.*

$$\begin{aligned}
&\frac{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}{4}\mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right) \supseteq \Delta_1 \supseteq \frac{m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} \supseteq \Delta_2 \\
&\supseteq \left\{m[\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\left[\frac{1}{2}+\frac{1}{H\left(\frac{1}{2}, \frac{1}{2}\right)}\right]\right\}\int_0^1 \frac{d\sigma}{H(\sigma, 1-\sigma)},
\end{aligned}$$

where

$$\begin{aligned}
\triangle_1 &= \frac{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}{4}\left[\mathcal{G}\left(\frac{4m\mathtt{cd}}{m\mathtt{c}+3\mathtt{d}}\right)+\mathcal{G}\left(\frac{4m\mathtt{cd}}{m\mathtt{d}+3\mathtt{c}}\right)\right], \\
\triangle_2 &= \left[\mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)+m\left(\frac{\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})}{2}\right)\right]\int_0^1 \frac{d\sigma}{H(\sigma, 1-\sigma)}.
\end{aligned}$$

Proof. Assume that $\mathcal{G} \in SGHX\left(\left(\frac{1}{h_1}, \frac{1}{h_2}\right), [\mathtt{c}, \mathtt{d}], R_I^+\right)$ and $\mathcal{G} \in IR_{[\mathtt{c},\mathtt{d}]}$. For $\left[m\mathtt{c}, \frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right]$, we have

$$\frac{\mathcal{G}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m\sigma\mathtt{c}+(1-\sigma)\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}+\frac{\mathcal{G}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m(1-\sigma)\mathtt{c}+\sigma\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]} \subseteq \mathcal{G}\left(\frac{4m\mathtt{cd}}{m\mathtt{c}+3\mathtt{d}}\right),$$

and we obtain

$$\frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}\left[\mathcal{G}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m\sigma\mathtt{c}+(1-\sigma)\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)+\mathcal{G}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m(1-\sigma)\mathtt{c}+\sigma\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)\right] \subseteq \mathcal{G}\left(\frac{4m\mathtt{cd}}{m\mathtt{c}+3\mathtt{d}}\right).$$

Integrating over $(0,1)$ with respect to "σ", we have

$$\begin{aligned}
&\frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}\left[\int_0^1 \underline{\mathcal{G}}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m\sigma\mathtt{c}+(1-\sigma)\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)d\sigma+\int_0^1 \underline{\mathcal{G}}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m(1-\sigma)\mathtt{c}+\sigma\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)d\sigma,\right. \\
&\quad\left.\int_0^1 \overline{\mathcal{G}}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m\sigma\mathtt{c}+(1-\sigma)\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)d\sigma+\int_0^1 \overline{\mathcal{G}}\left(\frac{m\mathtt{c}\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}{m(1-\sigma)\mathtt{c}+\sigma\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}}\right)d\sigma\right] \subseteq \mathcal{G}\left(\frac{4m\mathtt{cd}}{m\mathtt{c}+3\mathtt{d}}\right).
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]}\left[\frac{2m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}} \frac{\underline{\mathcal{G}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}+\frac{2m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}} \frac{\underline{\mathcal{G}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}, \frac{2m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}} \frac{\overline{\mathcal{G}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\right. \\
&\qquad\left.+\frac{2m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}} \frac{\overline{\mathcal{G}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\right] \subseteq \mathcal{G}\left(\frac{4m\mathtt{cd}}{m\mathtt{c}+3\mathtt{d}}\right), \quad (7)
\end{aligned}$$

i.e.,

$$\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\frac{4m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}\frac{\underline{\mathcal{G}}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u},\frac{4m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}\frac{\overline{\mathcal{G}}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}\right]\subseteq\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right).$$

