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Estimation of the Second-Order Hankel Determinant of Logarithmic Coefficients for Two Subclasses of Starlike Functions

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Abstract: In our present study, two subclasses of starlike functions which are symmetric about the origin are considered. These two classes are defined with the use of the sigmoid function and the trigonometric function, respectively. We estimate the first four initial logarithmic coefficients, the Zalcman functional, the Fekete–Szegö functional, and the bounds of second-order Hankel determinants with logarithmic coefficients for the first class $\mathcal{S}^*_{\text{seg}}$ and improve the obtained estimate of the existing second-order Hankel determinant of logarithmic coefficients for the second class $\mathcal{S}^*_{\sin}$. All the bounds that we obtain in this article are proven to be sharp.

Keywords: starlike functions; sigmoid function; logarithmic coefficient; Hankel determinant

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1. Introduction, Definitions and Preliminaries

To aid readers in interpreting the basics used throughout our reporting of these important results, certain fundamental knowledge from function theory is included here, starting with the letters $\mathcal{S}$ and $\mathcal{A}$, which stand for the normalized univalent (or Schlicht) functions class and the normalized holomorphic (or analytic) functions class, respectively. The subsequent set builder representations in the region of the open unit disc $\mathbb{O}_d = \{z \in \mathbb{C} : |z| < 1\}$ present these fundamental notions:

$$\mathcal{A} = \left\{ g \in \mathcal{H}(\mathbb{O}_d) : g(z) = \sum_{j=1}^{\infty} b_j z^j, \, (z \in \mathbb{O}_d) \right\}, \tag{1}$$

where $\mathcal{H}(\mathbb{O}_d)$ shows the class of holomorphic functions, and

$$\mathcal{S} = \{ g \in \mathcal{A} : g \text{ is schlicht in } \mathbb{O}_d \}.$$

A stunning interplay between univalent function theory and fluid dynamics has recently been shown by Aleman and Constantin [1]. In fact, they showed a straightforward technique for using a univalent harmonic map to obtain explicit solutions of incompressible two-dimensional Euler equations.

The formula below provides the logarithmic coefficients λ_n of $g \in \mathcal{S}$

$$G_g(z) := \log\left(\frac{g(z)}{z}\right) = 2\sum_{n=1}^{\infty} \lambda_n z^n \quad \text{for} \quad z \in \mathbb{O}_d.$$

These coefficients have a considerable impact on the theory of Schlicht functions in many estimations. In 1985, de Branges [2] deduced that, for $n \geq 1$,

$$\sum_{j=1}^{n} j(n-j+1)|\lambda_n|^2 \leq \sum_{j=1}^{n} \frac{n-j+1}{j},$$

and equality is achieved if g has the form $z/\left(1-e^{i\theta}z\right)^2$ for some $\theta \in \mathbb{R}$. It is evident that this inequality yields the most general version of the well-established Bieberbach–Robertson–Milin conjectures involving Taylor coefficients of g belonging to $\mathcal{S}$. For further details on the explanation of de Brange's assertion, see [3–5]. By taking into consideration the logarithmic coefficients, in 2005 Kayumov [6] was able to resolve Brennan's conjecture for conformal mappings. We include a few studies here that have made major contributions to the investigation of logarithmic coefficients [7–15].

From the definition provided above, it is not challenging to calculate that, for g belonging to $\mathcal{S}$, its logarithmic coefficients are provided by

$$\lambda_1 = \frac{1}{2}a_2, \tag{2}$$

$$\lambda_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{3}$$

$$\lambda_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{4}$$

$$\lambda_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \tag{5}$$

It is quite clear that the geometric interpretations of an analytic function depend on the bounds of the coefficients that appear in its Taylor series form. This is why researchers have shown keen interest in studying coefficient-related problems for various analytic functions in recent years. Among these problems, the Hankel determinant for $m, n \in \mathbb{N} = \{1, 2, \ldots\}$ and $g \in \mathcal{S}$,

$$H_{m,n}(g) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+m-1} & a_{n+m} & \cdots & a_{n+2m-2} \end{vmatrix} \tag{6}$$

created by Pommerenke [16,17] is perhaps the most challenging problem in this field, in particular the determination of the sharp bounds. Here, we cite several recent works on Hankel determinants of different orders in which the authors investigated sharp bounds for different subclasses of univalent functions; see [18–30].

In recent times, Kowalczyk and Lecko [31,32] have offered analyses of the Hankel determinant $H_{m,n}(G_g/2)$, the members of which are logarithmic coefficients of g, that is,

$$H_{m,n}(G_g/2) = \begin{vmatrix} \lambda_n & \lambda_{n+1} & \cdots & \lambda_{n+m-1} \\ \lambda_{n+1} & \lambda_{n+2} & \cdots & \lambda_{n+m} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_{n+m-1} & \lambda_{n+m} & \cdots & \lambda_{n+2m-2} \end{vmatrix}. \tag{7}$$

It has been noted that

$$H_{2,1}(G_g/2) = \lambda_1\lambda_3 - \lambda_2^2, \tag{8}$$

$$H_{2,2}(G_g/2) = \lambda_2\lambda_4 - \lambda_3^2. \tag{9}$$

In the present paper, our main focus is on finding the sharp upper bounds of logarithmic coefficient-related problems, including the Zalcman functional and Fekete–Szegö functional, along with (8) and (9) for the subclass $\mathcal{S}^*_{\text{seg}}$ of starlike functions established by Kumar and Goel [33], which is stated as

$$\mathcal{S}^*_{\text{seg}} := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \frac{2}{1+e^{-z}}, \qquad (z \in \mathbb{O}_d) \right\}.$$

In addition, for the following defined class $\mathcal{S}^*_{\sin}$ introduced by Cho et al. [34], we improve the obtained estimate of the existing second-order Hankel determinant of logarithmic coefficients:

$$\mathcal{S}^*_{\sin} := \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec 1 + \sin z, \qquad (z \in \mathbb{O}_d) \right\},$$

where "$\prec$" denotes the familiar subordination between analytic functions.

2. A Set of Lemmas

Here, we include the facts that are incorporated into our main problems. First, we define the class shown below.

$$\mathcal{P} := \left\{ q \in \mathcal{A} : q(z) = \sum_{n=1}^{\infty} c_n z^n \prec \frac{1+z}{1-z} \qquad (z \in \mathbb{O}_d) \right\} \tag{10}$$

The following Lemma consists of the widely used c_2 formula [35], the c_3 formula [36], and the c_4 formula illustrated in [37].

Lemma 1. *Let $q \in \mathcal{P}$ have the form (10). Then, for $x, \varrho, \delta \in \overline{\mathbb{O}_d} = \mathbb{O}_d \cup \{1\}$,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{11}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\varrho, \tag{12}$$

$$8c_4 = c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right)\left(1 - |x|^2\right) \left[c(x-1)\varrho + \overline{x}\varrho^2 - \left(1 - |\varrho|^2\right)\delta\right]. \tag{13}$$

Lemma 2 ([38]). *If $q \in \mathcal{P}$ is provided by (10) and if $E \in [0,1]$ with $E(2E-1) \leq F \leq E$, then we have*

$$\left| c_3 - 2Ec_1c_2 + Fc_1^3 \right| \leq 2. \tag{14}$$

Lemma 3. *Let $q \in \mathcal{P}$ be the series expansion (10). Then,*

$$|c_n| \leq 2 \qquad n \geq 1. \tag{15}$$

and

$$|c_{n+k} - \delta c_n c_k| \leq 2\max\{1, |2\delta - 1|\} = \begin{cases} 2 & for \quad 0 \leq \delta \leq 1; \\ 2|2\delta - 1| & \quad otherwise. \end{cases} \tag{16}$$

Inequalities (15) and (16) are studied in [35,39], respectively.

