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Oscillation Results of Third-Order Differential Equations with Symmetrical Distributed Arguments

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Abstract: This paper is concerned with the oscillation and asymptotic behavior of certain third-order nonlinear delay differential equations with distributed deviating arguments. By establishing sufficient conditions for the nonexistence of Kneser solutions and existing oscillation results for the studied equation, we obtain new criteria which ensure that every solution oscillates by using the theory of comparison with first-order delay equations and the technique of Riccati transformation. Some examples are presented to illustrate the importance of main results.

Keywords: delay; third-order differential equations; oscillation criteria

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1. Introduction

Since the beginning of the eighteenth century, scientists began to focus light on the study and development of the oscillation theory and with this rapid development, many results appeared related to the asymptotic behavior of first- and second-order differential equations, see [1–4]; while few results appeared for third-order equations. It is worth noting that fixed point theory and fractional calculus emerged as two indispensable and interrelated tools in the mathematical modelling of various experiments in nonlinear sciences and engineering over the last few decades, for example [5–9].

In recent years, the oscillation theory of different classes of third-order functional differential equations and dynamical functional equations on time scales has received great attention from researchers in various fields because they have wide applications in natural sciences and engineering, see [10–15]. In particular, the oscillation property for solutions of these equations plays an important role in explaining the various phenomena of life, which encouraged researchers to make greater efforts to achieve better results. We refer the reader to [16–22]. However, interest in third-order neutral differential equations has remained somewhat limited, for instance [23–43].

In this recent study, we focus on the oscillation of the third-order neutral differential equation of the following form:

$$\left(\Psi\left(\Omega''\right)^{\alpha}\right)'(\iota) + \int_a^b \varphi(\iota,s)F(\theta(\nu(\iota,s)))\mathrm{d}s = 0, \text{ for } \iota \geq \iota_0, \tag{1}$$

where

$$\Omega(\iota) = \theta(\iota) + m(\iota)\theta(\sigma(\iota)),$$

$F \in C(\mathbb{R}, \mathbb{R})$, $|F(\theta)| \geq \kappa|\theta^{\beta}|$, $\theta \neq 0$, $\kappa > 0$ and α and β are ratios of non-negative and non-zero odd and natural numbers. We consider the below assumptions:

(I$_1$) $m, \Psi \in C([\iota_0, \infty))$, $m(\iota) \leq m_0 < \infty$ and $\Psi > 0$ such that

$$\int^{\infty} \Psi^{-1/\alpha}(s)\mathrm{d}s = \infty; \tag{2}$$

(I$_2$) $\varphi \in C([\iota_0, \infty) \times (a, b), \mathbb{R})$, $\varphi(\iota, s) > 0$ does not vanish identically;
(I$_3$) $\sigma \in C([\iota_0, \infty), (0, \infty)), \sigma(\iota) < \iota, \sigma'(\iota) \geq \sigma_0 > 0$ and $\lim_{\iota \to \infty} \sigma(\iota) = \infty$;
(I$_4$) $\nu \in C([\iota_0, \infty) \times (a, b), \mathbb{R}), \nu(\iota, s) < \iota, \lim_{\iota \to \infty} \nu(\iota, s) = \infty$. Furthermore $\sigma \circ \nu = \nu \circ \sigma$;
(I$_{4a}$) $\sigma(\iota) = \iota - \delta_0$ for $\delta_0 \geq 0$, $m(\iota) = m_0 \neq 1$.

By a solution to (1), we mean a nontrivial function $\theta \in C([\iota_0, \infty), [0, \infty))$ with $\iota_\theta \geq \iota_0$, which has the property $\Psi(\Omega'')^{\alpha} \in C([\iota_a, \infty), [0, \infty))$ and satisfies (1) on $[\iota_b, \infty)$. We only consider those solutions of (1) which exist on some half-line $[\iota_c, \infty)$ such that it holds the property $\sup\{|\theta(\iota)| : \iota_c \leq \iota < \infty\} > 0$ for any $\iota_c \geq \iota_b$. Furthermore, if a solution θ of (1) is neither eventually positive nor eventually negative, then it is said to be oscillatory. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

Definition 1. *Let θ is positive solution and corresponding function $\Omega(\iota)$ and its second derivative are positive functions. If*
(i) $\Omega'(\iota) > 0$, *then we call the set of all solutions θ of (1) class C_1;*
(ii) $\Omega'(\iota) < 0$, *then we call the set of all solutions θ of (1) class C_2.*

Below, some motivation and some previous studies are provided. Grace et al. [21] considered a nonlinear differential equation

$$\Big(\Psi_2(\iota)\big(\Psi_1(\iota)\theta'(\iota)\big)'\Big)' + \varphi(\iota)\theta(\nu(\iota)) = 0, \ \text{ for } \iota \geq \iota_0, \tag{3}$$

under condition

$$\int_{t_0}^{\infty} \Psi_1^{-1}(t)\mathrm{d}s = \int_{t_0}^{\infty} \Psi_2^{-1}(t)\mathrm{d}s = \infty. \tag{4}$$

Saker et al. [44] studied nonlinear differential equation

$$\Big(\Psi(\iota)\big(\theta''(\iota)\big)^{\alpha}\Big)' + \varphi(\iota)\theta^{\alpha}(\nu(\iota)) = 0, \ \text{ for } \iota \geq \iota_0,$$

and established some sufficient conditions which guarantee that every solution of (1) oscillates or converges to zero under condition (2). By comparison with first order oscillatory differential equations, Elabbasy et al. [19] established oscillation criteria for third-order nonlinear differential equation

$$\left(\Psi_1(\iota)\Big(\big(\Psi_2(\iota)\theta'((\iota))^{\alpha_1}\big)'\Big)^{\alpha_2}\right)' + \varphi(\iota)f(\theta(\nu(\iota))) = 0.$$

Furthermore, Li et al. [34] extended some of their results to neutral differential equation where $\alpha_1 = \alpha_2 = 0$. Thandapani and Li [20] considered the oscillation of equation

$$\left(\Psi(\iota)\Big(\big(\theta(\iota) + m(\iota)\theta(\sigma(\iota))\big)''\Big)^{\alpha}\right)' + \varphi(\iota)\theta^{\alpha}(\nu(\iota)) = 0,$$

and assumed that $\int^{\infty} \Psi^{-1/\alpha}(t)\mathrm{d}s = \infty$ and $m(\iota) \geq 1$. For the same equation, Dzurina et al. [33] obtained different results under condition $0 \leq m(\iota) \leq m_0 < \infty$.