Thus,

$$\frac{4}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}\frac{\underline{\mathcal{G}}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u},\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}\frac{\overline{\mathcal{G}}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}\right]\subseteq\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right);$$

so,

$$\frac{4}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}\left[\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}\frac{\mathcal{G}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}\right]\subseteq\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right),$$

and consequently,

$$\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}\frac{\mathcal{G}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}\subseteq\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right). \tag{8}$$

Similarly, for the interval $\left[\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}},m\mathrm{d}\right]$, we can have

$$\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{\frac{2\mathrm{cd}}{\mathrm{c}+\mathrm{d}}}^{m\mathrm{d}}\frac{\mathcal{G}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}\subseteq\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{d}+3\mathrm{c}}\right). \tag{9}$$

Adding (8) and (9), we obtain

$$\triangle_1=\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\left[\mathcal{G}\left(\frac{4m\mathrm{cd}}{3\mathrm{c}+m\mathrm{d}}\right)+\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right)\right]\supseteq\left[\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\mathrm{d}}\frac{\mathcal{G}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}\right].$$

Consequently, we obtain

$$\begin{aligned}
\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathcal{G}\left(\frac{2m\mathrm{cd}}{m\mathrm{c}+\mathrm{d}}\right)&=\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathcal{G}\left(\frac{2\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\frac{4m\mathrm{cd}}{m\mathrm{d}+3\mathrm{c}}}{\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}+\frac{4m\mathrm{cd}}{m\mathrm{d}+3\mathrm{c}}}\right)\\
&\supseteq\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\left[\frac{\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{d}+3\mathrm{c}}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\\
&=\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4H\left(\frac{1}{2},\frac{1}{2}\right)}\left[\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right)+\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{d}+3\mathrm{c}}\right)\right]\\
&=\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\left[\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right)+\mathcal{G}\left(\frac{4m\mathrm{cd}}{m\mathrm{c}+3\mathrm{d}}\right)\right]\\
&=\triangle_1\\
&\supseteq\frac{m\mathrm{cd}}{\mathrm{d}-m\mathrm{c}}\int_{m\mathrm{c}}^{\mathrm{d}}\frac{\mathcal{G}(\mathrm{u})}{\mathrm{u}^2}d\mathrm{u}.
\end{aligned}$$

A minor modification of the argument in Theorem 1 will give

$$\begin{aligned}
&\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\\
&=\frac{1}{2}\left[\frac{2m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\frac{2\mathtt{c}\mathtt{d}}{\mathtt{c}+\mathtt{d}}}\frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}+\frac{2m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{\frac{2\mathtt{c}\mathtt{d}}{\mathtt{c}+\mathtt{d}}}^{m\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\right]\\
&\supseteq\frac{1}{2}\left[\left[m\mathcal{G}(\mathtt{c})+\mathcal{G}\left(\frac{2\mathtt{c}\mathtt{d}}{\mathtt{c}+\mathtt{d}}\right)\right]\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}+\left[m\mathcal{G}(\mathtt{d})+\mathcal{G}\left(\frac{2\mathtt{c}\mathtt{d}}{\mathtt{c}+\mathtt{d}}\right)\right]\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}\right]\\
&=\frac{1}{2}\left[\left\{m\mathcal{G}(\mathtt{c})+m\mathcal{G}(\mathtt{d})+2\mathcal{G}\left(\frac{2\mathtt{c}\mathtt{d}}{\mathtt{c}+\mathtt{d}}\right)\right\}\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}\right]\\
&=\left[\frac{m\mathcal{G}(\mathtt{c})+m\mathcal{G}(\mathtt{d})}{2}+\mathcal{G}\left(\frac{2\mathtt{c}\mathtt{d}}{\mathtt{c}+\mathtt{d}}\right)\right]\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}\\
&=\triangle_2\\
&\supseteq\left[\frac{m\mathcal{G}(\mathtt{c})+m\mathcal{G}(\mathtt{d})}{2}+\frac{m\mathcal{G}(\mathtt{c})}{H\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{m\mathcal{G}(\mathtt{d})}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}\\
&=\left[\frac{m\mathcal{G}(\mathtt{c})+m\mathcal{G}(\mathtt{d})}{2}+\frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}[m\mathcal{G}(\mathtt{c})+m\mathcal{G}(\mathtt{d})]\right]\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}\\
&=\left\{m[\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\left[\frac{1}{2}+\frac{1}{H\left(\frac{1}{2},\frac{1}{2}\right)}\right]\right\}\int_0^1\frac{d\sigma}{H(\sigma,1-\sigma)}.
\end{aligned}$$

