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Symmetries and Solutions for Some Classes of Advective Reaction–Diffusion Systems

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Abstract: In this paper, we consider some reaction–advection–diffusion systems in order to obtain exact solutions via a symmetry approach. We write the determining system of a general class. Then, for particular subclasses, we obtain special forms of the arbitrary constitutive parameters that allow us to extend the principal Lie algebra. In some cases, we write the corresponding reduced system and we find special exact solutions.

Keywords: reaction–diffusion–advection equations; symmetries; exact solutions

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1. Introduction

In several of our previous papers, we analysed, in the framework of symmetry methods, systems belonging to some subclasses of the following pde systems

$$\begin{cases} u_t = D_x(f(u,v)u_x) + g(u,v,u_x), \\ v_t = h(u,v) \end{cases} \tag{1}$$

with them in mind, generally, regarding models of population dynamics. With D_x, we denote the total differentiation with respect to x, while, usually, the subscripts denote the partial differentiations with respect to the indicated variables. Finally, in the following, the prime will denote the differentiation of functions with respect to their single argument.

The first equation, an advection–reaction–diffusion equation, is obtained by introducing a Fick–Fourier flux in the balance equation for the species of density u; the diffusion coefficient f could depend not only on u but also on density v [1,2], of a second species interacting with u. The function g may not always depend on the advection u_x if the species u does not detect external *stimuli* (usually water currents or wind); this occurs in the *swarming* cells in the *Proteus mirabilis* colonies.

The second equation is concerned with the species of density v that is assumed to be subject to neither diffusion nor advection, as, for instance, *Proteus mirabilis* [2–5] or mosquito models (*Aedes Aegypt* [6–9], *Anopheles* [10,11]), respectively, for *swinging cells* or the so-called *aquatic populations*.

Here, we restrict ourselves to the subclass of phenomena modelled by the following partial differential equation system:

$$\begin{cases} u_t = D_x(f(u)u_x) + g(u,v,u_x), \\ v_t = h(u,v), \end{cases} \tag{2}$$

where, in general, there is a population of mosquitos of density u interacting in more or less a weak manner with an aquatic population of density v. Moreover, analogous systems could be found in several other fields, for instance, in the framework of the models dealing with chemotaxis [12,13]. The motivation of the Lie symmetry analysis approach is twofold. The first one, well known, is that the Lie symmetries offer a methodological way to derive exact solutions by reducing the number of dependent variables. The second one, not negligible, is that the discussion of the equation determining the symmetries could bring about requests for special forms of some constitutive functions in order to admit additional symmetries. This request, sometimes, creates an interesting feedback between the different constitutive parameters of the model and its sensitivity.

In (2), the arbitrary functions $f(u)$, $g(u,v,u_x)$, and $h(u,v)$ appear. Once we have obtained the so-called determining system, we will obtain some restrictions for the coordinates of the symmetry generator, and three classifying equations involving the unknown coordinates and the arbitrary elements of the system. One equation involves only the function f and another one only the function h. The last condition is more complex, because it involves both f and g, having each different functional dependence. Therefore, we must introduce some simplifying constitutive assumptions. In the following, we consider some systems of the subclass (2) where, for the sake of simplicity, we assume

$$f(u) \in \{f_0, f_0 u\},$$

and

$$g = A(u)u_x + \Pi(u) + \Gamma(v),$$

with

$$\begin{aligned} &A(u) \in \{A_0, A_0 u\}, \\ &\Pi(u) \in \{\gamma_1 u(\gamma_2 - u), \gamma_1 u(\gamma_2 - \ln u)\}, \end{aligned}$$

while the constitutive functions $\Gamma(v)$ and $h(u,v)$ remain free to perform a suitable classification with respect to them, which allows us to obtain additional symmetries. Once known, the classifying equations of (2) are obtained with arbitrary f, g, and h; by introducing some specializations of them, it is possible to ascertain the existence of extensions with respect to the principal Lie algebra $\mathcal{L}_{\mathcal{P}}$ [14] for some special form of the functions $\Gamma(v)$ and $h(u,v)$.