Li et al. [36] extended the oscillation results in [21] for Equation (3) to be in the form

$$\left(\Psi_1(\iota)\left(\Psi_2(\iota)(\theta(\iota)+m(\iota)\theta(\sigma(\iota)))'\right)'\right)'+\varphi(\iota)\theta(\nu(\iota))=0,\ \ \text{for } \iota\geq\iota_0, \tag{5}$$

they established some sufficient condition for the nonexistence of a positive decreasing solution under the assumption

$$0\leq m(\iota)\leq m_0<1, \tag{6}$$

with three cases for Ψ_1 and Ψ_2 as follows:

$$\text{case } \int_{t_0}^{\infty}\Psi_1^{-1}(t)\mathrm{d}s<\infty \text{ and } \int_{t_0}^{\infty}\Psi_2^{-1}(t)\mathrm{d}s=\infty$$

and

$$\text{case } \int_{t_0}^{\infty}\Psi_1^{-1}(t)\mathrm{d}s<\infty \text{ and } \int_{t_0}^{\infty}\Psi_2^{-1}(t)\mathrm{d}s<\infty$$

in addition to the case (4).

Moreover, Baculikova and Dzurina [25] obtained new oscillation criteria and covered both cases when the term in neutrality is positive or negative and (6) holds. Furthermore, Candan in [26,27] examined the oscillation behavior of (5) under the condition (6).

Contrary to [19–21,33] which include conditions that guarantee that the solutions of the Equation (1) are almost oscillatory, we aim in this paper to establish two different conditions which ensure the oscillation of all solutions of Equation (1) by using the technique of comparison with first order delay differential equations and the technique of Riccati transformation. These results extend, simplify, and improve the results in [28–30].

2. Some Lemmas

In this section, we provide several lemmas that we will intensively use in the main results.

Lemma 1 ([3])**.** *Suppose that $d_1,d_2\in[0,\infty)$. Then*

$$(d_1+d_2)^{\gamma}\leq\mu\left(d_1^{\gamma}+d_2^{\gamma}\right), \tag{7}$$

where $\mu=1$ if $\gamma\in(0,1]$ and $\mu=2^{\gamma-1}$ if $\gamma\in(1,\infty)$.

Lemma 2 ([38], Lemma 2.2)**.** *Suppose that $\theta(\iota)>0$ and $\Omega(\iota)$ is a nonincreasing positive solution of (1), eventually. Then*

$$\Omega^{\beta-\alpha}(\iota)\geq\varsigma(\iota):=\begin{cases}1 & \text{if } \alpha=\beta\\ a_1 & \text{if } \alpha>\beta\\ a_2\pi^{\beta-\alpha}(\iota) & \text{if } \alpha<\beta\end{cases},\ \textit{where } a_1,a_2 \textit{ are positive constants.}$$

Lemma 3 ([16])**.** *Assume that $\theta\in C^{y}([\iota_0,\infty),(0,\infty))$, $\theta^{(y)}(\iota)$ is not identically zero on $[\iota_0,\infty)$ and fixed sign as well as $\theta^{(y)}(\iota)$ not a value of zero on $[\iota_0,\infty)$ in a way that*

$$\theta^{(y-1)}(\iota)\theta^{(y)}(\iota)\leq 0,\ \forall\iota\geq\iota_1.$$

If $\lim_{\iota\to\infty}\theta(\iota)\neq 0$, then

$$\theta(\iota)\geq\frac{\mu}{(y-1)!}\iota^{y-1}\left|\theta^{(y-1)}(\iota)\right|\ \forall\mu\in(0,1) \textit{ and } \iota\geq\iota_\mu\geq\iota_1.$$

Lemma 4 ([11]). *Let $\widetilde{u}$, $\widetilde{v} \in C([\iota_0,\infty),\mathbb{R})$, $\widetilde{u}(\iota) = \widetilde{v}(\iota) + D\widetilde{v}(\iota - B)$ for $\iota \geq \iota_0 + \max\{C,0\}$, and $D \neq 1$ as well as B are fixed values. Assume that $\exists\, l \in \mathbb{R}$ such that $\lim_{\iota\to\infty} \widetilde{u} = l$*

$$(\mathbf{a})\ If\ \liminf_{\iota\to\infty} \widetilde{v}(\iota) = \widetilde{v}_* \in \mathbb{R},\ then\ \widetilde{v}_* = l/(1+D);$$

$$(\mathbf{b})\ If\ \limsup_{\iota\to\infty} \widetilde{v}(\iota) = \widetilde{v}^* \in \mathbb{R},\ then\ \widetilde{v}^* = l/(1+D).$$

Now, for the sake of brevity, we will define the following operators:

$$£\Omega := \Psi\left(\Omega''\right)^{\alpha} \text{ and } \widehat{£\Omega} := £\Omega(\iota) + \frac{(m_0)^{\beta}}{\sigma_0} £\Omega(\sigma(\iota)).$$