This completes the proof. □

Theorem 3. *Let $h_1,h_2:(0,1)\to R^+$ and $\mathcal{G},\mathcal{V}:[\mathtt{c},\mathtt{d}]\to R_I^+$ be interval-valued functions. If $\mathcal{G},\mathcal{V}\in SGHX\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mathtt{c},\mathtt{d}],R_I^+\right)$, $m\in(0,1]$ and $\mathcal{G}\mathcal{V}\in IR_{[\mathtt{c},\mathtt{d}]}$, then, we have*

$$\begin{aligned}
&\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\\
&\qquad\supseteq\mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1\frac{1}{H^2(\sigma,1-\sigma)}d\sigma+\mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1\frac{1}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}d\sigma,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}(\mathtt{c},\mathtt{d})&=m^2\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{c})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{d}).\\
\mathcal{N}(\mathtt{c},\mathtt{d})&=m[\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{d})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{c})].
\end{aligned}$$

Proof. Assume that $\mathcal{G},\mathcal{V}\in SGHX\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mathtt{c},\mathtt{d}],R_I^+\right)$. Then, we have

$$\begin{aligned}
\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)&\supseteq\frac{m\mathcal{G}(\mathtt{c})}{h_1(\sigma)h_2(1-\sigma)}+\frac{\mathcal{G}(\mathtt{d})}{h_1(1-\sigma)h_2(\sigma)}.\\
\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)&\supseteq\frac{m\mathcal{V}(\mathtt{c})}{h_1(\sigma)h_2(1-\sigma)}+\frac{\mathcal{V}(\mathtt{d})}{h_1(1-\sigma)h_2(\sigma)}.
\end{aligned}$$

From the above hypotheses, we obtain

$$\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)$$
$$\supseteq \frac{m^2\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{c})}{H^2(\sigma,1-\sigma)}+\frac{m[\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{d})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{c})]}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}+\frac{\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{d})}{H^2(1-\sigma,\sigma)}.$$

Integrating both sides over $(0,1)$ with respect to "σ", we have

$$\begin{aligned}
&\int_0^1 \mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)d\sigma\\
=&\left[\int_0^1 \underline{\mathcal{G}}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\underline{\mathcal{V}}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)d\sigma,\right.\\
&\left.\int_0^1 \overline{\mathcal{G}}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\overline{\mathcal{V}}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)d\sigma\right]\\
=&\left[\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\underline{\mathcal{G}}(\mathtt{u})\underline{\mathcal{V}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u},\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\overline{\mathcal{G}}(\mathtt{u})\overline{\mathcal{V}}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\right]\\
=&\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\\
\supseteq&\int_0^1\frac{[m^2\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{c})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{d})]}{H^2(\sigma,1-\sigma)}d\sigma+\int_0^1\frac{m[\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{d})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{c})]}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}d\sigma.
\end{aligned}$$

It follows that

$$\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\supseteq\mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1\frac{d\sigma}{H^2(\sigma,1-\sigma)}+\mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1\frac{d\sigma}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}.$$