In this work, we analyse three systems of the type (2) and several classes of exact solutions are found. To the best of our knowledge, the literature regarding group analysis applications to similar systems is quite poor, so we believe that our results are new. The outline of the paper is the following. In Section 2, we write the determining system for the class (2) and we establish the principal Lie algebra. Systems with linear advection are studied in Section 3, while, in Section 4, we consider a system with diffusion and advection coefficients linear in u. Some remarks about the constitutive functions Γ and h are given in Section 5. Finally, in the last section, the conclusions are shown.

2. Symmetries: Classifying Equations

We look for a symmetry generator of the form

$$X = \xi^1(x,t,u,v)\partial_x + \xi^2(x,t,u,v)\partial_t + \eta^1(x,t,u,v)\partial_u + \eta^2(x,t,u,v)\partial_v, \tag{3}$$

that leaves the system (2) invariant. Then, we require that

$$\begin{cases} X^{(2)}(-u_t + D_x(f(u)u_x) + g(u,v,u_x))\,|_{(2)} = 0, \\ \\ X^{(2)}(-v_t + h(u,v))\,|_{(2)} = 0, \end{cases} \tag{4}$$

where the operator $X^{(2)}$ is

$$X^{(2)} \quad = \quad X + \zeta^1_t \partial_{u_t} + \zeta^1_x \partial_{u_x} + \zeta^1_{xx} \partial_{u_{xx}} + \zeta^2_t \partial_{v_t}. \tag{5}$$

As usual, the expressions of the coordinates ζ_t^1, ζ_x^1, ζ_{xx}^1, ζ_t^2 are (see, e.g., [15–22])

$$\zeta_t^1 = D_t(\eta_1) - u_t D_t(\xi_1) - u_x D_t(\xi_2), \tag{6}$$
$$\zeta_x^1 = D_x(\eta_1) - u_t D_x(\xi_1) - u_x D_x(\xi_2), \tag{7}$$
$$\zeta_{xx}^1 = D_x(\zeta_x^1) - u_{xt} D_x(\xi_1) - u_{xx} D_x(\xi_2), \tag{8}$$
$$\zeta_t^2 = D_t(\eta_2) - v_t D_t(\xi_1) - v_x D_t(\xi_2), \tag{9}$$

where the operators D_x, and D_t are the total derivatives with respect to x and t, respectively. The conditions (4) give us the system of determining equations for the unknown coordinates ξ_1, ξ_2, η_1, and η_2. By solving only the equations that do not depend on the form of the arbitrary elements f, g, and h, we obtain the following restrictions about the coordinates ξ_1, ξ_2, η_1, and η_2:

$$\xi_1 = \alpha(x), \quad \xi_2 = \beta(t), \quad \eta_1 = \phi(x,t,u), \quad \eta_2 = \psi(x,t,v). \tag{10}$$

The generator (3) with the coordinates (10) will be a symmetry generator for the system (2) if the functions α, β, ϕ, and ψ satisfy the following remaining determining equations, where the arbitrary elements f, g, and h appear:

$$\begin{cases} (2\alpha' - \beta')f - \phi f_u = 0, \\ (f'g + u_x^2 f'^2)\phi - \left[\phi_{uu}u_x^2 - (\alpha'' - 2\phi_{xu})u_x + \phi_{xx}\right]f^2 - \psi g_v f - \phi g_u f + \\ \quad (\phi_u - 2\alpha')gf - \phi_t f - \phi u_x^2 f'' f - (2\phi_x + \phi_u u_x)u_x f' f + \\ \quad ((\alpha' - \phi_u)u_x - \phi_x)g_{u_x} f = 0, \\ (\psi_v - \beta')h - \phi h_u - \psi h_v + \psi_t = 0. \end{cases} \tag{11}$$

It is a simple matter to derive that, for f, g, and h arbitrary, we obtain

$$\alpha' = 0, \; \beta' = 0, \; \phi = 0, \; \psi = 0. \tag{12}$$

Then, we can affirm that the principal Lie algebra $L_{\mathcal{P}}$ is spanned by the time translations and space translations

$$X_1 = \partial_x, \qquad X_2 = \partial_t.$$

3. Symmetries and Solutions for Systems with Linear Advection

Here, we assume $f = f_0$ and $g = Au_x + \Pi(u) + \Gamma(v)$, with $f_0 \neq 0$ and $\Gamma' \neq 0$, so the system (2) becomes

$$\begin{cases} u_t = f_0 u_{xx} + Au_x + \Pi(u) + \Gamma(v), \\ \\ v_t = H(u,v). \end{cases} \tag{13}$$