Lemma 4 ([40]). *Let τ, ψ, ϱ and ς satisfy the inequalities $0 < \tau < 1, 0 < \varsigma < 1$ and*

$$8\varsigma(1-\varsigma)\left((\tau\psi - 2\varrho)^2 + (\tau(\varsigma+\tau) - \psi)^2\right) + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 \leq 4\varsigma\tau^2(1-\tau)^2(1-\varsigma).$$

If $q \in \mathcal{P}$ is the series form (10), then

$$\left|\varrho c_1^4 + \varsigma c_2^2 + 2\tau c_1 c_3 - \frac{3}{2}\psi c_1^2 c_2 - c_4\right| \le 2.$$

3. Logarithmic Coefficient Inequalities for the Class $\mathcal{S}^*_{\mathbf{seg}}$

In this section, we study problems involving coefficients for the class $\mathcal{S}^*_{\text{seg}}$. The bound of the most difficult problem of the third-order Hankel determinant is under consideration here for this class as well. We begin by proving the sharp bounds of the initial coefficient of $g \in \mathcal{S}^*_{\text{seg}}$.

Theorem 1. *If the function $g \in \mathcal{S}^*_{seg}$ is defined by (1), then*

$$\begin{aligned} |\lambda_1| &\le \frac{1}{4}, \\ |\lambda_2| &\le \frac{1}{8}, \\ |\lambda_3| &\le \frac{1}{12}, \\ |\lambda_4| &\le \frac{1}{16}. \end{aligned}$$

The above inequalities are sharp.

Proof. From the definition of the class $\mathcal{S}^*_{\text{seg}}$ along with subordination principal, there is a Schwarz function $w(z)$ such that

$$\frac{zg'(z)}{g(z)} = \frac{2}{1+e^{-w(z)}}, \qquad (z \in \mathbb{O}_d).$$

Assuming that $q \in \mathcal{P}$, by writing q in terms of the Schwarz function $w(z)$, we have

$$q(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

which is equivalent to

$$w(z) = \frac{q(z)-1}{q(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \tag{17}$$

Using (1), we can easily obtain

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= 1 + (a_2)z + \left(-a_2^2 + 2a_3\right)z^2 + \left(-3a_2a_3 + 3a_4 + a_2^3\right)z^3 \\ &\quad + \left(-4a_2a_4 - a_2^4 + 4a_5 + 4a_2^2a_3 - 2a_3^2\right)z^4 + \cdots. \end{aligned} \tag{18}$$

From the series expansion of (17), we have

$$\begin{aligned} \frac{2}{1+e^{-w(z)}} &= 1 + \left(\frac{1}{4}c_1\right)z + \left(-\frac{1}{8}c_1^2 + \frac{1}{4}c_2\right)z^2 + \left(-\frac{1}{4}c_1c_2 + \frac{11}{192}c_1^3 + \frac{1}{4}c_3\right)z^3 \\ &\quad + \left(\frac{11}{64}c_1^2c_2 - \frac{1}{4}c_1c_3 - \frac{3}{128}c_1^4 + \frac{1}{4}c_4 - \frac{1}{8}c_2^2\right)z^4 + \cdots. \end{aligned} \tag{19}$$

By comparing (18) and (19), it follows that

$$\begin{aligned} a_2 &= \left(\frac{1}{4}\right)c_1, \\ a_3 &= -\frac{1}{32}c_1^2 + \frac{1}{8}c_2, \\ a_4 &= -\frac{5}{96}c_1c_2 + \frac{7}{1152}c_1^3 + \frac{1}{12}c_3, \\ a_5 &= \frac{7}{384}c_1^2c_2 - \frac{1}{24}c_1c_3 - \frac{17}{18432}c_1^4 + \frac{1}{16}c_4 - \frac{3}{128}c_2^2. \end{aligned} \tag{20}$$

By using (20) in (2)–(5), we can obtain

$$\lambda_1 = \frac{1}{8}c_1, \tag{21}$$

$$\lambda_2 = -\frac{1}{32}c_1^2 + \frac{1}{16}c_2, \tag{22}$$

$$\lambda_3 = -\frac{1}{24}c_1c_2 + \frac{11}{1152}c_1^3 + \frac{1}{24}c_3, \tag{23}$$

$$\lambda_4 = \frac{11}{512}c_1^2c_2 + \frac{1}{32}c_4 - \frac{1}{32}c_1c_3 - \frac{3}{1024}c_1^4 - \frac{1}{64}c_2^2. \tag{24}$$

Implementing (15) in (21), we obtain

$$|\lambda_1| \leq \frac{1}{4}.$$

Now, reshuffling (22), we obtain

$$\lambda_2 = \frac{1}{16}\left(c_2 - \frac{1}{2}c_1^2\right).$$

Applying (16), we obtain

$$|\lambda_2| \leq \frac{1}{8}.$$

From (23), we can deduce that

$$|\lambda_3| = \frac{1}{24}\left|\left(c_3 - 2\left(\frac{1}{2}\right)c_1c_2 + \left(\frac{11}{48}\right)c_1^3\right)\right|.$$

From (14), let

$$E = \frac{1}{2} \quad \text{and} \quad F = \frac{11}{48}.$$

It is clear that $0 \leq E \leq 1$, $E \geq F$, and

$$E(2E-1) = 0 \leq F.$$

Thus, all the conditions of Lemma 2 are satisfied. Hence, we have

$$|\lambda_3| \leq \frac{1}{12}.$$

From (24), we can deduce that

$$\lambda_4 = -\frac{1}{32}\left(\left(\frac{3}{32}\right)c_1^4 + \left(\frac{1}{2}\right)c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{11}{24}\right)c_1^2c_2 - c_4\right). \tag{25}$$

By comparing the right side of (25) with

$$\left|\varrho c_1^4+\varsigma c_2^2+2\tau c_1c_3-\frac{3}{2}\psi c_1^2c_2-c_4\right|,$$

we obtain the following values:

$$\varrho=\frac{3}{32},\quad \varsigma=\frac{1}{2},\quad \tau=\frac{1}{2},\quad \psi=\frac{11}{24}.$$

It follows that $0<\varsigma<1, 0<\tau<1$, and

$$8\varsigma(1-\varsigma)\left((\tau\psi-2\varrho)^2+(\tau(\varsigma+\tau)-\psi)^2\right)+\tau(1-\tau)(\psi-2\varsigma\tau)^2=\frac{17}{2304},$$

and

$$4\varsigma\tau^2(1-\tau)^2(1-\varsigma)=\frac{1}{16}.$$

Thus, all the conditions of Lemma 4 are satisfied. Hence, we have

$$|\lambda_4|\leq\frac{1}{16}.$$

The required inequalities are sharp and the equality is determined from (2)–(5) along with consideration of a function

$$g_n'(z)=\frac{2}{1+e^{-z^n}},\qquad n=1,2,3,4.$$

Thus, we have

$$\begin{aligned}
g_1(z) &= z\exp\left(\int_0^z\frac{2}{1+e^{-t}}dt\right)=z+\frac{1}{2}z^2+\frac{1}{8}z^3+\cdots,\\
g_2(z) &= z\exp\left(\int_0^z\frac{2}{1+e^{-t^2}}dt\right)=z+\frac{1}{4}z^3+\frac{1}{32}z^5+\cdots,\\
g_3(z) &= z\exp\left(\int_0^z\frac{2}{1+e^{-t^3}}dt\right)=z+\frac{1}{6}z^4+\frac{1}{72}z^7+\cdots,\\
g_4(z) &= z\exp\left(\int_0^z\frac{2}{1+e^{-t^4}}dt\right)=z+\frac{1}{8}z^5+\frac{1}{128}z^9+\cdots.
\end{aligned}$$

□

Theorem 2. *If the function $g\in\mathcal{S}^*_{seg}$ is defined by (1), then*

$$\left|\lambda_2-\delta\lambda_1^2\right|\leq\max\left\{\frac{1}{8},\frac{|\delta|}{16}\right\}.$$

Thus, the Fekete–Szegö functional is the best possible.