The Theorem is proved. □

Theorem 4. *Let $h_1,h_2:(0,1)\to R^+$ and $\mathcal{G},\mathcal{V}:[\mathtt{c},\mathtt{d}]\to R_I^+$ be interval-valued functions. If $\mathcal{G},\mathcal{V}\in SGHX\left(\left(\frac{1}{h_1},\frac{1}{h_2}\right),[\mathtt{c},\mathtt{d}],R_I^+\right)$, $m\in(0,1]$ and $\mathcal{G}\mathcal{V}\in IR_{[\mathtt{c},\mathtt{d}]}$, then, we have*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{1+m^2}\mathcal{G}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\supseteq\frac{m\mathtt{c}\mathtt{d}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}$$
$$+\mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1\frac{1+m^2}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}d\sigma+\mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1\frac{m}{H^2(\sigma,1-\sigma)}d\sigma.$$

Proof. By hypothesis, one has

$$\mathcal{G}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\supseteq\frac{\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}.$$
$$\mathcal{V}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\supseteq\frac{\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}+\frac{\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)}{H\left(\frac{1}{2},\frac{1}{2}\right)}.$$

Then,

$$
\begin{aligned}
\mathcal{G}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\frac{2m\mathtt{c}\mathtt{d}}{m\mathtt{c}+\mathtt{d}}\right) \supseteq{}& \frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\right.\\
&\left.+\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\right]\\
&+\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\right.\\
&\left.+\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\supseteq{}& \frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\right.\\
&\left.+\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\right]\\
&+\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\left(\frac{m\mathcal{G}(\mathtt{c})}{H(\sigma,1-\sigma)}+\frac{\mathcal{G}(\mathtt{d})}{H(1-\sigma,\sigma)}\right)\left(\frac{m\mathcal{V}(\mathtt{c})}{H(1-\sigma,\sigma)}+\frac{\mathcal{V}(\mathtt{d})}{H(\sigma,1-\sigma)}\right)\right.\\
&\left.+\left(\frac{\mathcal{G}(\mathtt{c})}{H(1-\sigma,\sigma)}+\frac{m\mathcal{G}(\mathtt{d})}{H(\sigma,1-\sigma)}\right)\left(\frac{\mathcal{V}(\mathtt{c})}{H(\sigma,1-\sigma)}+\frac{m\mathcal{V}(\mathtt{d})}{H(1-\sigma,\sigma)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
={}& \frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\right.\\
&\left.+\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\right]\\
&+\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\left(\frac{1+m^2}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}\right)[\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{c})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{d})]\right.\\
&\left.+m\left(\frac{1}{H^2(\sigma,1-\sigma)}+\frac{1}{H^2(1-\sigma,\sigma)}\right)[\mathcal{G}(\mathtt{c})\mathcal{V}(\mathtt{d})+\mathcal{G}(\mathtt{d})\mathcal{V}(\mathtt{c})]\right]
\end{aligned}
$$

$$
\begin{aligned}
={}& \frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{c}+(1-\sigma)\mathtt{d}}\right)\right.\\
&\left.+\mathcal{G}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\mathcal{V}\left(\frac{m\mathtt{c}\mathtt{d}}{m\sigma\mathtt{d}+(1-\sigma)\mathtt{c}}\right)\right]\\
&+\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\left(\frac{1+m^2}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}\right)\mathcal{M}(\mathtt{c},\mathtt{d})\right.\\
&\left.+m\left(\frac{1}{H^2(\sigma,1-\sigma)}+\frac{1}{H^2(1-\sigma,\sigma)}\right)\mathcal{N}(\mathtt{c},\mathtt{d})\right].
\end{aligned}
$$

Integrating with respect to "σ" over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \mathcal{G}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)d\sigma\\
&= \left[\int_0^1 \underline{\mathcal{G}}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\tfrac{2m\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right)d\sigma, \int_0^1 \overline{\mathcal{G}}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)d\sigma\right] = \mathcal{G}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\tfrac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\\
&\supseteq \frac{1+m^2}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\frac{m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}\right]\\
&+\frac{1}{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1 \frac{1+m^2 d\sigma}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)} + \mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1 \frac{m d\sigma}{H^2(\sigma,1-\sigma)}\right].
\end{aligned}$$