Furthermore, throughout this paper, we use the following notations for sufficiently large ι_1 with $\iota_1 \geq \iota_0$:

$$\begin{aligned}
\eta(\iota,u) \quad &: \quad = \int_u^{\iota} \frac{1}{\Psi^{\frac{1}{\alpha}}(H)} \mathrm{d}H,\ \widetilde{\eta}(\iota,u) = \int_u^{\iota}\left(\int_u^{H} \frac{1}{\Psi^{\frac{1}{\alpha}}(u)} \mathrm{d}u\right)\mathrm{d}H;\ \text{for } \iota \geq \iota_0;\\
\widetilde{\varphi}(\iota) \quad &: \quad = \min\{\varphi(\iota,H), \varphi(\sigma(\iota,H))\};\\
\varphi(\iota,H) \quad &: \quad = \int_a^b \widetilde{\varphi}(\iota,H)\mathrm{d}H,\ \ \widehat{\varphi}(\iota,H) := \frac{1}{\Psi(u)}\int_u^{\infty} \varphi(\iota,H)\mathrm{d}H
\end{aligned}$$

and

$$\tilde{O}(\iota,H) = \int_a^b \widetilde{\varphi}(\iota,H)(\widetilde{\eta}(\sigma,H))^{\beta}\mathrm{d}H.$$

Lemma 5. *Let $\theta \in C_2$. Then*

$$\Omega(u) \geq \widetilde{\eta}(v,u)£^{1/\alpha}\Omega(v), \tag{8}$$

for $v \geq u$, and

$$\left(\widehat{£\Omega}\right)' \leq -\frac{\kappa}{\mu}\int_a^b \widetilde{\varphi}(\iota,H)\Omega^{\beta}(\nu(\iota,H))\mathrm{d}H. \tag{9}$$

Proof. Let θ be solution of (1) and $\theta > 0$. Then, $\theta(\sigma(\iota))$ and $\theta(\nu(\iota,H))$ are positive functions for $\iota_1 \leq \iota$. By Lemma 1, (1) and $(\mathbf{I}_2)$, we obtain

$$\Omega^{\beta}(\iota) \leq \mu\left(\theta^{\beta}(\iota) + m_0^{\beta}\theta^{\beta}(\sigma(\iota))\right). \tag{10}$$

Since $£\Omega(\iota)$ is nonincreasing, we have

$$-\Omega'(u) \geq \int_u^{v} \frac{1}{\Psi^{1/\alpha}(H)}£^{1/\alpha}\Omega(H)\mathrm{d}H \geq £^{1/\alpha}\Omega(v)\int_u^{v}\frac{1}{\Psi^{1/\alpha}(H)}\mathrm{d}H,\ \text{for } u \leq v. \tag{11}$$

Integrating (from u to v), we get

$$\Omega(u) - \Omega(v) \geq £^{1/\alpha}\Omega(v)\int_u^{v}\left(\int_u^{\rho}\frac{1}{\Psi^{1/\alpha}(H)}\mathrm{d}H\right)\mathrm{d}\rho.$$

Thus,

$$\Omega(u) \geq \widetilde{\eta}(v,u)£^{1/\alpha}\Omega(v). \tag{12}$$

Now, from (1) and $(\mathbf{I}_3)$, we obtain

$$(£\Omega(\sigma(\iota)))'\frac{1}{\sigma'(\iota)} + \kappa\int_a^b \varphi(\sigma(\iota,H))\theta^{\beta}(\nu(\sigma(\iota,H)))\mathrm{d}H \leq 0. \tag{13}$$

Combining (10) and (13) then combining the resulting inequality with (1), we get

$$\begin{aligned} 0 \geq\ & (\pounds\Omega(\iota))' + \kappa \int_a^b \varphi(\iota,H)\theta^\beta(\nu(\iota,H))\mathrm{d}H + \frac{m_0^\beta}{\sigma_0}(\pounds\Omega(\sigma(\iota)))' \\ & + m_0^\beta \kappa \int_a^b \varphi(\sigma(\iota,H))\theta^\beta(\nu(\sigma(\iota,H)))\mathrm{d}H \\ \geq\ & (\pounds\Omega(\iota))' + \frac{m_0^\beta}{\sigma_0}(\pounds\Omega(\sigma(\iota)))' \\ & + \kappa \int_a^b \widetilde{\varphi}(\iota,H)\Big(\theta^\beta(\nu(\iota,H)) + m_0^\beta\theta^\beta(\nu(\sigma(\iota,H)))\Big)\mathrm{d}H. \end{aligned}$$

Thus,

$$\left(\pounds\widehat{\Omega}\right)' + \frac{\kappa}{\mu}\int_a^b \widetilde{\varphi}(\iota,H)\Omega^\beta(\nu(\iota,H))\mathrm{d}H \leq 0. \tag{14}$$

The proof of the lemma is complete. □

3. Main Results

The following theorems contain conditions that guarantee nonexistence of positive decreasing solutions and nonexistence of positive increasing solutions.

Theorem 1. *Suppose that* $\exists\Im \in C([\iota_0,\infty),(0,\infty)), \sigma(\iota) \leq \Im(\iota), \nu^{-1}(\Im(\iota)) < \iota$ *and* $\Im'(\iota,H) > 0$. *If one of the following statements is true:*
($\mathbf{b}_1$) $\beta = \alpha$ *and condition*

$$\liminf_{\iota\to\infty} \int_{\sigma^{-1}(\Im(\iota))}^{\iota} \tilde{O}(\iota,H)\mathrm{d}H > \frac{\mu\left(\sigma_0 + m_0^\beta\right)}{k\sigma_0 e}. \tag{15}$$

is hold.
($\mathbf{b}_2$) $\beta > \alpha$, *there exists a function* $\xi(\iota) \in C^1([\iota_0,\infty))$ *such that* $\xi'(\iota) > 0$, $\lim_{\iota\to\infty}\xi(\iota) = \infty$,

$$\limsup_{\iota\to\infty} \frac{\beta\xi'(\sigma^{-1}(\Im(\iota)))\left(\sigma^{-1}(\Im(\iota))\right)'}{\alpha\xi'(\iota)} < 1 \tag{16}$$

and

$$\liminf_{\iota\to\infty}\left[\frac{\kappa}{\mu\xi'(\iota)}\left(\frac{\sigma_0}{\sigma_0 + m_0^\beta}\right)^{\beta/\alpha} \tilde{O}(\iota,\Im)e^{-\xi(\iota)}\right] > 0. \tag{17}$$

Then $C_2 = \emptyset$.