Proof. By employing (21) and (22), we obtain

$$\left|\lambda_2-\delta\lambda_1^2\right|=\frac{1}{16}\left|c_2-\frac{1}{2}c_1^2-\frac{1}{4}\delta c_1^2\right|.$$

By applying (16) to the above equation, we obtain

$$\left|\lambda_2-\delta\lambda_1^2\right|\leq\frac{2}{16}\max\left\{1,\left|\frac{2+\delta}{2}-1\right|\right\}.$$

After simplification, we have

$$\left|\lambda_2 - \delta\lambda_1^2\right| \le \max\left\{\frac{1}{8}, \frac{|\delta|}{16}\right\}.$$

The required Fekete–Szegö functional is the best possible and is obtained using (2), (3), and

$$g_2(z) = z\exp\left(\int_0^z \frac{2}{1+e^{-t^2}}dt\right) = z + \frac{1}{4}z^3 + \frac{1}{32}z^5 + \cdots.$$

□

Theorem 3. *If the function $g \in \mathcal{S}_{seg}^*$ is of the form (1), then*

$$|\lambda_1\lambda_2 - \lambda_3| \le \frac{1}{12}.$$

This inequality is the best possible.

Proof. Using (21)–(23), we have

$$|\lambda_1\lambda_2 - \lambda_3| = \frac{1}{24}\left|c_3 - 2\left(\frac{19}{32}\right)c_1c_2 + \left(\frac{31}{96}\right)c_1^3\right|.$$

From (14), let

$$E = \frac{19}{32} \quad \text{and} \quad F = \frac{31}{96}.$$

It is clear that $0 \le E \le 1$, $E \ge F$, and

$$E(2E-1) = \frac{57}{512} \le F.$$

Thus, all the conditions of Lemma 2 are satisfied. Hence, we have

$$|\lambda_1\lambda_2 - \lambda_3| \le \frac{1}{12}.$$

The required inequality is the best possible and is determined using (2)–(4) and

$$g_3(z) = z\exp\left(\int_0^z \frac{2}{1+e^{-t^3}}dt\right) = z + \frac{1}{6}z^4 + \frac{1}{72}z^7 + \cdots.$$

□

Theorem 4. *If $g \in \mathcal{S}_{seg}^*$ is of the form (1), then*

$$\left|\lambda_4 - \lambda_2^2\right| \le \frac{1}{16}.$$

The Zalcman functional is sharp.

Proof. From (22) and (24), we obtain

$$\left|\lambda_4 - \lambda_2^2\right| = -\frac{1}{32}\left|\left(\frac{1}{8}\right)c_1^4 + \left(\frac{5}{8}\right)c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{13}{24}\right)c_1^2c_2 - c_4\right|. \tag{26}$$

By comparing the right side of (26) with

$$\left|\varrho c_1^4 + \varsigma c_2^2 + 2\tau c_1c_3 - \frac{3}{2}\psi c_1^2c_2 - c_4\right|,$$

we obtain the following values:

$$\varrho = \frac{1}{8}, \quad \varsigma = \frac{5}{8}, \quad \tau = \frac{1}{2}, \quad \psi = \frac{13}{24}.$$

It follows that $0 < \varsigma < 1, 0 < \tau < 1$, and

$$8\varsigma(1-\varsigma)\left((\tau\psi - 2\varrho)^2 + (\tau(\varsigma+\tau) - \psi)^2\right) + \tau(1-\tau)(\psi - 2\varsigma\tau)^2 = \frac{31}{9216},$$

and

$$4\varsigma\tau^2(1-\tau)^2(1-\varsigma) = \frac{15}{256}.$$

Thus, all the conditions of Lemma 4 are satisfied. Hence, we have

$$\left|\lambda_4 - \lambda_2^2\right| \leq \frac{1}{16}.$$

The required Zalcman functional is sharp and the equality is obtained using (3), (5), and

$$g_4(z) = z\exp\left(\int_0^z \frac{2}{1+e^{-t^4}}dt\right) = z + \frac{1}{8}z^5 + \frac{1}{128}z^9 + \cdots.$$

□

4. Second Hankel Determinant with Logarithmic Coefficients for Class $\mathcal{S}^*_{\mathbf{seg}}$

Theorem 5. *If the function $g \in \mathcal{S}^*_{seg}$ is defined by (1), then*

$$\left|H_{2,1}(G_g/2)\right| \leq \frac{1}{64}.$$

This inequality is the best possible.

Proof. The determinant $H_{2,1}(G_g/2)$ can be reconfigured as follows:

$$H_{2,1}(G_g/2) = \lambda_1\lambda_3 - \lambda_2^2.$$

From (21)–(23), we achieve

$$H_{2,1}(G_g/2) = -\frac{1}{768}c_1^2c_2 + \frac{1}{4608}c_1^4 + \frac{1}{192}c_1c_3 - \frac{1}{256}c_2^2.$$

Using (11) and (12) to express c_2 and c_3 in terms of c_1, and with $c_1 = c$, and $c \in [0,2]$, we obtain

$$\begin{aligned}\left|H_{2,1}(G_g/2)\right| &= \left|-\frac{1}{9216}c^4 - \frac{1}{768}c^2x^2\left(4-c^2\right) - \frac{1}{1024}x^2\left(4-c^2\right)^2\right. \\ &\quad \left. + \frac{1}{384}c\left(4-c^2\right)\left(1-|x|^2\right)\varrho\right|,\end{aligned}$$

By replacing $|\varrho| \leq 1$ and $|x| = t$, where $t \leq 1$, and using triangle inequality while taking $c \in [0,2]$, we have

$$\begin{aligned}\left|H_{2,1}(G_g/2)\right| &\leq \frac{1}{9216}c^4 + \frac{1}{768}c^2t^2\left(4-c^2\right) + \frac{1}{1024}t^2\left(4-c^2\right)^2 \\ &\quad + \frac{1}{384}c\left(4-c^2\right)\left(1-t^2\right) := \Theta(c,t).\end{aligned}$$

Now, differentiating $\Theta(c,t)$ with respect to t, we have

$$\frac{\partial\Theta(c,t)}{\partial t} = t\left(4-c^2\right)\left(\frac{1}{1536}c^2 - \frac{1}{192}c + \frac{1}{128}\right).$$

It is a simple exercise to show that $\frac{\partial\Theta(c,t)}{\partial t} \geq 0$ on $[0,1]$, thus, $\Theta(c,t) \leq \Theta(c,1)$. If $t=1$, we have

$$\left|H_{2,1}(G_g/2)\right| \leq \frac{1}{9216}c^4 + \frac{1}{768}c^2\left(4-c^2\right) + \frac{1}{1024}\left(4-c^2\right)^2 := K(c).$$

Clearly, $K'(c) < 0$; thus, it follows that $K(c)$ is a decreasing function. Hence, $K(c)$ achieves its maximum value at $c=0$. We can see that

$$\left|H_{2,1}(G_g/2)\right| \leq \frac{1}{64}.$$

The required $H_{2,1}(G_g/2)$ is the best possible and is determined using (2)–(4) and

$$g_2(z) = z\exp\left(\int_0^z \frac{2}{1+e^{-t^2}}dt\right) = z + \frac{1}{4}z^3 + \frac{1}{32}z^5 + \cdots.$$

□

Theorem 6. *If the function $g \in \mathcal{S}^*_{seg}$ is defined by (1), then*

$$\left|H_{2,2}(G_g/2)\right| \leq \frac{1}{144}.$$

The inequality is sharp.