Remark 1. *We recall that the following change in independent and dependent variables*

$$\tau = t, \qquad z = x + At, \qquad u = u(\tau, z) \qquad v = v(\tau, z) \tag{14}$$

brings the system (13) in the following form

$$\begin{cases} u_\tau = f_0 u_{zz} + \Pi(u) + \Gamma(v), \\ \\ v_\tau = -Av_z + H(u,v). \end{cases} \tag{15}$$

We can find the symmetry generator for the class (13) by solving (11) after having input the special forms of the functions f, g, and h. Taking into account that, for the system (13),

$$\begin{cases} f = f_0, \\ g = Au_x + \Pi(u) + \Gamma(v), \\ h = H(u,v), \end{cases} \tag{16}$$

the conditions (11) become

$$
\begin{aligned}
&(2\alpha' - \beta')f_0 = 0, && (17)\\
&-\left[\phi_{uu}u_x^2 - (\alpha'' - 2\phi_{xu})u_x + \phi_{xx}\right]f_0^2 - \psi\Gamma' f_0 - \phi\Pi' f_0 + \\
&(\phi_u - 2\alpha')(Au_x + \Pi(u) + \Gamma(v))f_0 + \phi_t f_0 + \\
&((\alpha' - \phi_u)u_x - \phi_x)Af_0 = 0, && (18)\\
&(\psi_v - \beta')H - \phi H_u - \psi H_v + \psi_t = 0. && (19)
\end{aligned}
$$

From (17), we have

$$\alpha(x) = c_0 + c_1 x, \quad \beta(t) = 2c_1 t + c_2, \tag{20}$$

while, taking into account that no function depends on u_x, from (18), we obtain

$$
\begin{aligned}
&\phi_{uu} = 0,\ c_1 A + 2\phi_{xu} f_0 = 0, && (21)\\
&\phi_{xx} f_0 + \psi\Gamma' + \phi\Pi' + (2c_1 - \phi_u)(\Pi + \Gamma) - \phi_t + \phi_x A = 0. && (22)
\end{aligned}
$$

From conditions (21), it follows that

$$\phi(x,t,u) = u\left(\phi_1(t) - \frac{c_1 A x}{2 f_0}\right) + \phi_2(x,t). \tag{23}$$

In the condition (22), the arbitrary elements $\Pi(u)$ and $\Gamma(v)$ appear.
We study (13) by specializing $\Pi(u)$ in these two special forms

$$\Pi(u) = \gamma_1 u(\gamma_2 - u),\ \Pi(u) = \gamma_1 u(\gamma_2 - \ln u).$$

3.1. Logistic u- Production

A typical choice for the function $\Pi(u)$ is given by the logistic form

$$\Pi(u) = \gamma_1 u(\gamma_2 - u), \tag{24}$$

but taking into account (23) and this form of $\Pi(u)$, from the condition (22), we obtain

$$c_1 = 0,\ \phi_1(t) = 0,\ \phi_2(x,t) = 0,\ \psi(x,t,v) = 0, \tag{25}$$

i.e., extensions of the principal Lie algebra do not exist.
Thus, we look for invariant solutions of the system

$$
\begin{cases}
u_t = f_0 u_{xx} + Au_x + \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\
v_t = H(u,v).
\end{cases} \tag{26}
$$

By using the generator

$$X = c_1 X_1 + c_2 X_2, \tag{27}$$

the solutions are

$$u = U(\sigma), \qquad v = V(\sigma),$$

where $\sigma = c_2 x - c_1 t$, and with U and V solutions of the following reduced system

$$
\begin{cases}
c_1 U' + f_0 U'' + c_2 AU' + \gamma_1 U(\gamma_2 - U) + \Gamma(V) = 0, \\
cV' = H(U,V).
\end{cases} \tag{28}
$$