Proof. Assume that the solution of (1) is $\theta > 0$ and the case **(ii)** holds. Then, $\theta(\nu(\iota,H))$ and $\theta(\sigma(\iota))$ are positive functions for $\iota_1 \leq \iota$, ι_1 sufficiently large. Thus, from (1), we see that

$$\left(\Psi(\Omega'')^\alpha\right)'(\iota) \leq 0 \text{ for } \iota \geq \iota_1.$$

Using Lemma 5, we arrive at (8) and (9). Combining (9) and (8) with replacing u and v with $\sigma(\iota)$ and $\Im(\iota)$, respectively, we find

$$\left(\pounds\widehat{\Omega}\right)' + \frac{\kappa}{\mu}\int_a^b \widetilde{\varphi}(\iota,H)(\widetilde{\eta}(\sigma,\Im))^\beta \pounds^{\beta/\alpha}\Omega(\Im(\iota,H))\mathrm{d}H \leq 0. \tag{18}$$

Since $\Im(\iota,H)$ is nonincreasing with respect to H, we get

$$\left(\pounds\widehat{\Omega}\right)' + \frac{\kappa}{\mu}\pounds^{\beta/\alpha}\Omega(\Im(\iota,a))\tilde{O}(\iota,H) \leq 0.$$

Using $\pounds\Omega(\iota)$ is nonincreasing, we find that $\pounds\Omega(\iota) \leq \pounds\Omega(\sigma(\iota))$, hence

$$\widehat{\pounds\Omega} \leq \left(1 + \frac{1}{\sigma_0} m_0^\beta\right) \pounds\Omega(\sigma(\iota)). \tag{19}$$

From (18) along with (19), we note that $v(\iota) := \widehat{\pounds\Omega}$ is a positive solution of inequality

$$v'(\iota) + \frac{\kappa}{\mu} \tilde{O}(\iota, H) \left(\frac{\sigma_0}{\sigma_0 + m_0^\beta} \right)^{\beta/\alpha} v^{\beta/\alpha}\left(\nu^{-1}(\Im(\iota, a))\right) \leq 0.$$

That is by ([13] Theorem 1), we note that the associated delay equation

$$v'(\iota) + \frac{\kappa}{\mu} \tilde{O}(\iota, H) \left(\frac{\sigma_0}{\sigma_0 + m_0^\beta} \right)^{\beta/\alpha} v^{\beta/\alpha}\left(\nu^{-1}(\Im(\iota, a))\right) = 0, \tag{20}$$

also has a positive solution. Hence, it is well-known from [1,18] that conditions (15)–(17) imply oscillation of (20), which is a contradiction. The proof is complete. $\square$

Theorem 2. *Let $\beta \geq \alpha$. If $\exists\, \aleph \in C([\iota_0, \infty), (0, \infty))$ and $\aleph(\iota) \leq \iota$, $\sigma(\iota) \leq \nu(\aleph(\iota))$ and*

$$\limsup_{\iota \to \infty} k l^{\beta - \alpha} \eta^\alpha(\sigma, \nu(\aleph)) \int_{\aleph(\iota)}^{\iota} \widetilde{\varphi}(\iota, H) \mathrm{d}H > \mu \left(1 + \frac{1}{\sigma_0} m_0^\beta\right), \; M > 0, \tag{21}$$

then $C_2 = \emptyset$.

Proof. As in the proof of Theorem 1, we get (19). By Lemma 5, we have (8) and (9). By integrating (9) from $\aleph(\iota)$ to ι, we find that

$$\begin{aligned} 0 \quad < \quad & \pounds\Omega(\iota) + \frac{1}{\sigma_0} m_0^\beta\, \pounds\Omega(\nu(\iota)) \leq \pounds\Omega(\aleph(\iota)) + \frac{1}{\sigma_0} m_0^\beta\, \pounds\Omega(\nu(\aleph(\iota))) \\ & - \frac{\kappa}{\mu} \int_{\aleph(\iota)}^{\iota} \widetilde{\varphi}(\iota, H) \Omega^\beta(\sigma(\iota, H)) \mathrm{d}H, \end{aligned}$$

which with (19) gives

$$\left(1 + \frac{1}{\sigma_0} m_0^\beta\right) \pounds\Omega(\nu(\aleph(\iota))) \geq \frac{\kappa}{\mu} \Omega^\beta(\sigma(\iota, H)) \int_{\aleph(\iota)}^{\iota} \widetilde{\varphi}(\iota, H) \mathrm{d}H. \tag{22}$$

Since $\Omega'(\iota) < 0$, there exists a positive constant $l > 0$ such that $\Omega(\iota) \geq l$ for $\iota \geq \iota_2$, that is, (22) becomes

$$\left(1 + \frac{1}{\sigma_0} m_0^\beta\right) \pounds\Omega(\nu(\aleph(\iota))) \geq \frac{k l^{\beta - \alpha}}{\mu} \Omega^\alpha(\sigma(\iota)) \int_{\aleph(\iota)}^{\iota} \widetilde{\varphi}(\iota, H) \mathrm{d}H.$$