Proof. The determinant $H_{2,2}(G_g/2)$ is described as follows:

$$H_{2,2}(G_g/2) = \lambda_2\lambda_4 - \lambda_3^2.$$

By virtue of (22)–(24), along with $c_1 = c \in [0,2]$, it can be determined that

$$\begin{aligned} H_{2,2}(G_g/2) &= \frac{1}{2654208}\Big(c^6 - 156c^4c_2 + 480c^3c_3 + 252c^2c_2^2 - 2592c^2c_4 \\ &\quad + 4032cc_2c_3 - 2592c_2^3 + 5184c_2c_4 - 4608c_3^2\Big). \end{aligned} \tag{27}$$

Let $j = 4 - c^2$ in (11)–(13). Now, using these lemmas, we obtain

$$\begin{aligned} 156c^4c_2 &= 78c^6 + 78c^4jx, \\ 480c^3c_3 &= 120\left(c^6 - c^4jx^2\right) + 240\left(c^4xj + c^3j\left(1-|x|^2\right)\varrho\right), \\ 252c^2c_2^2 &= 126c^4jx + 63\left(c^6 + c^2j^2x^2\right), \\ 2592c^2c_4 &= 324c^6 + 324c^4jx^3 - 972c^4jx^2 + 972c^4xj + 1296jc^2x^2 - 1296c^3j \\ &\quad \left(1-|x|^2\right)x\varrho - 1296c^2j\left(1-|x|^2\right)\overline{x}\varrho^2 + 1296c^2j\left(1-|x|^2\right)\left(1-|\varrho|^2\right)\delta \\ &\quad + 1296c^3j\left(1-|x|^2\right)\varrho, \\ 4032cc_2c_3 &= -504x^3j^2c^2 - 504c^4jx^2 + 1008cxj^2\left(1-|x|^2\right)\varrho + 1008x^2j^2c^2 \\ &\quad + 1008c^3j\left(1-|x|^2\right)\varrho + 1512c^4xj + 504c^6, \\ 2592c_2^3 &= 972\left(c^4jx + c^2j^2x^2\right) + 324\left(j^3x^3 + c^6\right), \end{aligned}$$

$$
\begin{aligned}
5184c_2c_4 &= 324c^6 + 324c^4jx^3 - 972c^4jx^2 + 1296c^4xj + 1296jc^2x^2 - 1296c^3j \\
&\left(1-|x|^2\right)x\varrho - 1296c^2j\left(1-|x|^2\right)\overline{x}\varrho^2 + 1296c^2j\left(1-|x|^2\right)\left(1-|\varrho|^2\right)\delta \\
&+1296c^3j\left(1-|x|^2\right)\varrho + 324x^4j^2c^2 - 972x^3j^2c^2 + 972x^2j^2c^2 \\
&+1296x^3j^2 - 1296x^2j^2\left(1-|x|^2\right)c\varrho - 1296xj^2\left(1-|x|^2\right)\overline{x}\varrho^2 \\
&+1296xj^2\left(1-|x|^2\right)\left(1-|\varrho|^2\right)\delta + 1296cxj^2\left(1-|x|^2\right)\varrho, \\
4608c_3^2 &= 288x^4j^2c^2 - 1152x^2j^2\left(1-|x|^2\right)c\varrho - 1152x^3j^2c^2 - 576c^4jx^2 + 1152 \\
&j^2\left(1-|x|^2\right)^2\varrho^2 + 2304cxj^2\left(1-|x|^2\right)\varrho + 1152x^2j^2c^2 + 1152c^4xj \\
&+288c^6 + 1152c^3j\left(1-|x|^2\right)\varrho.
\end{aligned}
$$

Inserting the above formulae into (27), we obtain

$$
\begin{aligned}
H_{2,2}(G_g/2) &= \frac{1}{2654208}\Big\{96c^3j\left(1-|x|^2\right)\varrho - 324x^3j^3 + 1296x^3j^2 - 48c^4jx^2 \\
&-81c^2x^2j^2 - 324x^3j^2c^2 + 36x^4j^2c^2 - 1152j^2\left(1-|x|^2\right)^2\varrho^2 \\
&-144x^2j^2\left(1-|x|^2\right)c\varrho - 1296xj^2\left(1-|x|^2\right)\overline{x}\varrho^2 + 1296xj^2 \\
&\left(1-|x|^2\right)\left(1-|\varrho|^2\right)\delta - 2c^6\Big\}.
\end{aligned}
$$

Because $j = 4 - c^2$,

$$
H_{2,2}(G_g/2) = \frac{1}{2654208}\Big(\mathfrak{q}_1(c,x) + \mathfrak{q}_2(c,x)\varrho + \mathfrak{q}_3(c,x)\varrho^2 + \omega(c,x,\varrho)\delta\Big),
$$

where $x, \varrho, \delta \in \overline{\mathbb{O}_d}$ and

$$
\begin{aligned}
\mathfrak{q}_1(c,x) &= -2c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(-81c^2x^2 + 36c^2x^4\right) - 48c^4x^2\right], \\
\mathfrak{q}_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-144cx^2\right) + 96c^3\right], \\
\mathfrak{q}_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-144|x|^2 - 1152\right)\right], \\
\omega(c,x,\varrho) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\varrho|^2\right)\left[1296x\left(4-c^2\right)\right].
\end{aligned}
$$

Let $|x| = x$ and $|\varrho| = y$. By taking $|\delta| \leq 1$, we achieve

$$
\begin{aligned}
\left|H_{2,2}(G_g/2)\right| &\leq \frac{1}{2654208}\Big(|\mathfrak{q}_1(c,x)| + |\mathfrak{q}_2(c,x)|y + |\mathfrak{q}_3(c,x)|y^2 + |\omega(c,x,\varrho)|\Big). \\
&\leq \frac{1}{2654208}(F(c,x,y)), \qquad (28)
\end{aligned}
$$

where

$$
F(c,x,y) = \mathfrak{s}_1(c,x) + \mathfrak{s}_2(c,x)y + \mathfrak{s}_3(c,x)y^2 + \mathfrak{s}_4(c,x)\left(1-y^2\right),
$$

with

$$
\begin{aligned}
\mathfrak{s}_1(c,x) &= 2c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(81c^2x^2 + 36c^2x^4\right) + 48c^4x^2\right], \\
\mathfrak{s}_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(144cx^2\right) + 96c^3\right], \\
\mathfrak{s}_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(144x^2 + 1152\right)\right], \\
\mathfrak{s}_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[1296x\left(4-c^2\right)\right].
\end{aligned}
$$

Now, we have to maximize $F(c,x,y)$ in the closed cuboid $\Gamma : [0,2] \times [0,1] \times [0,1]$.

For this, we have to discuss the maximum values of $F(c,x,y)$ in the interior of Γ, in the interior of its six faces, and on its twelve edges.

1. Interior points of cuboid Γ **:**

Let $(c,x,y) \in (0,2) \times (0,1) \times (0,1)$. By taking a partial derivative of $F(c,x,y)$ with respect to y, we obtain

$$\frac{\partial F}{\partial y} = 48\left(4-c^2\right)(1-x^2)\left[6y\left(4-c^2\right)(x-1)(x-8) + c\left(3x^2\left(4-c^2\right)+2c^2\right)\right].$$

Setting $\frac{\partial F}{\partial y} = 0$ yields

$$y = \frac{c\left(3x^2(4-c^2)+2c^2\right)}{6(4-c^2)(x-1)(8-x)} = y_0.$$

If y_0 is a critical point inside Γ, then $y_0 \in (0,1)$, which is possible only if

$$3cx^2\left(4-c^2\right) + 2c^3 < 6\left(4-c^2\right)(x-1)(8-x). \tag{29}$$

and

$$c^2 > 4. \tag{30}$$

For the existence of the critical points, we have to obtain the solutions which satisfy both inequalities (29) and (30).

As $c^2 > 4$, it is not hard to show that (29) does not hold true in this case for all values of $x \in (0,1)$. Thus, there is no critical point of $F(c,x,y)$ that exists in $(0,2) \times (0,1) \times (0,1)$.

2. Interior of all the six faces of cuboid Γ **:**

(i) On face $c = 0$, $F(c,x,y)$ yields

$$F(0,x,y) = \mathfrak{h}_1(x,y) = 2304y^2(1-x^2)(x-8)(x-1) + 20736x\left(1-x^2\right).$$

Taking the partial derivative with respect to y, we obtain

$$\frac{\partial \mathfrak{h}_1}{\partial y} = 4608y(1-x^2)(x-8)(x-1).$$

However, $\frac{\partial \mathfrak{h}_1}{\partial y} \neq 0$. Thus, $\mathfrak{h}_1(x,y)$ has no critical point in the interval $(0,1) \times (0,1)$.

