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The Fractional Hilbert Transform of Generalized Functions

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Abstract: The fractional Hilbert transform, a generalization of the Hilbert transform, has been extensively studied in the literature because of its widespread application in optics, engineering, and signal processing. In the present work, we expand the fractional Hilbert transform that displays an odd symmetry to a space of generalized functions known as Boehmians. Moreover, we introduce a new fractional convolutional operator for the fractional Hilbert transform to prove a convolutional theorem similar to the classical Hilbert transform, and also to extend the fractional Hilbert transform to Boehmians. We also produce a suitable Boehmian space on which the fractional Hilbert transform exists. Further, we investigate the convergence of the fractional Hilbert transform for the class of Boehmians and discuss the continuity of the extended fractional Hilbert transform.

Keywords: convolution; Boehmian; fractional Hilbert transform; Hilbert transform; equivalence class; delta sequences; compact support

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1. Introduction

The space of Boehmians is a class of generalized functions that include all regular operators and generalized functions or distributions, and other objects. The theory of Boehmians with two convergences, introduced by Mikusinski [1], broadens the concept of Boehme's regular operators [2]. In contrast to the theory of distributions in which generalized functions are treated as members of the dual space of any space of testing function, the space of Boehmians treats distributions more as algebraic objects. Several integral transforms for various spaces of Boehmians were studied and their properties were investigated in [3–13]. Currently, a large number of studies are available on the extension of classical integral transforms to Boehmians. Karunakaran and Roopkumar introduced the Hilbert transform as continuous linear mapping defined on some space of Boehmians into another space of Boehmians [7]. They also studied the Hilbert transform for the space of ultradistributions [8]. The pioneering work of Zayed [13], Al-Omari, and Agarwal [6] introduced an extension of fractional integral transform to Boehmians by extending the fractional Fourier and Sumudu transforms to the space of integrable Boehmians. The properties and generalizations of various quaternion integral transform [14] and fractional integral transforms were also studied from the perspective of q-calculus analysis [15,16] and rapidly decaying functions [17]. In recent years, the extension of fractional integral transforms to the space of Boehmians has been an active area of research. Many well-known fractional integral transforms have been extended to the space of Boehmians, but an extension of the fractional Hilbert transform (FHT) has not yet been reported. So, the goal of this paper is to extend the FHT to some space of Boehmians. Different definitions of FHT exist in the literature [18–20], but in the generalization of the classical Hilbert transform, it might rightly be said that the fractionalization of Hilbert transform is given by Zayed and is mathematically elaborated in [21]. The fractional Hilbert transform of a function $f(x)$, denoted by $H_\alpha[f(x)]$, is defined as [20]

$$H_\alpha[f(x)] = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-t^2}{2}\cot\alpha}}{x-t} f(t)dt \ \text{ for } \alpha \neq 0,\ \pi/2,\ \pi, \tag{1}$$

where the integral is taken in the sense of the Cauchy principal value. The special case $\alpha = \pi/2$ reduces FHT into the standard Hilbert transform. Indeed, the FHT allows for converting a real signal into a complex signal by suppressing the negative frequency. Such a signal has a wide variety of applications in optics, signal processing, and image processing [22–25]. It also does not flip the domain of the signal—the signal remains in the same domain. However, it lacks detailed mathematical analysis, so we require a thorough mathematical theory of FHT to understand its strengths and limitations. Consequently, we need to extend the existing theory on such a significant transformation in terms of generalized functions. An extension of FHT to some space of Boehmians may have applications in engineering and other sciences, as it may apply in converting functions with discontinuities into smooth functions that consequently lead to the description of various physical occurrences such as point charges [26].

The present paper is organized as follows: Section 1 covers the introduction. Section 2 covers the important definitions and theorems, and we also discuss the abstract construction of Boehmians to render the paper self-contained. Section 3 covers results that comprise a new convolutional operator and a new convolutional theorem for FHT, and proves auxiliary results required for the construction of two Boehmian spaces. Lastly, we extend the FHT to some spaces of Boehmians. Section 4 presents our conclusions.

2. Preliminaries

Let $\mathbb{R}$ be the set of all real numbers, $\mathcal{L}^1(\mathbb{R}) = \mathcal{L}^1$ be the collection of complex-valued measurable functions f defined on $\mathbb{R}$ for which

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|dx < \infty,$$

and $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R})$ be the set of all infinitely differentiable functions defined on $\mathbb{R}$, such that functions and their derivatives converge uniformly on compact sets in $\mathbb{R}$.