From (8) $[u = \sigma(\iota)$ and $v = \nu(\aleph(\iota))]$, we find

$$\left(1 + \frac{1}{\sigma_0} m_0^\beta\right) \geq \frac{k l^{\beta - \alpha}}{\mu} \eta^\alpha(\sigma, \nu(\aleph)) \int_{\aleph(\iota)}^{\iota} \widetilde{\varphi}(\iota, H) \mathrm{d}H. \tag{23}$$

Taking the lim sup of (23), we obtain a contradiction to (21). The proof is complete. $\square$

Theorem 3. *Suppose that $\sigma(\iota) \leq \nu(\iota)$ and $\nu'(\iota, a) > 0$. If $\exists\, \jmath(\iota) \in C^1([\iota_0, \infty), (0, \infty))$ and $\iota_1 \geq \iota_0$ such that*

$$\limsup_{\iota \to \infty} \int_{\iota_1}^{\iota} \left[\jmath(\iota) \frac{\varsigma(\nu(\iota, a))\kappa}{\mu} \varphi(\iota, H) - \frac{\sigma_0 + m_0^\beta}{(\alpha+1)^{\alpha+1} \sigma_0} \frac{(\jmath'(\iota))^{\alpha+1}}{(\jmath(\iota) \eta(\nu(\iota), \iota_1) \nu'(\iota))^\alpha} \right] \mathrm{d}H = \infty, \tag{24}$$

then $C_1 = \emptyset$.

Proof. Let $\theta > 0$ is a solution of (1) and satisfy case **(i)**. Define a positive function by

$$\omega(\iota) = \frac{\jmath(\iota)\pounds\Omega(\iota)}{\Omega^{\alpha}(\nu(\iota))}. \tag{25}$$

Hence, by differentiating (25), we get

$$\omega'(\iota) = \jmath'(\iota)\frac{\pounds\Omega(\iota)}{\Omega^{\alpha}(\nu(\iota))} + \jmath(\iota)\frac{(\pounds\Omega(\iota))'}{\Omega^{\alpha}(\nu(\iota))} - \frac{\alpha\rho(\iota)\pounds\Omega(\iota)\Omega^{\alpha-1}(\nu(\iota))\Omega'(\nu(\iota))\nu'(\iota)}{\Omega^{2\alpha}(\nu(\iota))}. \tag{26}$$

Substituting (25) into (26) implies

$$\omega'(\iota) = \jmath(\iota)\frac{(\pounds\Omega(\iota))'}{\Omega^{\alpha}(\nu(\iota))} + \frac{\jmath'(\iota)}{\jmath(\iota)}\omega(\iota) - \frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}\omega^{\frac{\alpha+1}{\alpha}}(\iota). \tag{27}$$

Now, define function v by

$$v(\iota) = \jmath(\iota)\frac{\pounds\Omega(\sigma(\iota))}{\Omega^{\alpha}(\nu(\sigma(\iota)))}. \tag{28}$$

By differentiating (28), we get

$$v'(\iota) = \jmath'(\iota)\frac{\pounds\Omega(\sigma(\iota))}{\Omega^{\alpha}(\nu(\iota))} + \jmath(\iota)\frac{(\pounds\Omega(\sigma(\iota)))'}{\Omega^{\alpha}(\nu(\iota))} - \frac{\alpha\rho(\iota)\pounds\Omega(\sigma(\iota))\Omega^{\alpha-1}(\nu(\iota))\Omega'(\nu(\iota))\nu'(\iota)}{\Omega^{2\alpha}(\nu(\iota))}. \tag{29}$$

Substituting (28) into (29) implies

$$v'(\iota) = \jmath(\iota)\frac{(\pounds\Omega(\sigma(\iota)))'}{\Omega^{\alpha}(\nu(\iota))} + \frac{\jmath'(\iota)}{\jmath(\iota)}v(\iota) - \frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}v^{\frac{\alpha+1}{\alpha}}(\iota). \tag{30}$$

From (27) and (30), we have

$$\begin{aligned} \omega'(\iota) + \frac{m_0^{\beta}}{\sigma_0}v'(\iota) \leq\ & \frac{\jmath(\iota)\left((\pounds\Omega(\iota))' + \frac{m_0^{\beta}}{\sigma_0}(\pounds\Omega(\sigma(\iota)))'\right)}{\Omega^{\alpha}(\nu(\iota))} \\ & + \frac{\jmath'(\iota)}{\jmath(\iota)}\omega(\iota) - \frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}\omega^{\frac{\alpha+1}{\alpha}}(\iota) \\ & + \frac{m_0^{\beta}}{\sigma_0}\left(\frac{\jmath'(\iota)}{\jmath(\iota)}v(\iota) - \frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}v^{\frac{\alpha+1}{\alpha}}(\iota)\right). \end{aligned} \tag{31}$$

From Lemma 2 and $\nu(\iota,a)$ is increasing, and (9) becomes

$$\begin{aligned} \left(\widehat{\pounds\Omega}\right)' &\leq -\frac{\kappa}{\mu}\Omega^{\beta}(\nu(\iota,a))\varphi(\iota,H) \\ &\leq -\frac{\varsigma(\nu(\iota,a))\kappa}{\mu}\Omega^{\alpha}(\nu(\iota,a))\varphi(\iota,H). \end{aligned} \tag{32}$$

Take into account Lemma 1, (32) and (31), we obtain

$$\begin{aligned}\omega'(\iota)+\frac{m_0^\beta}{\sigma_0}v'(\iota) \quad\leq\quad & -\jmath(\iota)\frac{\varsigma(\nu(\iota,a))\kappa}{\mu}\varphi(\iota,H)\\ & +\frac{\jmath'(\iota)\omega(\iota)}{\jmath(\iota)}-\frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)\omega^{\frac{\alpha+1}{\alpha}}(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}\\ & +\frac{m_0^\beta}{\sigma_0}\left(\frac{\jmath'(\iota)v(\iota)}{\jmath(\iota)}-\frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)v^{\frac{\alpha+1}{\alpha}}(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}\right).\end{aligned}$$