(ii) On face $c = 2$, $F(c,x,y)$ reduces to

$$F(2,x,y) = 128.$$

(iii) On face $x = 0$, $F(c,x,y)$ is equivalent to

$$F(c,0,y) = \mathfrak{h}_2(c,y) = 2c^6 + 96c^3y(4-c^2) + 1152y^2\left(4-c^2\right)^2.$$

Taking the derivative of $\mathfrak{h}_2(c,y)$ partially with respect to y, we have

$$\frac{\partial \mathfrak{h}_2}{\partial y} = 96c^3(4-c^2) + 2304y\left(4-c^2\right)^2.$$

Again, taking derivative of $\mathfrak{h}_2(c,y)$ partially with respect to c, we have

$$\frac{\partial \mathfrak{h}_2}{\partial c} = 12c^5 - 192c^4y + 288c^2y\left(4-c^2\right) - 4608cy^2\left(4-c^2\right).$$

A numerical calculation shows that the system of equations

$$\frac{\partial \mathfrak{h}_2}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \mathfrak{h}_2}{\partial c} = 0,$$

has no solution in $(0,2) \times (0,1)$. Hence, $\mathfrak{h}_2(c,y)$ has no optimal point in the interval $(0,2) \times (0,1)$.

(iv) On face $x = 1$, $F(c,x,y)$ takes the form

$$F(c,1,y) = \mathfrak{h}_3(c,y) = 71c^6 - 744c^4 + 1872c^2$$

Clearly,

$$\frac{\partial \mathfrak{h}_3}{\partial c} = 426c^5 - 2976c^3 + 3744c.$$

By solving $\frac{\partial \mathfrak{h}_3}{\partial c} = 0$, we can find that the only critical point in $(0,2)$ is $c = \frac{2}{71}\sqrt{4402 - 355\sqrt{43}}$, at which $\mathfrak{h}_3(c,y)$ achieves its maximum value, provided by

$$F(c,1,y) \leq \frac{345551959}{250000}.$$

(v) On face $y = 0$, $F(c,x,y)$ becomes

$$\begin{aligned} F(c,x,0) \quad = \quad & \mathfrak{h}_4(c,x) = 36c^6x^4 + 33c^6x^2 - 288c^4x^4 - 1296c^4x^3 + 2c^6 - 456c^4x^2 \\ & + 576c^2x^4 + 1296c^4x + 10368c^2x^3 + 1296c^2x^2 - 10368c^2x \\ & - 20736x^3 + 20736x. \end{aligned}$$

Taking the derivative partially with respect to x and then simplifying, with respect to c we have

$$\begin{aligned} \frac{\partial \mathfrak{h}_4}{\partial x} \quad = \quad & 144c^6x^3 + 66c^6x - 1152c^4x^3 - 3888c^4x^2 - 912c^4x + 2304c^2x^3 + 1296c^4 \\ & + 31104c^2x^2 + 2592c^2x - 10368c^2 - 62208x^2 + 20736. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathfrak{h}_4}{\partial c} \quad = \quad & 216c^5x^4 + 198c^5x^2 - 1152c^3x^4 - 5184c^3x^3 + 12c^5 - 1824c^3x^2 + 1152cx^4 \\ & + 5184c^3x + 20736cx^3 + 2592cx^2 - 20736cx. \end{aligned}$$

Thus, after a few basic calculations we can find that the system of equations has no solution

$$\frac{\partial \mathfrak{h}_4}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial \mathfrak{h}_4}{\partial c} = 0,$$

in the interval $(0,2) \times (0,1)$. Hence, $\mathfrak{h}_4(c,x)$ has no optimal solution in the interval $(0,2) \times (0,1)$.

(vi) On face $y = 1$, $F(c,x,y)$ yields

$$\begin{aligned} F(c,x,1) \quad = \quad & \mathfrak{h}_5(c,x) = 36c^6x^4 - 144c^5x^4 + 33c^6x^2 - 432c^4x^4 + 240c^5x^2 + 1152c^3x^4 \\ & + 2c^6 - 1464c^4x^2 + 1728c^2x^4 - 96c^5 - 1536c^3x^2 - 2304cx^4 + 1152c^4 \\ & + 9360c^2x^2 - 2304x^4 + 384c^3 + 2304cx^2 - 9216c^2 - 16128x^2 + 18432. \end{aligned}$$

Taking the partial derivative of $\mathfrak{h}_5(c,x)$ with respect to x, with respect to c we have

$$\begin{aligned} \frac{\partial \mathfrak{h}_5}{\partial x} \quad = \quad & 144c^6x^3 - 576c^5x^3 - 1728c^4x^3 + 480c^5x + 4608c^3x^3 - 2928c^4x - 9216x^3 \\ & + 6912c^2x^3 - 3072c^3x - 9216cx^3 + 18720c^2x + 4608cx - 32256x + 66c^6x. \end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathfrak{h}_5}{\partial c} &= 216c^5x^4 - 720c^4x^4 + 198c^5x^2 - 1728c^3x^4 + 1200c^4x^2 + 3456c^2x^4 + 12c^5 \\ &\quad -5856c^3x^2 + 3456cx^4 - 480c^4 - 4608c^2x^2 - 2304x^4 + 4608c^3 + 1152c^2 \\ &\quad +18720cx^2 + 2304x^2 - 18432c.\end{aligned}$$

As in the above case, we can obtain the same result for face $y = 0$, that is, that there is no existing solution for the system of equations

$$\frac{\partial \mathfrak{h}_5}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \mathfrak{h}_5}{\partial c} = 0,$$

in the interval $(0,2) \times (0,1)$.

3. On the Edges of Cuboid Γ :

(i) On edge $x = 0$ and $y = 0$, $F(c, x, y)$ reduces to

$$F(c, 0, 0) = 2c^6 = \mathfrak{h}_6(c).$$

By simple computation, it follows that $\mathfrak{h}_6(c)$ achieves its maximum value at $c = 2$, provided by

$$F(c, 0, 0) \leq 128.$$

(ii) On edge $x = 0$ and $y = 1$, $F(c, x, y)$ is equivalent to

$$F(c, 0, 1) = 2c^6 - 96c^5 + 1152c^4 + 384c^3 - 9216c^2 + 18432 = \mathfrak{h}_7(c).$$

Clearly,

$$\mathfrak{h}_7'(c) = 12c^5 - 480c^4 + 4608c^3 + 1152c^2 - 18432c.$$

We can see that $\mathfrak{h}_7'(c) < 0$ in $[0,2]$ shows that $\mathfrak{h}_7(c)$ is decreasing over $[0,2]$. Thus, $\mathfrak{h}_7(c)$ achieves its maximum at $c = 0$. Hence,

$$F(c, 0, 1) \leq 18432.$$

(iii) On edge $c = 0$ and $x = 0$, $F(c, x, y)$ reduces to

$$F(0, 0, y) = 18432y^2 = \mathfrak{h}_8(y).$$

Note that $\mathfrak{h}_8'(y) > 0$ in $[0,1]$ follows $\mathfrak{h}_8(y)$ increasing over $[0,1]$. Thus, $\mathfrak{h}_8(y)$ achieve its maxima at $y = 1$, and we have

$$F(0, 0, y) \leq 18432.$$

(iv) As we can see that $F(c, 1, y)$ is independent of y, we have

$$F(c, 1, 0) = F(c, 1, 1) = \mathfrak{h}_9(c).$$

$$\mathfrak{h}_9(c) = 71c^6 - 744c^4 + 1872c^2.$$

Taking the derivative with respect to c, we obtain

$$\mathfrak{h}_9'(c) = 426c^5 - 2976c^3 + 3744c.$$

By setting $\mathfrak{h}_9'(c) = 0$, we obtain the critical point $c = \frac{2}{71}\sqrt{4402 - 355\sqrt{43}}$ at which $\mathfrak{h}_9(c)$ achieves its maximum value, that is,

$$F(c, 1, 0) \leq \frac{345551959}{250000}.$$

(v) On edge $c = 0$ and $x = 1$, $F(c, x, y)$ yields

$$F(0, 1, y) = 0$$

(vi) On edge $c = 2$, $F(c, x, y)$ takes the form

$$F(2, x, y) = 128.$$

As $F(2, x, y)$ is independent of c, x and y, we have

$$F(2, 0, y) = F(2, 1, y) = F(2, x, 0) = F(2, x, 1) = 128.$$

(vii) On edge $c = 0$ and $y = 1$, $F(c, x, y)$ is equivalent to

$$F(0, x, 1) = -2304x^4 - 16128x^2 + 18432 = \mathfrak{h}_{10}(x).$$

It is clear that

$$\mathfrak{h}'_{10}(x) = -9216x^3 - 32256x.$$

Note that $\mathfrak{h}'_{10}(x) < 0$ in $[0, 1]$, therefore, $\mathfrak{h}_{10}(x)$ is decreasing in $[0, 1]$. Hence, $\mathfrak{h}_{10}(x)$ achieves its maximum at $x = 0$, which is provided by

$$F(0, x, 1) \leq 18432.$$

(viii) On edge $c = 0$ and $y = 0$, $F(c, x, y)$ becomes

$$F(0, x, 0) = -20736x^3 + 20736x = \mathfrak{h}_{11}(x).$$

Clearly,

$$\mathfrak{h}'_{11}(x) = -62208x^2 + 20736.$$

We know that $\mathfrak{h}'_{11}(x) = 0$ provides the critical point $x = \frac{1}{\sqrt{3}}$ at which $\mathfrak{h}_{11}(x)$ achieves its maximum value. Thus, we have

$$F(0, x, 0) \leq 4608\sqrt{3}.$$

Hence, from the above cases we can deduce that

$$F(c, x, y) \leq 18432 \quad \text{on} \quad [0, 2] \times [0, 1] \times [0, 1].$$