Theorem 1 ([27] Theorem 9.5)**.** *For any function f on $\mathbb{R}$ and for all $t \in \mathbb{R}$, let f_t be defined by*

$$f_t(x) = f(x-t).$$

If $p \geq 1$ and $f \in \mathcal{L}^p$, then mapping $t \to f_t$ is uniformly continuous from $\mathbb{R}$ into $\mathcal{L}^p(\mathbb{R})$.

Definition 1. *Let f and g be any two functions on $\mathbb{R}$; their convolution, denoted by $f * g$, is defined as*

$$f * g = \int_{-\infty}^{\infty} f(t)g(x-t)dt. \tag{2}$$

The Hilbert transform of convolutional operation $*$ is given as follows:

Theorem 2. *If $f,\ g \in \mathcal{L}^1(\mathbb{R})$ with Hilbert transforms Hf, Hg respectively, so that $Hf, Hg \in \mathcal{L}^1(\mathbb{R})$, then*

$$H[f * g] = Hf * g = f * Hg.$$

The FHT may not act as agreeably with the classical convolutional operator as the classical Hilbert transform (Theorem 2).

Boehmian Space

The members of Boehmian spaces are called Boehmians, which are equivalence classes of "quotients of sequences". These equivalence classes are formulated from an integral domain of continuous functions. The integral domain operations for Boehmians are addition and convolution. This convolutional operation may differ from the standard convolutional operation given in Definition 2.

We now present a brief introduction to Boehmians.

Let G be a complex linear space, $(H, .)$ is a commutative semigroup, and let $\otimes : G \times H \to G$, so that the conditions given below hold:

- $(f \otimes \phi) \otimes \psi = f \otimes (\phi.\psi), \quad \forall f \in G, \forall \phi, \psi \in H;$
- $(f + g) \otimes \phi = f \otimes \phi + g \otimes \phi, \quad \forall f, g \in G, \forall \phi \in H;$
- $\lambda(f \otimes \phi) = (\lambda f \otimes \phi) \; \forall f \in G, \quad \forall \phi \in H, \lambda \in \mathbb{C};$
- If $f_n \to f$ as $n \to \infty$ and $\phi \in H$ then $f_n \otimes \phi \to f \otimes \phi$ as $n \to \infty$.

Let Δ be a collection of sequences on H, so that

- If $\{\phi_n\}, \{\psi_n\} \in \Delta$ then $\{\phi_n.\psi_n\} \in \Delta$;
- If $f_n \to f$ as $n \to \infty$ and $\{\phi_n\} \in \Delta$ then $f_n \otimes \phi_n \to f$ as $n \to \infty$.

A pair of sequences $\{f_n, \phi_n\}$ with $f_n \in G$ for all $n \in \mathbb{N}$ and $\{\phi_n\} \in \Delta$ are a quotient of sequences, denoted by $\frac{f_n}{\phi_n}$, if

$$f_n \otimes \phi_m = f_m \otimes \phi_n \quad \forall m, n \in \mathbb{N}.$$

Two quotients of sequences $\frac{f_n}{\phi_n}$ and $\frac{g_n}{\psi_n}$ are equivalent ($\sim$) if, for every $n \in \mathbb{N}$

$$f_n \otimes \psi_n = g_n \otimes \phi_n.$$

The equivalence class of $\frac{f_n}{\phi_n}$ induced by "$\sim$" is denoted by $\left[\frac{f_n}{\phi_n}\right]$. Every equivalence class is called a Boehmian. The space of all Boehmians is denoted by $\mathcal{B} = \mathcal{B}(G, H, \otimes, \Delta)$. $\mathcal{B}$ is a vector space under the operations of addition and scalar multiplication defined as follows:

- $\lambda\left[\frac{f_n}{\phi_n}\right] = \left[\frac{\lambda f_n}{\phi_n}\right];$
- $\left[\frac{f_n}{\phi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n \otimes \phi_n + g_n \otimes \psi_n}{\phi_n \otimes \psi_n}\right].$

If we define an isomorphism $f \to \left[\frac{f \otimes \phi_n}{\phi_n}\right]$, then G is a subspace of $\mathcal{B}$. Therefore, every element of G can be expressed uniquely as a Boehmian.