Applying the following inequality

$$Bu-Au^{\frac{\alpha+1}{\alpha}}\leq\frac{\alpha^\alpha B^{\alpha+1}}{(\alpha+1)^{\alpha+1}A^\alpha},\quad A>0,$$

with

$$A=\frac{\alpha\eta(\nu(\iota),\iota_1)\nu'(\iota)}{\jmath^{\frac{1}{\alpha}}(\iota)}\text{ and } B=\frac{1}{\jmath(\iota)}\jmath'(\iota),$$

we get

$$\begin{aligned}\omega'(\iota)+\frac{m_0^\beta}{\sigma_0}v'(\iota) \quad\leq\quad & -\jmath(\iota)\kappa\frac{\varsigma(\nu(\iota,a))}{\mu}\varphi(\iota,H)+\frac{(\jmath'(\iota))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\jmath(\iota)\eta(\nu(\iota),\iota_1)\nu'(\iota))^\alpha}\\ & +\frac{\frac{m_0^\beta}{\sigma_0}(\jmath'(\iota))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\jmath(\iota)\eta(\nu(\iota),\iota_1)\nu'(\iota))^\alpha}.\end{aligned}$$

Integrating (from ι_1 to ι), we see that

$$\int_{\iota_1}^{\iota}\left[\frac{\jmath(\iota)\varsigma(\nu(\iota,a))\kappa}{\mu}\varphi(\iota,H)-\frac{\sigma_0+m_0^\beta}{(\alpha+1)^{\alpha+1}\sigma_0}\frac{(\jmath'(\iota))^{\alpha+1}}{(\jmath(\iota)\eta(\nu(\iota),\iota_1)\nu'(\iota))^\alpha}\right]\mathrm{d}H\leq\omega(\iota_2)+\frac{m_0^\beta}{\sigma_0}v(\iota_2).$$

The proof is complete. □

Oscillation criteria

The following theorem provides some criteria that guarantee all solutions of Equation (1) oscillate.

Theorem 4. *If all assumptions of Theorem 1 or Theorem 2 and assumptions of Theorem (3) and are satisfied, then Equation (1) is oscillatory.*

Remark 1. *It is clear that the results we obtained under the condition*

$$m(\iota)\leq m_0<\infty.$$

So, our results are an improvement of results in [21,25,34].

For a special case of the Equation (1), we present the following results under condition $(\mathbf{I}_{4a})$, and for the sake of brevity, we define

$$\widetilde{\varphi}_1(\iota,H)=\int_a^b\varphi(\iota,H)(\widetilde{\eta}(\iota,H))^\beta\mathrm{d}H$$

and

$$\widehat{\varphi}_1(\iota,H)=\int_a^b\widetilde{\varphi}(\iota,H)\nu^{2\beta}(\iota,H)\mathrm{d}H.$$

Lemma 6. *Let $\theta(\iota) > 0$ is a solution of equation (1) the corresponding function belongs to* ***class*** C_2. *If*

$$\int_{\iota_0}^{\infty} \int_{v}^{\infty} (\widetilde{\varphi}(\iota, H))^{1/\alpha} \mathrm{d}u \mathrm{d}v = \infty, \tag{33}$$

then

$$\lim_{\iota \to \infty} \theta(\iota) = 0. \tag{34}$$

Proof. Since $\Omega(\iota)$ is nonincreasing positive function, there exists a constant $\Omega_0 \geq 0$ such that $\lim_{\iota \to \infty} g(\iota) = \Omega_0 \geq 0$. We claim that $\Omega_0 = 0$. Otherwise, using Lemma, we conclude that $\lim_{\iota \to \infty} \theta(\iota) = \Omega_0/(1 + m_0) > 0$. Therefore, there exists a $\iota_2 \geq \iota_0$ such that, for all $\iota \geq \iota_2$

$$\theta(\nu(\iota)) > \frac{\Omega_0}{2(1 + m_0)} > 0. \tag{35}$$

From (1) and (35), we see that

$$(\pounds\Omega((\iota)))' \leq -\kappa \int_a^b \varphi(\iota, H) \left(\frac{\Omega_0}{2(1 + m_0)} \right)^{\beta} \mathrm{d}H.$$

Integrating the above inequality from ι to ∞, we have

$$\pounds\Omega((\iota)) \geq \kappa \left(\frac{\Omega_0}{2(1 + m_0)} \right)^{\beta} \int_{\iota}^{\infty} \varphi(u, H) \mathrm{d}u.$$

It follows that

$$\Omega''(\iota) \geq \kappa \left(\frac{\Omega_0}{2(1 + m_0)} \right)^{\frac{\beta}{\alpha}} \left(\frac{1}{\Psi(\iota)} \int_{\iota}^{\infty} \varphi(u, H) \mathrm{d}u \right)^{\frac{1}{\alpha}}. \tag{36}$$

Integrating (36) from ι to ∞, yields

$$-\Omega'(\iota) \geq \kappa \left(\frac{\Omega_0}{2(1 + m_0)} \right)^{\frac{\beta}{\alpha}} \int_{\iota}^{\infty} \left(\frac{1}{\Psi(\vartheta)} \int_{\vartheta}^{\infty} \varphi(u, H) \mathrm{d}u \right)^{1/\alpha} \mathrm{d}\vartheta.$$

Integrating again from ι_2 to ∞, we obtain

$$\Omega(\iota_2) \geq \kappa \left(\frac{\Omega_0}{2(1 + m_0)} \right)^{\frac{\beta}{\alpha}} \int_{\iota_2}^{\infty} \int_{v}^{\infty} \left(\frac{1}{\Psi(\vartheta)} \int_{\iota}^{\infty} \varphi(u, H) \mathrm{d}u \right)^{1/\alpha} \mathrm{d}\vartheta dv,$$

which contradicts (33). Therefore, $\lim_{\iota \to \infty} \Omega(\iota) = 0$, that is $0 < \theta(\iota) \leq \Omega(\iota)$. Thus we have property (34). The proof of the lemma is complete. □