From (28), we have

$$\left|H_{2,2}(G_g/2)\right| \leq \frac{1}{2654208}(F(c, x, y)) \leq \frac{1}{144}.$$

If $g \in \mathcal{S}^*_{seg}$, then the sharp bound for the second Hankel determinant is achieved using (3)–(5) and

$$g_3(z) = z \exp\left(\int_0^z \frac{2}{1 + e^{-t^3}} dt\right) = z + \frac{1}{6}z^4 + \frac{1}{72}z^7 + \cdots.$$

□

5. Second Hankel Determinant with Logarithmic Coefficients for the Class $\mathcal{S}^*_{\sin}$

Theorem 7. *If the function $g \in \mathcal{S}^*_{\sin}$ is defined by (1), then*

$$\left|H_{2,2}(G_g/2)\right| \leq \frac{1}{36}.$$

The inequality is sharp.

Proof. From the definition of the class $\mathcal{S}^*_{\sin}$ along with the subordination principal, there is a Schwarz function $w(z)$ such that

$$\frac{zg'(z)}{g(z)} = 1 + \sin(w(z)), \qquad (z \in \mathbb{O}_d).$$

Assuming that $q \in \mathcal{P}$, by writing q in terms of the Schwarz function $w(z)$, we have

$$q(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

which is equivalent to

$$w(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}. \tag{31}$$

Using (1), we can easily obtain

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= 1 + a_2 z + \left(-a_2^2 + 2a_3\right) z^2 + \left(3a_4 - 3a_2 a_3 + a_2^3\right) z^3 \\ &\quad + \left(-a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2 + 4a_5\right) z^4 + \cdots. \end{aligned} \tag{32}$$

From the series expansion of (31), we have

$$\begin{aligned} 1 + \sin(w(z)) &= 1 + \left(\frac{1}{2}c_1\right) z + \left(-\frac{1}{4}c_1^2 + \frac{1}{2}c_2\right) z^2 + \left(-\frac{1}{2}c_1 c_2 + \frac{5}{48}c_1^3 + \frac{1}{2}c_3\right) z^3 \\ &\quad + \left(-\frac{1}{4}c_2^2 - \frac{1}{2}c_1 c_3 - \frac{1}{32}c_1^4 + \frac{1}{2}c_4 + \frac{5}{16}c_1^2 c_2\right) z^4 + \cdots. \end{aligned} \tag{33}$$

By comparing (32) and (33), it follows that

$$\begin{aligned} a_2 &= \left(\frac{1}{2}\right) c_1, \\ a_3 &= \left(\frac{1}{4}\right) c_2, \\ a_4 &= -\frac{1}{24}c_1 c_2 - \frac{1}{144}c_1^3 + \frac{1}{6}c_3, \\ a_5 &= -\frac{1}{32}c_2^2 - \frac{1}{24}c_1 c_3 + \frac{5}{1152}c_1^4 + \frac{1}{8}c_4 - \frac{1}{192}c_1^2 c_2. \end{aligned} \tag{34}$$

By using (34) in (2)–(5), we achieve

$$\lambda_1 = \left(\frac{1}{4}\right) c_1, \tag{35}$$

$$\lambda_2 = -\frac{1}{16}c_1^2 + \frac{1}{8}c_2, \tag{36}$$

$$\lambda_3 = -\frac{1}{12}c_1 c_2 + \frac{5}{288}c_1^3 + \frac{1}{12}c_3, \tag{37}$$

$$\lambda_4 = -\frac{1}{32}c_2^2 - \frac{1}{256}c_1^4 + \frac{1}{16}c_4 - \frac{1}{16}c_1 c_3 + \frac{5}{128}c_1^2 c_2. \tag{38}$$

The determinant $H_{2,2}(G_g/2)$ is described as follows:

$$H_{2,2}(G_g/2) = \lambda_2 \lambda_4 - \lambda_3^2.$$

By virtue of (36)–(38) along with $c_1 = c \in [0,2]$, we can find that

$$H_{2,2}(G_g/2) = \frac{1}{331776}\Big(-19c^6 - 12c^4c_2 + 336c^3c_3 - 36c^2c_2^2 - 1296c^2c_4 + 2016cc_2c_3 - 1296c_2^3 + 2592c_2c_4 - 2304c_3^2\Big). \quad (39)$$

Let $j = 4 - c^2$ in (11)–(13). Now, using the aforementioned lemmas, we obtain

$$\begin{aligned}
12c^4c_2 &= 6c^6 + 6c^4jx,\\
336c^3c_3 &= 84\Big(c^6 - c^4jx^2\Big) + 168\Big(c^4xj + c^3j\Big(1-|x|^2\Big)\varrho\Big),\\
36c^2c_2^2 &= 18c^4jx + 9\Big(c^6 + c^2j^2x^2\Big),\\
1296c^2c_4 &= 162c^6 + 162c^4jx^3 - 486c^4jx^2 + 486c^4xj + 648jc^2x^2\\
&\quad -648c^3j\Big(1-|x|^2\Big)x\varrho - 648c^2j\Big(1-|x|^2\Big)\overline{x}\varrho^2 + 648c^2j\Big(1-|x|^2\Big)\\
&\quad \Big(1-|\varrho|^2\Big)\delta + 648c^3j\Big(1-|x|^2\Big)\varrho,\\
2016cc_2c_3 &= -252x^3j^2c^2 - 252c^4jx^2 + 504cxj^2\Big(1-|x|^2\Big)\varrho + 504x^2j^2c^2\\
&\quad +504c^3j\Big(1-|x|^2\Big)\varrho + 756c^4xj + 252c^6,\\
1296c_2^3 &= 486\Big(c^4jx + c^2j^2x^2\Big) + 162\Big(j^3x^3 + c^6\Big),\\
2592c_2c_4 &= 162c^6 + 162c^4jx^3 - 486c^4jx^2 + 648c^4xj + 648jc^2x^2\\
&\quad -648c^3j\Big(1-|x|^2\Big)x\varrho - 648c^2j\Big(1-|x|^2\Big)\overline{x}\varrho^2 + 648c^2j\Big(1-|x|^2\Big)\\
&\quad \Big(1-|\varrho|^2\Big)\delta + 648c^3j\Big(1-|x|^2\Big)\varrho + 162x^4j^2c^2 - 486x^3j^2c^2\\
&\quad +486x^2j^2c^2 + 648x^3j^2 - 648x^2j^2\Big(1-|x|^2\Big)c\varrho - 648xj^2\Big(1-|x|^2\Big)\overline{x}\varrho^2\\
&\quad +648xj^2\Big(1-|x|^2\Big)\Big(1-|\varrho|^2\Big)\delta + 648cxj^2\Big(1-|x|^2\Big)\varrho,\\
2304c_3^2 &= 144x^4j^2c^2 - 576x^2j^2\Big(1-|x|^2\Big)c\varrho - 576x^3j^2c^2 - 288c^4jx^2 + 576j^2\\
&\quad \Big(1-|x|^2\Big)^2\varrho^2 + 1152cxj^2\Big(1-|x|^2\Big)\varrho + 576x^2j^2c^2 + 576c^3j\Big(1-|x|^2\Big)\varrho\\
&\quad +576c^4xj + 144c^6.
\end{aligned}$$