If we specialize

$$\Gamma(v) = \gamma_2 v, \qquad h(u,v) = \frac{\gamma_2 v - \gamma_3}{\gamma_2}(u - 2\gamma_0),$$

the previous reduced system becomes

$$\begin{cases} c_1 U' + f_0 U'' + c_2 A U' + \gamma_1 U(\gamma_2 - U) - \gamma_2 V = 0, \\ \\ cV' = (V - \frac{\gamma_3}{\gamma_2})(U - 2\gamma_0). \end{cases} \tag{29}$$

If $U = 2\gamma_0$, V must be constant too and we obtain $V = 2\gamma_0\gamma_1\left(\gamma_2 - \frac{2\gamma_0}{\gamma_2}\right)$.
Instead, if $V = \dfrac{\gamma_3}{\gamma_2}$, then U must satisfy the following *autonomous equation*:

$$f_0 U'' + (c_1 + c_2 A)U' + \gamma_1 U(\gamma_2 - U) + \gamma_3 = 0. \tag{30}$$

After having performed the substitution $Z(U) = U'$, equation (30) becomes the following Abel equation of the second kind in the canonical form [23,24]:

$$f_0 Z Z' + (c_1 + c_2 A)Z + \gamma_1 U(\gamma_2 - U) + \gamma_3 = 0. \tag{31}$$

From this, after a suitable choice of arbitrary constants c_1 and c_2, we obtain the following separable variable equation:

$$f_0 Z Z' + \gamma_1 U(\gamma_2 - U) + \gamma_3 = 0. \tag{32}$$

3.2. Log-Logistic u- Production

If we want to obtain some extensions of the principal Lie algebra but at the same time a form of $\Pi(u)$ with the characteristics of the logistic form, we can choose

$$\Pi(u) = \gamma_1 u(\gamma_2 - \ln u). \tag{33}$$

Then, we consider

$$\begin{cases} u_t = f_0 u_{xx} + A u_x + \gamma_1 u(\gamma_2 - \ln u) + \Gamma(v), \\ \\ v_t = H(u, v). \end{cases} \tag{34}$$

We recall that for the system (13), we have already obtained

$$\alpha(x) = c_0 + c_1 x, \quad \beta(t) = 2c_1 t + c2, \quad \phi(x,t,u) = u\left(\phi_1(t) - \frac{c_1 A x}{2 f_0}\right) + \phi_2(x,t), \tag{35}$$

and the following conditions

$$(Ac_1 x + 4c_1 f_0 - 2f_0\phi_1)(\Pi + \Gamma) + (2\phi_2 f_0 + 2\phi_1 u f_0 - Auc_1 x)\Pi' + \\ 2A\phi_{2x} f_0 - (A^2 c_1 + 2\phi_1' f_0)u + \phi_{2xx} f_0^2 + 2\psi\Gamma' f_0 - 2\phi_{2t} f_0 = 0, \tag{36}$$

$$(\psi_v - \beta')H - \left(u\left(\phi_1 - \frac{c_1 A x}{2 f_0}\right) + \phi_2\right)H_u - \psi H_v + \psi_t = 0. \tag{37}$$

Substituting (33), the condition (36) reads

$$(Ac_1\gamma_1 x + 4c_1\gamma_1\gamma_2 f_0 - A^2 c_1 - 2f_0\gamma_1\phi_1 - 2\phi_1' f_0)U - 4c_1 f_0\gamma_1 u \ln u + \\ -2\phi_2 f_0\gamma_1 \ln u + Ac_1 x\Gamma + 2\phi_2 f_0\gamma_1\gamma_2 + 2A\phi_{2x} f_0 + 2\phi_{2xx} f_0^2 + \\ -2\phi_2 f_0\gamma_1 - 2\phi_1\Gamma f_0 + 2\psi\Gamma' f_0 + 4c_1 f_0\Gamma - 2\phi_{2t} f_0 = 0. \tag{38}$$

Taking into account that in the condition (38), no function depends on u, we obtain

$$c_1 = 0,\ \gamma_1\phi_1 - \phi_1' = 0,\ \phi_2 = 0,\ \phi_1\Gamma f_0 - \psi\Gamma' f_0 + \phi_{2t} f_0 = 0, \tag{39}$$

i.e.,

$$\phi_1 = c_3 e^{-\gamma_1 t}, \ \psi = c_3 \frac{e^{-\gamma_1}\Gamma}{\Gamma'}. \tag{40}$$