Theorem 5. *Assume (33) hold, $\exists\ W \in C(I, \mathbb{R})$ such that $W(\iota) \leq \sigma(\iota)$, $W(\iota) < \iota$ and $\lim_{\iota \to \infty} W(\iota) = \infty$. If one of the first-order delay differential equations*

$$y'(\iota) + \frac{\widetilde{\varphi}_1(\iota, H)}{(1 + m_0)^{\beta}} y^{\frac{\beta}{\alpha}}(W(\iota, a)) = 0$$

or

$$w'(\iota) + \frac{k\lambda^{\beta}}{2^{\beta}\mu} \left(\frac{\sigma_0}{\sigma_0 + m_0^{\beta}} \right) \widehat{\varphi}_1(\iota, H)\, w^{\beta/\alpha}(\sigma(\iota, a)) = 0$$

is oscillatory, then every solution $\theta(\iota)$ of (1) is almost oscillatory.

Proof. Let $\theta(\iota) > 0$ be a solution of (1), eventually. Then, $\exists\, \iota_1 \geq \iota_o$ such that $(\mathbf{i})$ or $(\mathbf{ii})$ hold $\forall\, \iota \geq \iota_1$. Let $(\mathbf{ii})$ hold. By Lemma 6, we see that (34) satisfies it. Now, if $(\mathbf{i})$ holds, since $\Omega''(\iota) \geq 0$ and $\Omega(\iota) > 0$, $\exists$ a constant c_0 such that

$$\lim_{\iota\to\infty} \Omega'(\iota) = c_0 > 0 \text{ (or } c_0 = \infty).$$

By Lemma 4, we have

$$\lim_{\iota\to\infty} \theta'(\iota) = c_0/(1+m_0) > 0,$$

this implies that $\theta(\iota) \geq 0$ and, taking into account $\delta_0 \geq 0$, we get

$$\Omega(\iota) = \theta(\iota) + m_0\theta(\iota - \delta_0) \leq (1+m_0)\theta(\iota),$$

that is

$$\theta(\iota) \geq \frac{1}{1+m_0}\Omega(\iota).$$

We conclude that

$$\theta(\sigma(\iota)) \geq \theta(W(\iota)) \geq \frac{1}{1+m_0}\Omega(W(\iota)), \text{ for } W(\iota) \leq \sigma(\iota).$$

Substituting into (1), we have

$$(\pounds\Omega(\iota))' + \frac{\kappa \int_a^b \varphi(\iota, H)\Omega^{\beta}(W(\iota, H))\mathrm{d}H}{(1+m_0)^{\beta}} \leq 0. \tag{37}$$

Using (8) and (37), we arrive at

$$\begin{aligned} 0 &\geq (\pounds\Omega(\iota))' + \frac{(\pounds\Omega(W(\iota,a)))^{\frac{\beta}{\alpha}}}{(1+m_0)^{\beta}} \int_a^b \varphi(\iota, H)(\widetilde{\eta}(v,u))^{\beta} \\ &= (\pounds\Omega(\iota))' + \frac{1}{(1+m_0)^{\beta}}(\pounds\Omega(W(\iota,a)))^{\frac{\beta}{\alpha}}\widetilde{\varphi}_1(\iota, H). \end{aligned}$$

Hence, one could have that $y = \pounds\Omega(\iota)$ is a non-zero and non-negative solution of

$$y'(\iota) + \frac{\widetilde{\varphi}_1(\iota, H)}{(1+m_0)^{\beta}} y^{\frac{\beta}{\alpha}}(W(\iota, a)) \leq 0.$$

Using the same approach of Lemma 1, as well as from (1), (10) and (13), one could get that (14) holds. Similarly, using the result of Lemma 3, one could get that

$$\Omega(\iota) > \frac{\lambda}{2}\iota^2\Omega''(\iota). \tag{38}$$

Since $\frac{\mathrm{d}}{\mathrm{d}\iota}\pounds\Omega(\iota) \leq 0$ and $\sigma(\iota) \leq \iota$, we obtain $\pounds\Omega(\sigma(\iota)) \geq \pounds\Omega(\iota)$, and so

$$\pounds\Omega(\iota) + \frac{1}{\sigma_0}m_0^{\beta}\pounds\Omega(\sigma(\iota)) \leq \left(1 + \frac{1}{\sigma_0}m_0^{\beta}\right)\pounds\Omega(\iota),$$

which with (14) gives

$$(\pounds\Omega(\iota))' + \frac{\kappa}{\mu}\left(\frac{\sigma_0}{\sigma_0 + m_0^{\beta}}\right)\int_a^b \widetilde{\varphi}(\iota, H)\Omega^{\beta}(\sigma(\iota, H))\mathrm{d}H \leq 0.$$

Thus, from (38) and $\sigma(\iota, H)$ is increasing with respect to H, we find

$$(£\Omega(\iota))' + \frac{k\lambda^\beta}{2^\beta\mu}\left(\frac{\sigma_0}{\sigma_0 + m_0^\beta}\right)(\Omega''(\sigma(\iota,a)))^\beta \int_a^b \widetilde{\varphi}(\iota,H)\sigma^{2\beta}(\iota,H)\mathrm{d}H \le 0.$$

If we set $w := £\Omega(\iota) = \Psi(\Omega'')^\alpha$, then one could get that $w > 0$ is one of the solutions to the following delay inequality

$$w'(\iota) + \frac{k\lambda^\beta}{2^\beta\mu}\left(\frac{\sigma_0}{\sigma_0 + m_0^\beta}\right)\widehat{\varphi}_1(\iota,H)\, w^{\beta/\alpha}(\sigma(\iota,a)) \le 0.$$