3. Results

In this section, we define a new convolutional operation for FHT that yields a generalized result for Theorem 2. Moreover, to extend the FHT to the class of Boehmians, we define two classes of Boehmians. Two convergences of FHT are proved on $\mathcal{C}^\infty$. Lastly, an extension of FHT on Boehmians is introduced.

3.1. Convolutional Structure for Fractional Hilbert Transform

The idea of convolutional operation makes it evident that, given any integral transform, we can associate a convolutional operation to it [28]. So, we introduce a new fractional convolutional operator that helps us in extending FHT to the space of Boehmians.

Definition 2. *Let $f, g \in \mathcal{L}^1(\mathbb{R})$. We define a fractional convolution $(f *_\alpha g)$ as*

$$(f *_\alpha g)(x) = \int_{-\infty}^{\infty} f(x - t) g(t) e^{-it(x-t)\cot\alpha} dt. \tag{3}$$

Lemma 1. *Let $f, g \in \mathcal{L}^1$. Then, $(f *_\alpha g)$ is also in $\mathcal{L}^1$.*

Proof. To prove that $f *_\alpha g \in \mathcal{L}^1$, we consider its $\mathcal{L}^1$ norm.

$$\begin{aligned}\|f *_\alpha g\|_1 &= \int_{-\infty}^{\infty} |f *_\alpha g| dx \\ &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-t)||g(t)| dt dx.\end{aligned}$$

By using Fubini's theorem, we have

$$\|f *_\alpha g\|_1 \leq \int_{-\infty}^{\infty} |f(x-t)| dx \int_{-\infty}^{\infty} |g(t)| dt.$$

Since the $\mathcal{L}^1$ norm is translation invariance, so $\int_{-\infty}^{\infty} |f(x-t)| dx = \|f_t\|_1 = \|f\|_1$. Therefore,

$$\|f *_\alpha g\|_1 \leq \|f\|_1 \|g\|_1.$$

Since $f, g \in \mathcal{L}^1$,

$$\|f *_\alpha g\|_1 \leq \|f\|_1 \|g\|_1 < \infty,$$

which proves that $f *_\alpha g \in \mathcal{L}^1$. □

To extend the FHT to the case of Boehmians, the essential step is to prove the convolutional theorem, and suitable Boehmian spaces can then be constructed by proving the supplementary results. Now, we state and prove the convolutional theorem for FHT.

Theorem 3. *(convolutional Theorem) Assume that $f, g \in \mathcal{L}^1$. Then,*

$$H_\alpha[f *_\alpha g] = H_\alpha[f] *_\alpha g = f *_\alpha H_\alpha[g]. \tag{4}$$

In addition, $(f *_\alpha g) = -(H_\alpha[f] *_\alpha H_\alpha[g])$.

Proof.

$$\begin{aligned}H_\alpha[(f *_\alpha g)(x)] &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-t^2}{2}\cot\alpha}}{x-t}(f *_\alpha g)(t)\, dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-t^2}{2}\cot\alpha}}{x-t}\int_{-\infty}^{\infty} f(t-y)g(y)e^{-iy(t-y)\cot\alpha} dy dt.\end{aligned}$$

By changing variables $t - y = \nu$, the above equation can be simplified to

$$\begin{aligned}H_\alpha[(f *_\alpha g)(x)] &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\frac{x^2-2xy+y^2-\nu^2}{2}\cot\alpha}}{(x-y)-\nu} f(\nu)g(y)e^{-i(yx-y^2)\cot\alpha} d\nu dy \\ &= \int_{-\infty}^{\infty} H_\alpha[f(x-y)]g(y)e^{-iy(x-y)\cot\alpha} dy \\ &= (H_\alpha[f] *_\alpha g)(x).\end{aligned}$$

Similarly,

$$H_\alpha[(f *_\alpha g)(x)] = H_\alpha[(g *_\alpha f)(x)] = (H_\alpha[g] *_\alpha f)(x) = (f *_\alpha H_\alpha[g])(x). \tag{5}$$