Multiplying both sides of the above equation by $\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}$, we obtain

$$\begin{aligned}
&\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{1+m^2}\mathcal{G}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\frac{2m\mathtt{cd}}{m\mathtt{c}+\mathtt{d}}\right)\\
&\qquad \supseteq \frac{m\mathtt{cd}}{\mathtt{d}-m\mathtt{c}}\int_{m\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} + \mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1 \frac{1+m^2}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}d\sigma\\
&\qquad\qquad + \mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1 \frac{m d\sigma}{H^2(\sigma,1-\sigma)}.
\end{aligned}$$

This completes the proof. □

4. Examples

Example 1. *Let $m = 1$, $h_1(\sigma) = \frac{1}{\sigma}$, and $h_2(\sigma) = 1$ where $\sigma \in (0,1), [\mathtt{c},\mathtt{d}] = \left[\frac{1}{2},1\right]$ and $\mathcal{G} : [\mathtt{c},\mathtt{d}] \to R_I^+$ is defined by $\mathcal{G}(\mathtt{u}) = \left[\mathtt{u}^2, 5-e^{\mathtt{u}}\right]$. Then,*

$$\begin{aligned}
&\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2} f\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right) = \mathcal{G}\left(\frac{2}{3}\right) \approx \left[\frac{4}{9}, 5-e^{\frac{2}{3}}\right].\\
&\frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}} \frac{\mathcal{G}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u} = \left[\int_{\frac{1}{2}}^1 d\mathtt{u}, \int_{\frac{1}{2}}^1 \frac{(5-e^{\mathtt{u}})}{\mathtt{u}^2}d\mathtt{u}\right] \approx \left[\frac{1}{2}, 2.98\right].\\
&[\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_{\frac{1}{2}}^1 \frac{d\sigma}{H(\sigma,1-\sigma)} = [\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})]\int_{\frac{1}{2}}^1 \sigma d\sigma \approx \left[\frac{15}{32}, \frac{3(10-\sqrt{e}-e)}{8}\right].
\end{aligned}$$

Thus, we obtain

$$\left[\frac{4}{9}, 5-e^{\frac{2}{3}}\right] \supseteq \left[\frac{1}{2}, 2.98\right] \supseteq \left[\frac{15}{32}, \frac{3(10-\sqrt{e}-e)}{8}\right].$$

This verifies Theorem 1.

Example 2. *For $m = 1$, $h_1(\sigma) = \frac{1}{\sigma}$, and $h_2(\sigma) = 1$, where $\sigma \in (0,1), [\mathtt{c},\mathtt{d}] = \left[\frac{1}{2},1\right]$ and $\mathcal{G} : [\mathtt{c},\mathtt{d}] \to R_I^+$ is defined as $\mathcal{G}(\mathtt{u}) = \left[\mathtt{u}^2, 4-e^{\mathtt{u}}\right]$. Then,*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{4}\mathcal{G}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right)=\mathcal{G}\left(\frac{2}{3}\right)\approx\left[\frac{4}{9},4-e^{\frac{2}{3}}\right].$$