Finally, by substituting these results into the condition (37), it becomes

$$c_3 e^{-\gamma_1 t}\Big[(H_u u - H)\Gamma'^2 + (H_v + \gamma_1)\Gamma\Gamma' + h\Gamma\Gamma''\Big] = 0. \tag{41}$$

We obtain the following extension of $L_{\mathcal{P}}$

$$X_3 = e^{-\gamma_1 t}\left(u\partial_u + \frac{\Gamma}{\Gamma'}\partial_v\right) \tag{42}$$

only when

$$H(u,v) = (\Phi(\omega) - \gamma_1 \ln \Gamma)\frac{\Gamma}{\Gamma'} \tag{43}$$

with $\omega = \dfrac{u}{\Gamma}$.

Remark 2. *We obtain the same result even if $A = 0$, i.e., without advection.*

If we use the generator

$$X = c_1 X_2 + c_2 X_1 + c_3 X_3, \tag{44}$$

with $c_1 \neq 0$, the invariant solutions are

$$\begin{aligned} u(t,x) &= e^{-\frac{c_3}{c_1\gamma_1}e^{-\gamma_1 t}} U(\sigma), \\ \Gamma(v(t,x))) &= e^{-\frac{c_3}{c_1\gamma_1}e^{-\gamma_1 t}} V(\sigma), \end{aligned} \tag{45}$$

where $\sigma = c_1 x - c_2 t$. The functions U and V are solutions of the reduced system

$$\begin{cases} -c_2 U' = c_1^2 f_0 U'' + c_1 A U' + \gamma_1 U(\gamma_2 - \ln U) + V, \\ -c_2 V' = \left(\Phi\left(\frac{U}{V}\right) - \gamma_1 \ln V\right) V. \end{cases} \tag{46}$$

In order to have a reduced system of the first order, we use the generator

$$X = c_2 X_1 + c_3 X_3, \tag{47}$$

with $c_2 \neq 0$. In this case, the invariant solutions are

$$\begin{aligned} u(t,x) &= e^{\frac{c_3}{c_2}x e^{-\gamma_1 t}} U(t), \\ \Gamma(v(t,x))) &= e^{\frac{c_3}{c_2}x e^{-\gamma_1 t}} V(t). \end{aligned} \tag{48}$$

The functions U and V are solutions of the reduced system

$$\begin{cases} U' = \frac{c_3^2}{c_2^2} e^{-2\gamma_1 t} f_0 U + \frac{c_3}{c_2} e^{-\gamma_1 t} A U + \gamma_1 U(\gamma_2 - \ln U) + V, \\ V' = \left(\Phi\left(\frac{U}{V}\right) - \gamma_1 \ln V\right) V. \end{cases} \tag{49}$$

In this case, if we consider

$$\Phi\left(\frac{U}{V}\right) = p_0 + p_1\frac{V}{U} + p_2\frac{\sqrt{V}}{\sqrt{U}} + p_3\ln\left(\frac{U}{V}\right), \tag{50}$$

we can look for solutions of type

$$U(t) = u_0 e^{f_1(t)} \quad \text{and} \quad V(t) = v_0 e^{-2\gamma_1} e^{f_1(t)}. \tag{51}$$

If we take

$$\begin{aligned} p_0 &= \gamma_1(\gamma_2 - 2), \\ p_1 &= \frac{f_0 u_0^2 p_2^2}{A^2 v_0^2} - 1, \\ p_3 &= -\gamma_1, \end{aligned} \tag{52}$$

by choosing

$$c_2 = \frac{A\sqrt{v_0}c_3}{\sqrt(u_0)p_2}, \tag{53}$$

we obtain

$$f_1(t) = \gamma_2 - \ln(u_0) + \left(\frac{\sqrt{u_0}}{\sqrt{v_0}}p_2 t + k\right)e^{-\gamma_1 t} - \frac{A^2v_0^2 + f_0p_2^2u_0^2}{A^2\gamma_1 u_0 v_0}e^{-2\gamma_1 t}, \tag{54}$$

with k arbitrary constant. Once we have obtained $U(t)$ and $V(t)$, from (48), we obtain directly $u(t,x)$. To find $v(t,x)$, we must invert the assigned function Γ (numerically or analytically).