If we substitute g by $H_\alpha[g]$ in (4), we can write

$$\begin{aligned}H_\alpha[(f *_\alpha H_\alpha[g])(x)] &= (H_\alpha[f] *_\alpha H_\alpha[g])(x), \\ (f *_\alpha H_\alpha[H_\alpha[g]])(x) &= (H_\alpha[f] *_\alpha H_\alpha[g])(x), \quad \text{(by (5))} \\ f *_\alpha g &= -(H_\alpha[f] *_\alpha H_\alpha[g]),\end{aligned}$$

where $H_\alpha^2 = -I$, and this proves the theorem. □

3.2. Abstract Construction of Boehmians

Now, we construct the Boehmian space required for extending the theory of the fractional Hilbert transform to some space of Boehmians. Here, we refer to only two spaces of Boehmians needed to develop the theory of FHT. Now to define the space of Boehmians, we introduce a class of identities as follows: Let space $\mathcal{D}$ constitute all infinitely differentiable functions with compact support in $\mathbb{R}$. Let

$$S = \{\phi \in \mathcal{D} : \phi \geq 0 \text{ and } \int_{\mathbb{R}} \phi = 1\}.$$

Then, the space of Boehmians is given by

$$\mathcal{B}_1 = \mathcal{B}_1(\mathcal{L}^1(\mathbb{R}), S, *_\alpha, \Delta),$$

where Δ is the collection of all sequences of real-valued functions $\{\phi_n(x)\} \subset S$, such that

1. $\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}\phi_n(x)dx = 1, \forall\, n \in \mathbb{N}$;
2. $\|\phi_n\|_1 \leq M, \forall\, n \in \mathbb{N}$ for some $M > 0$;
3. $\lim_{n\to\infty} \int_{|t|>\epsilon} |\phi_n(t)|dt = 0, \epsilon > 0$.

These sequences are *delta sequences*. We now state and prove the results that are needed to build the desired space for Boehmians.

Lemma 2. *The operation $*_\alpha$ is both commutative and associative.*

Proof. To prove that $*_\alpha$ is commutative, consider

$$(f *_\alpha g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)e^{-i(x-t)\cot\alpha}dt.$$

By changing variable $x - t = \tau$, we can simplify the above equation to

$$(f *_\alpha g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)e^{-i(x-\tau)\tau\cot\alpha}d\tau = (g *_\alpha f)(x).$$

To prove the associativity, let us consider

$$\begin{aligned}((f *_\alpha g) *_\alpha h)(x) &= \int_{-\infty}^{\infty} (f *_\alpha g)(x-t)h(t)e^{-i(x-t)\cot\alpha}dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-t-u)g(u)h(t)e^{-iu(x-t-u)\cot\alpha}e^{-it(x-t)\cot\alpha}dtdu.\end{aligned}$$

By changing variables $t + u = y$, we can write the above equation as

$$((f *_\alpha g) *_\alpha h)(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y-t)h(t)e^{-i(y-t)(x-y)\cot\alpha}e^{-it(x-t)\cot\alpha}dtdy.$$

As an application of Fubini's theorem, we have

$$\begin{aligned}((f *_\alpha g) *_\alpha h)(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y-t)h(t)e^{-i(-tx+yt+tx-t^2)\cot\alpha}f(x-y)e^{-iy(x-y)\cot\alpha}dtdy \\ &= \int_{-\infty}^{\infty} f(x-y)(g *_\alpha h)(y)e^{-iy(x-y)\cot\alpha}dy \\ &= (f *_\alpha (g *_\alpha h))(x).\end{aligned}$$

Thus, $((f *_\alpha g) *_\alpha h)(x) = (f *_\alpha (g *_\alpha h))(x)$. □

Lemma 3. *Assume that $\{\phi_n\}$ and $\{\psi_n\}$ are in Δ. Then, their convolution $\{\phi_n *_\alpha \psi_n\}$ is also in Δ.*

Proof. To prove that $\{\phi_n *_\alpha \psi_n\} \in \Delta$, we must show that the three conditions for delta sequences are fulfilled.