The proof is complete. □

By choosing $\alpha = \beta$, we obtain the following corollary:

Corollary 1. *Let (33),*

$$\int_\iota^{\iota+\sigma} \widehat{\varphi}_1(\iota,H)\mathrm{d}\zeta > \frac{2^\beta\mu\left(\sigma_0 + m_0^\beta\right)}{k\lambda^\beta\sigma_0} \tag{39}$$

and

$$\frac{k\lambda^\beta\sigma_0}{2^\beta\mu\left(\sigma_0 + m_0^\beta\right)}\int_{\iota_0}^\infty \tilde{O}(\iota,H)\ln\left(e\int_\iota^{\iota+\sigma}\frac{k\lambda^\beta\sigma_0}{2^\beta\mu\left(\sigma_0 + m_0^\beta\right)}\widehat{\varphi}_1(\zeta,H)\mathrm{d}\zeta\right)\mathrm{d}\iota = \infty. \tag{40}$$

Then every solution $\theta(\iota)$ of (1) is either oscillatory or satisfies (34).

Proof. In view of [2], conditions (39) and (40) imply oscillation of the delay differential Equation (20). □

Corollary 2. *Let (33) be satisfied. Assume $\exists\, W \in C(I,\mathbb{R})$ such that $W(\iota) \le \sigma(\iota)$, $W(\iota) < \iota$ and $\lim_{\iota\to\infty} W(\iota) = \infty$. If*

$$(1+m_0)\int_{W(\iota,a)}^{\iota} \liminf \widetilde{\varphi}_1(\iota,H)\mathrm{d}u > \frac{1}{e}.$$

Then every solution $y(\iota)$ of

$$y'(\iota) + \frac{\widetilde{\varphi}_1(\iota,H)}{(1+m_0)}y(W(\iota,a)) = 0$$

is oscillatory. Therefore, every solution $\theta(\iota)$ of (1) is either oscillatory or satisfies (34).

Example 1. *Take into consideration the following third-order delay differential equation*

$$\left[\left([\theta(\iota) + p\theta(\lambda t)]''\right)^\alpha\right]' + \frac{\varphi_0}{\iota^{\alpha(m-1)+1}}\theta^\alpha(\gamma t) = 0, \text{ such that } \gamma,\lambda \in (0,1). \tag{41}$$

Setting $\jmath(\iota) = \iota^2$, $\zeta(\iota) = 0.5(\gamma+\lambda)\iota$, we see condition (24) is satisfied if

$$\varphi_0 > \frac{1}{2^{1-\beta}\gamma^{2\alpha}}\left(\frac{2\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\sigma_0 + \jmath_0^\beta}{\sigma_0}\right).$$

Moreover, when $\alpha = 1$, condition (15) is satisfied if

$$\frac{\varphi_0(\gamma-\lambda)^2}{8}\ln\left(\frac{2\gamma}{\lambda+\gamma}\right) > \frac{\sigma_0 + m_0}{\sigma_0 e}.$$

By Theorem 4, if

$$\varphi_0 > \max\left\{ \frac{1}{\gamma^2}\left(\frac{\sigma_0 + \jmath_0}{\sigma_0}\right), \frac{8}{\sigma_0 e}\frac{(\sigma_0 + m_0)}{(\gamma - \lambda)^2 \ln\left(\frac{2\gamma}{\lambda+\gamma}\right)} \right\},$$

then Equation (41) is oscillatory.

Example 2. *Take into consideration the following third-order delay differential equation*

$$\left[\left([\theta(\iota) + p\theta(0.5t)]''\right)\right]' + \frac{\varphi_0}{t^3}\theta(0.75t) = 0, \ \ t \geq 1. \tag{42}$$

Set $\jmath(\iota) = \iota^2$, $\zeta(\iota) = 3/8\iota$. By Theorem 4, we see that Equation (42) is oscillatory if

$$\varphi_0 > 16\left(\frac{\sigma_0 + \jmath_0}{\sigma_0}\right)$$

and

$$\varphi_0 > \frac{128}{e\ln(2/3)}\frac{(\sigma_0 + m_0)}{\sigma_0}.$$

That is, (42) is oscillatory if

$$\varphi_0 > \max\left\{16\left(\frac{\sigma_0 + \jmath_0}{\sigma_0}\right), \frac{128}{e\ln(2/3)}\frac{(\sigma_0 + m_0)}{\sigma_0}\right\}.$$

Example 3. *By choosing $W(t) = t - 1$ in 2, it follows that every solution to a third-order neutral differential equation*

$$[\theta(\iota) + p_0\theta(t-1)]''' + \varphi_0(t)\theta(t+2) = 0, \tag{43}$$

where

$$\varphi_0(t) = \left(e^2 + p_0 e^3\right)$$

is either oscillatory or satisfies (34). (It is worth noting that $x(t) = e^{-t}$ is an exact solution satisfying (43)).

4. Conclusions

In the present work, we focus on filling the gap by establishing various sufficient criteria for eliminating positive decreasing solutions of (1) under the above conditions. Therefore, the criteria contained in this paper ensure that all the solutions of the Equation (1) oscillate, while the conditions in [19,20,33] and [44] provide conditions that guarantee that solutions of the Equation (1) are either oscillatory or converge to zero. As an extension of the results of [25,36], we create a new criterion for oscillation by combining the newly obtained results with the results obtained in the literature, which in turn is a simplification of the previous results in [28–30].

For further research, another interesting problem is to obtain new criteria for nonexistence of decreasing positive solutions of (1) without requiring

$$\sigma \circ \nu = \nu \circ \sigma \ \text{ or } \ \left(\sigma^{-1}(\iota)\right)' \geq \sigma.$$

Moreover,

$$\sigma'(\iota) \geq \sigma_0 > 0.$$

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