$$\begin{aligned}\triangle_1&=\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{4}\left[\mathcal{G}\left(\frac{4\mathtt{cd}}{\mathtt{c}+3\mathtt{d}}\right)+\mathcal{G}\left(\frac{4\mathtt{cd}}{\mathtt{d}+3\mathtt{c}}\right)\right]\\&=\frac{1}{2}\left[\left[\frac{16}{49},4-e^{\frac{4}{7}}\right]+\left[\frac{16}{25},4-e^{\frac{4}{5}}\right]\right]\approx\left[\frac{592}{1225},\frac{8-e^{\frac{4}{7}}-e^{\frac{4}{5}}}{2}\right].\\\triangle_2&=\left[f\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right)+\left(\frac{\mathcal{G}(\mathtt{c})+\mathcal{G}(\mathtt{d})}{2}\right)\right]\int_{\frac{1}{2}}^{1}\frac{d\sigma}{H(\sigma,1-\sigma)}\\&=\frac{3}{8}\left[\mathcal{G}\left(\frac{2}{3}\right)+\left(\frac{\mathcal{G}\left(\frac{1}{2}\right)+\mathcal{G}(1)}{2}\right)\right]\approx\left[\frac{77}{192},\frac{-3e-3\sqrt{e}-6e^{\frac{2}{3}}+48}{16}\right].\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\left[\tfrac{4}{9},4-e^{\frac{2}{3}}\right]\quad&\supseteq\left[\tfrac{592}{1225},\tfrac{8-e^{\frac{4}{7}}-e^{\frac{4}{5}}}{2}\right]\supseteq\left[\tfrac{1}{2},2.98\right]\\&\supseteq\left[\tfrac{77}{192},\tfrac{-3e-3\sqrt{e}-6e^{\frac{2}{3}}+48}{16}\right].\end{aligned}$$

This verifies Theorem 2.

Example 3. *Let $m=1$, $h_1(\sigma)=\frac{1}{\sigma}$, and $h_2(\sigma)=1$, where $\sigma\in(0,1),[\mathtt{c},\mathtt{d}]=\left[\frac{1}{2},1\right]$, and $\mathcal{G},\mathcal{V}:[\mathtt{c},\mathtt{d}]\to R_I^+$ are defined as $\mathcal{G}(\mathtt{u})=\left[\mathtt{u}^2,4-e^{\mathtt{u}}\right]$ and $\mathcal{V}(\mathtt{u})=\left[\mathtt{u},3-\mathtt{u}^2\right]$. Then,*

$$\begin{aligned}&\frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}=\left[\int_{\frac{1}{2}}^{1}\mathtt{u}d\mathtt{u},\int_{\frac{1}{2}}^{1}\frac{(4-e^{\mathtt{u}})(3-\mathtt{u}^2)}{\mathtt{u}^2}d\mathtt{u}\right]\approx\left[\frac{3}{8},5.0094\right].\\&\mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1\frac{1}{H^2(\sigma,1-\sigma)}d\sigma=\mathcal{M}\left(\frac{1}{2},1\right)\int_0^1\sigma^2d\sigma\approx\left[\frac{9}{24},\frac{19}{3}-\frac{2\sqrt{e}}{3}-\frac{11e}{12}\right].\\&\mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1\frac{1}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}d\sigma=\mathcal{N}\left(\frac{1}{2},1\right)\int_0^1\left(\sigma-\sigma^2\right)d\sigma\approx\left[\frac{3}{24},\frac{19}{6}-\frac{11\sqrt{e}}{24}-\frac{e}{3}\right].\end{aligned}$$

It follows that

$$\begin{aligned}\left[\frac{3}{8},5.0094\right]\supseteq&\left[\frac{9}{24},\frac{19}{3}-\frac{2\sqrt{e}}{3}-\frac{11e}{12}\right]+\left[\frac{3}{24},\frac{19}{6}-\frac{11\sqrt{e}}{24}-\frac{e}{3}\right]\\=&\left[\frac{1}{2},\frac{19}{6}+\frac{-30e-27\sqrt{e}+152}{24}\right].\end{aligned}$$

This verifies Theorem 3.