1. $\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}(\phi_n *_\alpha \psi_n)(x)dx = \int_{\mathbb{R}} e^{it(x-t)\cot\alpha} \int_{-\infty}^{\infty} \left(\phi_n(x-t)\psi_n(t)e^{-it(x-t)\cot\alpha}\right)dtdx.$
   By using Fubini's theorem, we can write
$$\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}(\phi_n *_\alpha \psi_n)(x)dx = \int_{\mathbb{R}} e^{it(x-t)\cot\alpha} e^{-it(x-t)\cot\alpha}\phi_n(x-t)dx \int_{-\infty}^{\infty} \psi_n(t)dt.$$
   Since $\{\phi_n\}, \{\psi_n\} \in \Delta$, then
$$\int_{\mathbb{R}} e^{it(x-t)\cot\alpha}(\phi_n *_\alpha \psi_n)(x)dx = 1.$$
2.
$$\begin{aligned}
\|\phi_n *_\alpha \psi_n\|_1 &= \int_{-\infty}^{\infty} |(\phi_n *_\alpha \psi_n)(x)|dx \\
&= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \phi_n(x-t)\psi_n(t)e^{-it(x-t)\cot\alpha}dt \right| dx \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \phi_n(x-t)\psi_n(t)e^{-it(x-t)\cot\alpha}dt \right| dx \\
&= \|\phi_n\|_1 \|\psi_n\|_1 \\
&\leq M^2, \quad \forall n \in \mathbb{N}.
\end{aligned}$$
   Thus, $\|\phi_n *_\alpha \psi_n\|_1 \leq M^2$.
3.
$$\begin{aligned}
\lim_{n\to\infty} \int_{|t|>\epsilon} |(\phi_n *_\alpha \psi_n)(x)|dx &\leq \lim_{n\to\infty} \int_{|t|>\epsilon} \int_{-\infty}^{\infty} |\phi_n(x-t)\psi_n(t)|dtdx \\
&= \|\phi_n\|_1 \lim_{n\to\infty} \int_{|t|>\epsilon} |\psi_n(t)|dt.
\end{aligned}$$
   Since $\{\psi_n\} \in \Delta$, then
$$\lim_{n\to\infty} \int_{|t|>\epsilon} |\psi_n(t)|dt = 0, \qquad \text{for } \epsilon > 0.$$
   Hence,
$$\int_{|t|>\epsilon} |(\phi_n *_\alpha \psi_n)(x)|dx \to 0 \quad \text{as } n \to \infty, \text{ for } \epsilon > 0.$$
   This completes the proof. □

Lemma 4. *If $f \in \mathcal{L}^1$ and $\phi_n \in \Delta$ then the convolution $f *_\alpha \phi_n \in \mathcal{L}^1$.*

Proof. Let $f \in \mathcal{L}^1$ and $\phi_n \in \Delta$. To show that $f *_\alpha \phi_n \in \mathcal{L}^1$, we consider the $\mathcal{L}^1$-norm.
$$\begin{aligned}
\|f *_\alpha \phi_n\|_1 &= \int_{\mathbb{R}} |(f *_\alpha \phi_n)(x)|dx, \\
&= \int_{\mathbb{R}} \left| \int_{-\infty}^{\infty} f(x-t)\phi_n(t)e^{-it(x-t)\cot\alpha}dt \right| dx, \\
&\leq \int_{\mathbb{R}} \int_{-\infty}^{\infty} \left| f(x-t)\phi_n(t)e^{-it(x-t)\cot\alpha} \right| dtdx, \\
&= \int_{-\infty}^{\infty} |f(x-t)|dx \int_{-\infty}^{\infty} |\phi_n(t)|dt, \\
&= \|f\|_1 \|\phi_n\|_1.
\end{aligned}$$

Since $f \in \mathcal{L}^1$ and $\{\phi_n\} \in \Delta$, $\|f *_\alpha \phi_n\|_1 \leq \|f\|_1 \|\phi_n\|_1 < \infty$, which proves that $f *_\alpha \phi_n \in \mathcal{L}^1$. □

Lemma 5. *If $f, g \in \mathcal{L}^1$, $\phi \in S$, then $(f+g) *_\alpha \phi = f *_\alpha \phi + g *_\alpha \phi$.*

The proof of this lemma is straightforward. Therefore, we omitted the details.

Lemma 6. *Let $f_n \to f$ in $\mathcal{L}^1$ as $n \to \infty$ and $\phi \in S$. Then $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{L}^1$.*

Proof. From Lemma 4, we can write

$$\begin{aligned}\|(f_n *_\alpha \phi) - (f *_\alpha \phi)\|_1 &= \|(f_n - f) *_\alpha \phi\|_1 \\ &\leq \|f_n - f\|_1 \|\phi\|_1 \\ &\leq M\|f_n - f\|_1 \to 0 \text{ as } n \to \infty \text{ for } M > 0.\end{aligned}$$

Hence, $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{L}^1$ whenever $f_n \to f$ in $\mathcal{L}^1$. □