4. Symmetries and Solutions for Systems with Non-Linear Advection

We consider here the following class A-RD systems

$$\begin{cases} u_t = (f_0 u u_x)_x + A u u_x + \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\ \\ v_t = H(u,v), \end{cases} \tag{55}$$

with $f_0 \neq 0$ and $\Gamma' \neq 0$. In this case, we have

$$\begin{cases} f = f_0 u, \\ g = A u u_x + \gamma_1 u(\gamma_2 - u) + \Gamma(v), \\ h = H(u,v). \end{cases} \tag{56}$$

As in the previous case, we can find the symmetry generator for the class (55) by solving (11) after having input the special forms of the functions f, g, and h, i.e.,

$$(2\alpha' - \beta')f_0 u - \phi f_0 = 0, \tag{57}$$

$$\begin{aligned} &(f_0 g + u_x^2 f_0^2)\phi - \left[\phi_{uu}u_x^2 - (\alpha'' - 2\phi_{xu})u_x + \phi_{xx}\right]f_0^2u^2 - \psi\Gamma' f_0 u + \\ &\quad -\phi(Au_x + \gamma_1\gamma_2 - 2\gamma_1\gamma_2 u)f_0 u - \phi_t f_0 u - (2\phi_x + \phi_u u_x)u_x f_0^2 u + \\ &\quad (\phi_u - 2\alpha')(Auu_x + \gamma_1 u(\gamma_2 - u) + \Gamma)f_0 u + \\ &((\alpha' - \phi_u)u_x - \phi_x)Af_0u^2 = 0, \end{aligned} \tag{58}$$

$$(\psi_v - \beta')H - \phi H_u - \psi H_v + \psi_t = 0. \tag{59}$$

For arbitrary $H(u,v)$, we obtain only the principal Lie algebra

$$X_1 = \partial_t,\ X_2 = \partial_x. \tag{60}$$

For

$$H(u,v) = \left(2\gamma_1\gamma_2 + \sqrt{\Gamma}\Phi(\omega)\right)\frac{\Gamma}{\Gamma'} \tag{61}$$

with $\omega = \dfrac{u}{\sqrt{\Gamma}}$, we obtain also the following generator

$$X_3 = e^{-\gamma_1\gamma_2 t}\left(\partial_t + \gamma_1\gamma_2 u\partial_u + 2\gamma_1\gamma_2\frac{\Gamma}{\Gamma'}\partial_v\right). \tag{62}$$

By using the generator

$$X = c_1X_3 + c_2X_2, \tag{63}$$

the invariant solutions are

$$u(t,x) = e^{\gamma_1\gamma_2 t}U(\sigma),\ \Gamma(v(t,x))) = e^{2\gamma_1\gamma_2 t}V(\sigma) \tag{64}$$

where $\sigma = \dfrac{c_2}{\gamma_1\gamma_2}e^{\gamma_1\gamma_2 t} - c_1x$. The functions U and V are solutions of the reduced system

$$\begin{cases} c_2U' = c_1^2 f_0(U'^2 + UU'') - c_1AUU' - \gamma_1U^2 + V, \\ c_2V' = V^{\frac{3}{2}}\Phi\left(\frac{U}{\sqrt{V}}\right). \end{cases} \tag{65}$$

If we choose

$$\Phi(\omega) = 2\frac{\sqrt{\gamma_1}}{A^2}\left(A^2 + 4f_0\gamma_1\right)(1 - \sqrt{\gamma_1}\omega)^2 \tag{66}$$

we find the following particular solutions for system (65)

$$U(\sigma) = 2\frac{c_2(2\gamma_1\sigma - c_1A)}{c_1^2(A^2 + 4f_0\gamma_1)}, \tag{67}$$

$$V(\sigma) = 16\frac{\gamma_1^3c_2^2\sigma^2}{c_1^4(A^2 + 4f_0\gamma_1)^2}. \tag{68}$$

From (64), we obtain directly $u(t,x)$. To find $v(t,x)$, we must invert the function Γ (numerically or analytically).