Example 4. *Let $m=1$, $h_1(\sigma)=\frac{1}{\sigma}$, and $h_2(\sigma)=1$, where $\sigma\in(0,1),[\mathtt{c},\mathtt{d}]=\left[\frac{1}{2},1\right]$, and $\mathcal{G},\mathcal{V}:[\mathtt{c},\mathtt{d}]\to R_I^+$ are defined as $\mathcal{G}(\mathtt{u})=\left[\mathtt{u}^2,4-e^{\mathtt{u}}\right]$ and $\mathcal{V}(\mathtt{u})=\left[\mathtt{u},3-\mathtt{u}^2\right]$. Then,*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathcal{G}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right)\mathcal{V}\left(\frac{2\mathtt{cd}}{\mathtt{c}+\mathtt{d}}\right)=2\mathcal{G}\left(\frac{2}{3}\right)\mathcal{V}\left(\frac{2}{3}\right)\approx\left[\frac{16}{27},\frac{46\left(4-e^{\frac{2}{3}}\right)}{9}\right].$$

$$\frac{\mathtt{cd}}{\mathtt{d}-\mathtt{c}}\int_{\mathtt{c}}^{\mathtt{d}}\frac{\mathcal{G}(\mathtt{u})\mathcal{V}(\mathtt{u})}{\mathtt{u}^2}d\mathtt{u}=\left[\int_{\frac{1}{2}}^{1}\mathtt{u}d\mathtt{u},\int_{\frac{1}{2}}^{1}\frac{(4-e^{\mathtt{u}})(3-\mathtt{u}^2)}{\mathtt{u}^2}d\mathtt{u}\right]\approx\left[\frac{3}{8},5.0094\right].$$

$$\mathcal{M}(\mathtt{c},\mathtt{d})\int_0^1\frac{1}{H(\sigma,\sigma)H(1-\sigma,1-\sigma)}d\sigma=\mathcal{M}\left(\frac{1}{2},1\right)\int_0^1\left(\sigma-\sigma^2\right)d\sigma=\left[\frac{9}{48},\frac{19}{6}-\frac{\sqrt{e}}{3}-\frac{11e}{24}\right].$$

$$\mathcal{N}(\mathtt{c},\mathtt{d})\int_0^1\frac{1}{H^2(\sigma,1-\sigma)}d\sigma=\mathcal{N}\left(\frac{1}{2},1\right)\int_0^1\sigma^2d\sigma\approx\left[\frac{3}{12},\frac{19}{3}-\frac{11\sqrt{e}}{12}-\frac{2e}{3}\right].$$

It follows that

$$\begin{aligned}\left[\frac{16}{27},\frac{184-46e^{\frac{2}{3}}}{9}\right]&\supseteq\left[\frac{3}{8},5.0094\right]+\left[\frac{9}{48},\frac{19}{6}-\frac{\sqrt{e}}{3}-\frac{11e}{24}\right]\\&\quad+\left[\frac{3}{12},\frac{19}{3}-\frac{11\sqrt{e}}{12}-\frac{2e}{3}\right]\\&=\left[\frac{13}{16},\frac{19}{6}+\frac{-27e-30\sqrt{e}+152}{24}+5.0094\right].\end{aligned}$$

This verifies Theorem 4.

5. Conclusions

Interval-valued functions may be a valuable alternative for incorporating uncertainty into the prediction processes. We first introduced a new notion of interval-valued harmonic convexity, i.e., a harmonically interval-valued (h_1, h_2)-Godunova–Levin function and established the Hermite–Hadamard and Pachpatt- type inequalities employing this new notion. Our new definition generalized many existing definitions present in the literature. We thereby added to the extension of many classical integral inequalities in the set-valued setting. Some numerical examples were provided to further explain the results.

Future presentations of various inequalities, including those of the Hermite–Hadamard, Ostrowski, Hadamard–Mercer, Simpson, Fejér, and Bullen types, could make use of this novel idea. The inequalities indicated above can be demonstrated for a variety of interval-valued LR convexities, fuzzy interval convexities, and CR convexities. These findings will also be generalized in relation to quantum calculus, coordinated interval-valued functions, fractional calculus, etc. Due to the fact that these are the most active areas of study in the subject of integral inequalities, many mathematicians will be interested in investigating how different types of interval-valued analysis might be applied to the integral inequalities.

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