Lemma 7. *Let $f_n \to f$ in $\mathcal{L}^1$ and $\{\phi_n\} \in \Delta$. Then $f_n *_\alpha \phi_n \to f$ in $\mathcal{L}^1$.*

Proof. Let $\{\phi_n\} \in \Delta$ then $\int_{-\infty}^{\infty} \phi_n(t) e^{it(x-t)} dt = 1$; therefore, we can write

$$\begin{aligned}(f_n *_\alpha \phi_n)(x) - f(x) &= \int_{-\infty}^{\infty} f_n(x-t)\phi_n(t) e^{-it(x-t)\cot\alpha} dt - f(x) \int_{-\infty}^{\infty} \phi_n(t) e^{it(x-t)\cot\alpha} dt \\ &= \int_{-\infty}^{\infty} \Big(f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x)\Big) e^{it(x-t)\cot\alpha} \phi_n(t) dt.\end{aligned}$$

Now, we consider the L^1-norm of the above equation:

$$\begin{aligned}\|f_n *_\alpha \phi_n - f\|_1 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \Big(f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x)\Big) e^{it(x-t)\cot\alpha} \phi_n(t) dt \right| dx \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x)||\phi_n(t)| dt dx.\end{aligned}$$

As an application of Fubini's theorem and via Property 2 of delta sequences, we have

$$\begin{aligned}\|f_n *_\alpha \phi_n - f\|_1 &\leq \int_{-\infty}^{\infty} |\phi_n(t)| dt \int_{-\infty}^{\infty} |f_n(x-t) e^{-2it(x-t)\cot\alpha} - f(x)| dx \\ &\leq M\|(f_n)_t e^{-2it(x-t)\cot\alpha} - f\|_1, \quad (M > 0).\end{aligned}$$

Using the triangular inequality of normed spaces,

$$\begin{aligned}\|f_n *_\alpha \phi_n - f\|_1 &\leq M\|(f_n)_t e^{-2it(x-t)\cot\alpha} - f_t e^{-2it(x-t)\cot\alpha}\|_1 + \|f_t e^{-2it(x-t)\cot\alpha} - f\|_1 \\ &\leq M\|(f_n)_t e^{-2it(x-t)\cot\alpha} - f_t e^{-2it(x-t)\cot\alpha}\|_1 + M\|f_t e^{-2it(x-t)\cot\alpha} - f\|_1.\end{aligned}$$

By using the convergence of $f_n \in \mathcal{L}^1$ and Theorem 1, we have

$$\|(f_n)_t e^{-2it(x-t)\cot\alpha} - f_t e^{-2it(x-t)\cot\alpha}\|_1 \to 0 \text{ as } n \to \infty,$$

and

$$\|f_t e^{-2it(x-t)\cot\alpha} - f\|_1 \to 0 \text{ as } t \to 0.$$

Therefore, $\|f_n *_\alpha \phi_n - f\|_1 \to 0$ as $n \to \infty$, hence, $f_n *_\alpha \phi_n \to f$ in $\mathcal{L}^1$. □

In order to extend the FHT to the class of Boehmians, we define another class of Boehmians (as the codomain of the extended fractional Hilbert transform) $\mathcal{B}_2 = \mathcal{B}_2(\mathcal{C}^\infty, S, *_\alpha, \Delta)$ [7]. The notion of delta sequences, quotients, and their equivalence classes remains the same as that in the prior case. We also retain the definitions of addition and scalar multiplication. Now, we define

$$D^m\left[\frac{f_n}{\phi_n}\right] = \left[\frac{D^m f_n}{\phi_n}\right] \text{ for any } \left[\frac{f_n}{\phi_n}\right] \in \mathcal{B}_2.$$

In addition,

$$\left[\frac{f_n}{\phi_n}\right] *_\alpha \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n *_\alpha g_n}{\phi_n *_\alpha \psi_n}\right].$$

Since a concept of convergence is required to construct a Boehmian space, we prove two convergences on $\mathcal{C}^\infty$.

Lemma 8. *Let $f_n \to f$ as $n \to \infty$ in $\mathcal{C}^\infty$ then $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{C}^\infty$ for all $\phi \in D$; further, for each delta sequence $\{\delta_n\}$, $f_n *_\alpha \delta_n \to f$ as $n \to \infty$ in $\mathcal{C}^\infty$.*

Proof. Let $K \subset \mathbb{R}$ be any compact set, such that $x \in K$. To prove the convergence of a sequence of functions in $\mathcal{C}^\infty$, we must show that the functions and their derivatives converge uniformly on compact sets.