5. Some Remarks about the Constitutive Functions Γ and H

We stress that, in all systems that we have considered, the extension of $L_{\mathcal{P}}$ does not imply a constraint on Γ form, while it influences the functional dependence of H that must be

$$H = H(\Gamma, \Phi(\omega)),$$

where Φ is an arbitrary function of $\omega = \omega(u,\Gamma)$, being ω a specific function of its arguments. After having recalled that $\Gamma'(v) \neq 0$, we consider the following equivalence transformation:

$$w = \Gamma(v). \tag{69}$$

The system (34) with H given by (43), after (69), reads as follows:

$$\begin{cases} u_t = f_0u_{xx} + Au_x + \gamma_1u(\gamma_2 - \ln u) + w, \\ w_t = \left(\Phi\left(\frac{u}{w}\right) - \gamma_1\ln w\right)w. \end{cases} \tag{70}$$

A symmetry generator for the system (70) has the form

$$X = \xi^1\partial_x + \xi^2\partial_t + \eta^1\partial_u + \eta^*\partial_w, \tag{71}$$

where the infinitesimal η^* is linked to the infinitesimal η^2 by the relation

$$\eta^* = \Gamma' \eta^2 \tag{72}$$

obtained by requiring the invariance of the transformation (69) with respect to the generator (see, e.g., [25])

$$X^* = \xi^1 \partial_x + \xi^2 \partial_t + \eta^1 \partial_u + \eta^2 \partial_v + \eta^* \partial_w. \tag{73}$$

and then its extension of $L_{\mathcal{P}}$ is

$$X_3 = e^{-\gamma_1 t}(u\partial_u + w\partial_w). \tag{74}$$

The system (55) with $H(u,v)$ given by (61) can be written in the following equivalent form:

$$\begin{cases} u_t = (f_0 u u_x)_x + A u u_x + \gamma_1 u(\gamma_2 - u) + w \\ w_t = \left(2\gamma_1\gamma_2 + \sqrt{w}\,\Phi\left(\dfrac{u}{\sqrt{w}}\right)\right) w. \end{cases} \tag{75}$$

Analogously, for the system (75), the extension of $L_{\mathcal{P}}$ assumes the following form:

$$X_3 = e^{-\gamma_1\gamma_2 t}(\partial_t + \gamma_1\gamma_2 u\partial_u + 2\gamma_1\gamma_2 w\partial_w). \tag{76}$$

Actually, the new forms (70), (75) of the systems (34) and (55) could be more useful in the search for additional exact solutions, once we have fixed the form of arbitrary constitutive function $\Phi(\omega)$, or they could suggest new ways to solve the systems. However, we must not forget that when we return to the original biological variable v, we must fix the form of the still arbitrary function $\Gamma(v)$.

6. Conclusions

In this paper, we continue our studies dealing with the extension of the principal Lie algebra and with our search for solutions of some subclasses of systems (2), begun in [7] and recently continued in [26,27]. In all the cases analysed here, *a priori*, we have considered the two constitutive functions of production $\Gamma(v)$ and $h(u,v)$ arbitrary in order to obtain a classification with respect to each of them.

In the case of linear advection, we obtain, for $\Pi(u) = \gamma_1 u(\gamma_2 - \ln u)$, the extension by one of $L_{\mathcal{P}}$ that leaves $\Gamma(v)$ arbitrary while h is, as expected as a solution of a PDE, an arbitrary function depending on $\omega = \dfrac{u}{\Gamma}$. For special forms of h, wide classes of exact solutions are derived. We remark that, in this case, the function $\Pi(u)$ has the same numerical features, at least in a suitable interval, of the classical logistic form $\Pi(u) = \gamma_1 u(\gamma_2 - u)$ for which we do not obtain extensions. For this form of $\Pi(u)$, however, looking for travelling wave solutions, in some cases, we bring the reduced system, with appropriate transformations, to an Abel equation. For the same system without advection, we obtained additional extensions in [27].

In the system analysed in Section 4, we assumed diffusion and advection coefficients both linear in u. Additionally, in this case, we obtain extensions with Γ arbitrary, while H is still dependent on an arbitrary function $\Phi(\omega)$ with $\omega = \dfrac{u}{\sqrt{\Gamma}}$. For an appropriate choice of $\Phi(\omega)$, classes of exact solutions are derived.

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