Inserting the above formulae into (39), we obtain

$$\begin{aligned}
H_{2,2}(G_g/2) = \frac{1}{331776}\Big\{&96c^3j\Big(1-|x|^2\Big)\varrho - 162x^3j^3 + 648x^3j^2 - 48c^4jx^2\\
&-81c^2x^2j^2 - 162x^3j^2c^2 + 18x^4j^2c^2 - 576j^2\Big(1-|x|^2\Big)^2\varrho^2\\
&-72x^2j^2\Big(1-|x|^2\Big)c\varrho - 648xj^2\Big(1-|x|^2\Big)\overline{x}\varrho^2 + 648xj^2\\
&\Big(1-|x|^2\Big)\Big(1-|\varrho|^2\Big)\delta - 4c^6\Big\}.
\end{aligned}$$

Because $j = 4 - c^2$,

$$H_{2,2}(G_g/2) = \frac{1}{331776}\Big(q_1(c,x) + q_2(c,x)\varrho + q_3(c,x)\varrho^2 + \omega(c,x,\varrho)\delta\Big),$$

where $x, \varrho, \delta \in \overline{\mathbb{O}_d}$ and

$$
\begin{aligned}
q_1(c,x) &= -4c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(-81c^2x^2+18c^2x^4\right) - 48c^4x^2\right],\\
q_2(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-72cx^2\right) - 96c^3\right],\\
q_3(c,x) &= \left(4-c^2\right)\left(1-|x|^2\right)\left[\left(4-c^2\right)\left(-72|x|^2 - 576\right)\right],\\
\omega(c,x,\varrho) &= \left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\varrho|^2\right)\left[648x\left(4-c^2\right)\right].
\end{aligned}
$$

Let $|x| = x$ and $|\varrho| = y$. Taking $|\delta| \leq 1$, we obtain

$$
\begin{aligned}
\left|H_{2,2}\left(G_g/2\right)\right| &\leq \frac{1}{331776}\left(|q_1(c,x)| + |q_2(c,x)|y + |q_3(c,x)|y^2 + |\omega(c,x,\varrho)|\right).\\
&\leq \frac{1}{331776}(\mathcal{E}(c,x,y)),
\end{aligned} \tag{40}
$$

where

$$
\mathcal{E}(c,x,y) = v_1(c,x) + v_2(c,x)y + v_3(c,x)y^2 + v_4(c,x)\left(1-y^2\right),
$$

with

$$
\begin{aligned}
v_1(c,x) &= 4c^6 + \left(4-c^2\right)\left[\left(4-c^2\right)\left(81c^2x^2+18c^2x^4\right) + 48c^4x^2\right],\\
v_2(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(72cx^2\right) + 96c^3\right],\\
v_3(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)\left(72x^2 + 576\right)\right],\\
v_4(c,x) &= \left(4-c^2\right)\left(1-x^2\right)\left[648x\left(4-c^2\right)\right].
\end{aligned}
$$

Now, we have to maximize $\mathcal{E}(c,x,y)$ in the closed cuboid $\Gamma : [0,2] \times [0,1] \times [0,1]$.

For this, we have to discuss the maximum values of $\mathcal{E}(c,x,y)$ in the interior of Γ, in the interior of its six faces, and on its twelve edges.

1. Interior points of cuboid Γ **:**

Let $(c,x,y) \in (0,2) \times (0,1) \times (0,1)$. By taking a partial derivative of $\mathcal{E}(c,x,y)$ with respect to y, we obtain

$$
\frac{\partial \mathcal{E}}{\partial y} = 24\left(4-c^2\right)(1-x^2)\left[6y\left(4-c^2\right)(x-1)(x-8) + c\left(3x^2\left(4-c^2\right) + 4c^2\right)\right].
$$

Setting $\frac{\partial \mathcal{E}}{\partial y} = 0$ yields

$$
y = \frac{c\left(3x^2(4-c^2) + 4c^2\right)}{6(4-c^2)(x-1)(8-x)} = y_0.
$$

If y_0 is a critical point inside Γ, then $y_0 \in (0,1)$, which is possible only if

$$
3cx^2\left(4-c^2\right) + 4c^3 < 6\left(4-c^2\right)(x-1)(8-x). \tag{41}
$$

and

$$
c^2 > 4. \tag{42}
$$

For the existence of the critical points, we have to obtain the solutions which satisfy both inequalities (41) and (42).

As $c^2 > 4$, it is not hard to show that (41) does not hold true in this case for all values of $x \in (0,1)$. Thus, there is no critical point of $\mathcal{E}(c,x,y)$ in $(0,2) \times (0,1) \times (0,1)$.

2. Interior of all six faces of cuboid Γ **:**

(i) On face $c = 0$, $\mathcal{E}(c, x, y)$ yields

$$\mathcal{E}(0, x, y) = I_1(x, y) = 1152y^2(1 - x^2)(x - 8)(x - 1) + 10368x\left(1 - x^2\right).$$

Taking the partial derivative with respect to y, we obtain

$$\frac{\partial I_1}{\partial y} = 2304y(1 - x^2)(x - 8)(x - 1).$$

However, $\frac{\partial I_1}{\partial y} \neq 0$. Hence, $I_1(x, y)$ has no critical point in the interval $(0, 1) \times (0, 1)$.
(ii) On face $c = 2$, $\mathcal{E}(c, x, y)$ reduces to

$$\mathcal{E}(2, x, y) = 256.$$

(iii) On face $x = 0$, $\mathcal{E}(c, x, y)$ is equivalent to

$$\mathcal{E}(c, 0, y) = I_2(c, y) = 4c^6 + 96c^3y(4 - c^2) + 576y^2\left(4 - c^2\right)^2.$$

Taking the derivative of $I_2(c, y)$ partially with respect to y, we have

$$\frac{\partial I_2}{\partial y} = 96c^3(4 - c^2) + 1152y\left(4 - c^2\right)^2.$$

Again, taking the derivative of $I_2(c, y)$ partially with respect to c, we have

$$\frac{\partial I_2}{\partial c} = 24c^5 - 192c^4y + 288c^2y\left(4 - c^2\right) - 2304cy^2\left(4 - c^2\right).$$

A numerical calculation shows that the system of equations

$$\frac{\partial I_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial I_2}{\partial c} = 0,$$

has no solution in $(0, 2) \times (0, 1)$. Hence, $I_2(c, y)$ has no optimal point in the interval $(0, 2) \times (0, 1)$.
(iv) On face $x = 1$, $\mathcal{E}(c, x, y)$ takes the form

$$\mathcal{E}(c, 1, y) = I_3(c, y) = 55c^6 - 600c^4 + 1584c^2$$

Clearly,

$$\frac{\partial I_3}{\partial c} = 330c^5 - 2400c^3 + 3168c.$$

By solving $\frac{\partial I_3}{\partial c} = 0$, we can find that the only critical point in $(0, 2)$ is $c = \frac{2}{55}\sqrt{2750 - 55\sqrt{685}}$, at which $I_3(c, y)$ achieves its maximum value, which is provided by

$$\mathcal{E}(c, 1, y) \leq \frac{284800 + 17536\sqrt{685}}{605}.$$

Now, using (40) along with the last obtained value, we can conclude that $|H_{2,2}(G_g/2)| < \frac{1}{36}$.
(v) On face $y = 0$, $\mathcal{E}(c, x, y)$ becomes

$$\begin{aligned} \mathcal{E}(c, x, 0) \quad = \quad & I_4(c, x) = 18c^6x^4 + 33c^6x^2 - 144c^4x^4 - 648c^4x^3 + 4c^6 - 456c^4x^2 \\ & + 288c^2x^4 + 648c^4x + 5184c^2x^3 + 1296c^2x^2 - 5184c^2x \\ & - 10368x^3 + 10368x. \end{aligned}$$

Taking the derivative partially with respect to x and then simplifying, with respect to c we have

$$\begin{aligned}\frac{\partial I_4}{\partial x} &= 72c^6x^3 + 66c^6x - 576c^4x^3 - 1944c^4x^2 - 912c^4x + 1152c^2x^3 + 648c^4 \\ &\quad +15552c^2x^2 + 2592c^2x - 5184c^2 - 31104x^2 + 10368.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial I_4}{\partial c} &= 108c^5x^4 + 198c^5x^2 - 576c^3x^4 - 2592c^3x^3 + 24c^5 - 1824c^3x^2 + 576cx^4 \\ &\quad +2592c^3x + 10368cx^3 + 2592cx^2 - 10368cx.\end{aligned}$$

Thus, after a few basic calculations we can find that the system of equations has no solution

$$\frac{\partial I_4}{\partial x} = 0 \quad \text{and} \quad \frac{\partial I_4}{\partial c} = 0,$$

in the interval $(0,2) \times (0,1)$. Hence, $I_4(c,x)$ has no optimal solution in the interval $(0,2) \times (0,1)$.