First, we prove that $f_n *_\alpha \phi \to f *_\alpha \phi$ in $\mathcal{C}^\infty$. For this, consider

$$|(f_n *_\alpha \phi - f *_\alpha \phi)(x)| = |((f_n - f) *_\alpha \phi)(x)| \leq \int_{-\infty}^{\infty} |(f_n - f)(x-t)|\phi(t)dt.$$

Since t varies over the compact support of ϕ; therefore, $x - t$ also varies over a compact set in $\mathbb{R}$. So, $|((f_n - f) *_\alpha \phi)(x)| \to 0$ as $n \to \infty$ uniformly on compact sets. Then,

$$|(f_n *_\alpha \phi - f *_\alpha \phi)(x)| \to 0 \text{ as } n \to \infty,$$

or we can write

$$f_n *_\alpha \phi \to f *_\alpha \phi \text{ as } n \to \infty, \tag{6}$$

uniformly on compact sets.

In addition,

$$D^m((f_n *_\alpha \phi) - (f *_\alpha \phi)) = (D^m f_n *_\alpha \phi) - (D^m f *_\alpha \phi). \tag{7}$$

Replacing $D^m f_n$ by f_n and $D^m f$ by f in (7), we have

$$D^m((f_n *_\alpha \phi) - (f *_\alpha \phi)) = (f_n *_\alpha \phi) - (f *_\alpha \phi), \tag{8}$$

the right-hand side of (8) approaches zero by (6). Thus,

$$D^m(f_n *_\alpha \phi) \to D^m(f *_\alpha \phi)$$

uniformly on compact sets. Hence, $f_n *_\alpha \phi \to f *_\alpha \phi$ as $n \to \infty$ in $\mathcal{C}^\infty$.

Next, without any loss of generality, let us suppose that $\{\delta_n\} \in \Delta$ is such that it has a compact support. Then,

$$\begin{aligned}|(f_n *_\alpha \delta_n - f)(x)| &= \left|\int_{-\infty}^{\infty} f_n(x-t)\delta_n(t)e^{-it(x-t)\cot\alpha}dt - f(x)\int_{-\infty}^{\infty} e^{it(x-t)\cot\alpha}\delta_n(t)dt\right| \\ &\leq \int_{-\infty}^{\infty} |f_n(x-t)e^{-2it(x-t)\cot\alpha} - f(x)|\delta_n(t)dt, \\ &\leq \int_{-\infty}^{\infty} \Big(|f_n(x-t)e^{-2it(x-t)\cot\alpha} - f(x-t)e^{-2it(x-t)\cot\alpha}| + |f(x-t)e^{-2it(x-t)\cot\alpha} - f(x)|\Big)\delta_n(t)dt.\end{aligned}$$

Now, both x and t vary over compact sets; therefore, $x - t$ also varies over a compact set. Thus,

$$\int_{-\infty}^{\infty}\Big(|f_n(x-t)e^{-2it(x-t)\cot\alpha}-f(x-t)e^{-2it(x-t)\cot\alpha}|+|f(x-t)e^{-2it(x-t)\cot\alpha}-f(x)|\Big)\delta_n(t)dt\to 0$$

as $n\to\infty$ and $t\to 0$.

We have $f_n *_\alpha \delta_n \to f$ uniformly on compact sets.

Similarly, $D^m(f_n *_\alpha \delta_n) \to D^m(f)$ uniformly on compact sets.

Hence, $f_n *_\alpha \delta_n \to f$ as $n\to\infty$ in $\mathcal{C}^\infty$. □

Lemma 9. *If $f_n \to f$ as $n\to\infty$ in $\mathcal{L}^1$, then $f_n *_\alpha \delta \to f *_\alpha \delta$ as $n\to\infty$ in $\mathcal{C}^\infty$ for every $\delta\in S$.*

Proof. To show the convergence in $\mathcal{C}^\infty$, we assume that x varies over a compact set K.

$$\begin{aligned}|(f_n *_\alpha \delta - f *_\alpha \delta)(x)| &= |((f_n * -f) *_\alpha \delta)(x)|\\ &= \left|\int_{-\infty}^{\infty}(f_n-f)(x-t)\delta(t)e^{-it(x-t)\cot\alpha}dt\right|\\ &\le \int_{-\infty}^{\infty}|(f_n-f)(x-t)||\delta(t)|dt\\ &\le \|f_n-f\|_1\|\delta\|_\infty.\end{aligned}$$

Since $f_n\to f$ in $\mathcal{L}^1$ and $\delta\in S$ has a compact support, $x-t$ varies over a compact set, and $|(f_n *_\alpha \delta - f *_\alpha \delta)(x)|\to 0$ as $n\to\infty$ on compact sets. Similarly, we have

$$|D^m[(f_n *_\alpha \delta - f *_\alpha \delta)](x)| \le \|f_n-f\|_1\|D^m\delta\|_\infty.$$

Thus, $D^m(f_n *_\alpha \delta)\to D^m(f *_\alpha \delta)$ on compact sets.

Hence, $f_n *_\alpha \delta \to f *_\alpha \delta$ as $n\to\infty$ in $\mathcal{C}^\infty$. □

3.3. Fractional Hilbert Transform on Boehmians

The following result is very important in the aftermath. The proof of the following theorem is similar to the proof of convolution theorem for FHT as in Theorem 2; we omitted the details.

Theorem 4. *If $f\in\mathcal{L}^1$ and $\delta\in\Delta$, then $H_\alpha[f *_\alpha \delta] = H_\alpha[f] *_\alpha \delta$.*

Definition 3. *The fractional Hilbert transform $\mathcal{H}_\alpha : \mathcal{B}_1\to\mathcal{B}_2$ on Boehmians is defined by*

$$\mathcal{H}_\alpha\left[\frac{f_n}{\phi_n}\right] = \left[\frac{\mathcal{H}_\alpha f_n}{\phi_n}\right],$$

where $\frac{f_n}{\phi_n}$ is an arbitrary representative of any given Boehmian $B\in\mathcal{B}_1$. Since

$$f_n *_\alpha \phi_m = f_m *_\alpha \phi_n \quad \forall m,n\in\mathbb{N}.$$

By Theorem 4, we can write $\mathcal{H}_\alpha[f_n] *_\alpha \phi_m = \mathcal{H}_\alpha[f_m] *_\alpha \phi_n \quad \forall m,n\in\mathbb{N}$.

Therefore, $\frac{\mathcal{H}_\alpha[f_n]}{\phi_n}$ represents a Boehmian in $\mathcal{B}_2$. In a similar manner, let $\frac{g_n}{\psi_n}$ be another representative of B; then, again, with an application of Theorem 4,

$$\frac{\mathcal{H}_\alpha[f_n]}{\phi_n}\sim\frac{\mathcal{H}_\alpha[g_n]}{\psi_n},$$

thus the extended FHT on Boehmians $\mathcal{H}_\alpha : \mathcal{B}_1\to\mathcal{B}_2$ is well-defined.

Theorem 5. *Let $\mathcal{H}_\alpha : \mathcal{B}_1\to\mathcal{B}_2$ be the extended FHT; then,*

1. *If $\frac{f_n}{\phi_n}\in\mathcal{B}_1$ then $\frac{\mathcal{H}_\alpha f_n}{\phi_n}\in\mathcal{B}_2$.*

2. $\mathcal{H}_\alpha$ *is well-defined.*
3. $\mathcal{H}_\alpha$ *is a continuous linear map.*
4. $\mathcal{H}_\alpha$ *is an injective map.*

Proof. The proof of the above theorem is similar to those of Hilbert transform on Boehmians; we omitted the details. For details, the reader is referred to [7]. □

4. Conclusions

This paper gave an extension of the fractional Hilbert transform to a class of generalized functions known as Boehmians. It introduces a new convolutional operator, and the consequent convolutional theorem was also presented. In addition, the extended fractional Hilbert transform is a well-defined map between the spaces of Boehmians having properties, such as continuity and linearity, identical to the classical properties of their corresponding classical versions. Lastly, convergence concerning δ and Δ was also examined.

The methods of this paper can also be utilized to extend FHT to the space of ultradistributions. We suggest that readers consider the expansion of the fractional Hilbert transform to q-calculus and develop the theory of the quaternion fractional Hilbert transform.

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Abbreviations

The following abbreviation is used in this manuscript:

FHT Fractional Hilbert transform

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