(vi) On face $y = 1$, $\mathcal{E}(c,x,y)$ yields

$$\begin{aligned}\mathcal{E}(c,x,1) &= I_5(c,x) = 18c^6x^4 - 72c^5x^4 + 33c^6x^2 - 216c^4x^4 + 168c^5x^2 + 576c^3x^4 \\ &\quad +4c^6 - 960c^4x^2 + 864c^2x^4 - 96c^5 - 960c^3x^2 - 1152cx^4 + 576c^4 \\ &\quad +5328c^2x^2 - 1152x^4 + 384c^3 + 1152cx^2 - 4608c^2 - 8064x^2 + 9216.\end{aligned}$$

Taking the partial derivative of $I_5(c,x)$ with respect to x, with respect to c we have

$$\begin{aligned}\frac{\partial I_5}{\partial x} &= 72c^6x^3 - 288c^5x^3 - 864c^4x^3 + 336c^5x + 2304c^3x^3 - 1920c^4x - 4608x^3 \\ &\quad +3456c^2x^3 - 1920c^3x - 4608cx^3 + 10656c^2x + 2304cx - 16128x + 66c^6x.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial I_5}{\partial c} &= 108c^5x^4 - 360c^4x^4 + 198c^5x^2 - 864c^3x^4 + 840c^4x^2 + 1728c^2x^4 + 24c^5 \\ &\quad -3840c^3x^2 + 1728cx^4 - 480c^4 - 2880c^2x^2 - 1152x^4 + 2304c^3 \\ &\quad +10656cx^2 + 1152c^2 + 1152x^2 - 9216c.\end{aligned}$$

As in the above case, we can obtain the same result for face $y = 0$, that is, that there is no existing solution for the system of equations

$$\frac{\partial I_5}{\partial x} = 0 \quad \text{and} \quad \frac{\partial I_5}{\partial c} = 0,$$

in the interval $(0,2) \times (0,1)$.

3. On the Edges of Cuboid Γ **:**

(i) On edge $x = 0$ and $y = 0$, $\mathcal{E}(c,x,y)$ reduces to

$$\mathcal{E}(c,0,0) = 4c^6 = I_6(c).$$

Clearly, the function $I_6(c)$ achieves its maximum value at $c = 2$, as provided by

$$\mathcal{E}(c,0,0) \leq 256.$$

(ii) On edge $x = 0$ and $y = 1$, $\mathcal{E}(c,x,y)$ is equivalent to

$$\mathcal{E}(c,0,1) = 4c^6 - 96c^5 + 576c^4 + 384c^3 - 4608c^2 + 9216 = I_7(c).$$

Clearly,

$$I_7'(c) = 24c^5 - 480c^4 + 2304c^3 + 1152c^2 - 9216c.$$

We can see that $I_7'(c) < 0$ in $[0,2]$ shows that $I_7(c)$ is decreasing over $[0,2]$. Thus, $I_7(c)$ achieves its maxima at $c = 0$. Hence,

$$\mathcal{E}(c,0,1) \leq 9216.$$

(iii) On edge $c = 0$ and $x = 0$, $\mathcal{E}(c,x,y)$ reduces to

$$\mathcal{E}(0,0,y) = 9216y^2 = I_8(y).$$

Note that from $I_8'(y) > 0$ in $[0,1]$ it follows that $I_8(y)$ is increasing over $[0,1]$. Thus, $I_8(y)$ achieves its maxima at $y = 1$. Thus, we have

$$\mathcal{E}(0,0,y) \leq 9216.$$

(iv) As we can see that $\mathcal{E}(c,1,y)$ is independent of y, we have

$$\mathcal{E}(c,1,0) = \mathcal{E}(c,1,1) = I_9(c).$$

$$I_9(c) = 55c^6 - 600c^4 + 1584c^2.$$

Taking the derivative with respect to c, we have

$$I_9'(c) = 330c^5 - 2400c^3 + 3168c.$$

Setting $I_9'(c) = 0$, we obtain the critical point $c = \frac{2}{55}\sqrt{2750 - 55\sqrt{685}}$ at which $I_9(c)$ achieves its maximum value, which is provided by

$$\mathcal{E}(c,1,0) \leq \frac{284800 + 17536\sqrt{685}}{605}.$$

(v) On the edge $c = 0$ and $x = 1$, $\mathcal{E}(c,x,y)$ yields

$$\mathcal{E}(0,1,y) = 0$$

(vi) On edge $c = 2$, $\mathcal{E}(c,x,y)$ takes the form

$$\mathcal{E}(2,x,y) = 256.$$

As $\mathcal{E}(2,x,y)$ is independent of c, x and y, we have

$$\mathcal{E}(2,0,y) = \mathcal{E}(2,1,y) = \mathcal{E}(2,x,0) = \mathcal{E}(2,x,1) = 256.$$

(vii) On edge $c = 0$ and $y = 1$, $\mathcal{E}(c,x,y)$ is equivalent to

$$\mathcal{E}(0,x,1) = -1152x^4 - 8064x^2 + 9216 = I_{10}(x).$$

It is clear that

$$I_{10}'(x) = -4608x^3 - 16128x.$$

Note that $I_{10}'(x) < 0$ in $[0,1]$; therefore, $I_{10}(x)$ is decreasing in $[0,1]$. Hence, $I_{10}(x)$ achieves its maxima at $x = 0$, which is provided by

$$\mathcal{E}(0,x,1) \leq 9216.$$

(viii) On edge $c = 0$ and $y = 0$, $\mathcal{E}(c,x,y)$ becomes

$$\mathcal{E}(0,x,0) = -10368x^3 + 10368x = I_{11}(x).$$

Clearly,

$$I'_{11}(x) = -31104x^2 + 10368.$$

We know that $I'_{11}(x) = 0$ yields the critical point $x = \frac{1}{\sqrt{3}}$ at which $I_{11}(x)$ achieve its maximum value. Thus, we have

$$\mathcal{E}(0, x, 0) \leq 2304\sqrt{3}.$$

Hence, from the above cases we can deduce that

$$\mathcal{E}(c, x, y) \leq 9216 \quad \text{on} \quad [0, 2] \times [0, 1] \times [0, 1].$$

From (40), we have

$$|H_{2,2}(G_g/2)| \leq \frac{1}{331776}(\mathcal{E}(c, x, y)) \leq \frac{1}{36}.$$

If $g \in \mathcal{S}^*_{\sin}$, then the sharp bound for the second Hankel determinant can be achieved using (3)–(5) and

$$g(z) = z \exp\left(\int_0^z \frac{1 + \sin(t^3) - 1}{t} dt\right) = z + \frac{1}{3}z^4 + \cdots.$$

□

6. Conclusions

Calculating the third-order Hankel determinant sharp bound is a challenging task in spite of the extensive literature on the Hankel determinants in the area of geometric function theory. In the present article, two subfamilies of starlike functions connected to special functions are taken into consideration. For the stated classes, we achieve sharp bounds on the coefficient-related problems. In particular, by transforming the third Hankel determinant to a real function with three variables defined on a cuboid, we determine the exact bound of the third Hankel determinant with logarithmic coefficient entries. This makes it easier to comprehend the additional geometric characteristics of these function classes. By upgrading the current methodologies, it could be feasible to obtain more results for other univalent or analytic function subfamilies.

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