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Bell's Polynomials and Laplace Transform of Higher Order Nested Functions

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Abstract: Using Bell's polynomials it is possible to approximate the Laplace Transform of composite functions. The same methodology can be adopted for the evaluation of the Laplace Transform of higher-order nested functions. In this case, a suitable extension of Bell's polynomials, as previously introduced in the scientific literature, is used, namely higher order Bell's polynomials used in the representation of the derivatives of multiple nested functions. Some worked examples are shown, and some of the polynomials used are reported in the Appendices.

Keywords: Laplace transform; Bell's polynomials; nested functions

MSC: 44A10; 05A10; 11B65; 11B83

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Citation: Caratelli, D.; Ricci, P.E. Bell's Polynomials and Laplace Transform of Higher Order Nested Functions. *Symmetry* **2022**, *14*, 2139. https://doi.org/10.3390/sym14102139

Academic Editor: Manuel Manas

Received: 25 July 2022
Accepted: 8 October 2022
Published: 13 October 2022

1. Introduction

In this study, we illustrate a procedure for the evaluation of the Laplace Transform (LT) of multi-nested analytic functions. To this end, we make use of Bell's polynomials [1–5], which constitute the essential tool for computing the subsequent derivatives of composite functions.

The Bell's polynomials appear in many different fields, ranging from number theory [6–8] to operator theory [9], and from differential equations [4] to integral transforms [10,11]. It is worth noting here that Bell's polynomials are closely related to and can be written in terms of symmetric functions in combinatorial Hopf algebras [12].

The importance of the LT [13,14] is well known and it is redundant to remind it here.

We use the classic definition of the LT:

$$\mathcal{L}(f) := \int_0^\infty \exp(-s\,t) f(t)\,dt = L(s)\,.$$

The LT converts a function of a real variable t (usually representing the time) to a function of a complex variable s (which represents the complex frequency). The LT holds for locally integrable functions on $[0, +\infty)$. It is convergent in every half-plane $Re(s) > a$, where a is the so-called convergence abscissa, depending on the growth rate at infinity of $f(t)$.

Our procedure is as follows: we use Taylor's expansion of the considered analytic function, and express the relevant coefficients in terms of Bell's polynomials; then, we approximate the LT of the given nested function by a series expansion, which provides an asymptotic representation of the LT when that exists.

We start from the easier case of the LT of a nested exponential function, considering the first few values of the complete Bell's polynomials. The result is a Laurent expansion which approximates the relevant LT.

Then, we consider the case of the LT of general nested functions. The main problem is to provide a table of Bell's polynomials. These exhibit higher complexity, but their evaluation can be easily performed through a dedicated computer code.

Our results can be compared with the LT of nested functions appearing in the literature only in a few cases [15], but the results we have obtained in these cases are completely satisfactory.

All the computations reported in this study have been performed using the computer algebra program Mathematica©.

The second-order Bell's polynomials $Y_n^{[2]}$, representing the derivatives of nested functions of the type $f(g(h(t))$ are then introduced, and two examples of LT of these functions are given.

In Appendix A a table of the second-order Bell's polynomials is reported.

Lastly, we give some examples to show that the same methodology can be used even for the LT of higher-order nested functions. The first few terms of the corresponding generalized Bell's polynomials, of order 4, $Y_n^{[4]}$, are shown in Appendix B The polynomials $Y_n^{[7]}$ have been computed in the same way but are not reported here owing to the lack of space.

It is worth noting that more general extensions of Bell's polynomials have been introduced in the past, including those appearing in the two-variable case [16], as well as the multi-variable case [17]. Since all the aforementioned extensions have been proven through the classical case, more general results could be obtained by applying the methods described in this article.

2. Definition of Bell's Polynomials

The n-th derivative of the composite (differentiable) function $\Phi(t) := f(g(t))$, as evaluated by the chain rule, is expressed by Bell's polynomials as follows

$$\Phi_n := D_t^n \Phi(t) = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) f_k, \tag{1}$$

where

$$f_h := D_x^h f(x)|_{x=g(t)}, \quad g_k := D_t^k g(t). \tag{2}$$

The coefficients $B_{n,k}$, for all $k = 1, \ldots, n$, are polynomials of the variables $g_1, g_2, \ldots, g_{n-k+1}$, that are homogeneous of degree k and *isobaric* of weight n (i.e., they are a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$); in the literature, they are also referred to as partial Bell's polynomials.

Bell's polynomials satisfy the recursion

$$\begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \ldots; f_n, g_n; f_{n+1}, g_{n+1}) = \\ \qquad = \displaystyle\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}) g_{k+1} \,. \end{cases} \tag{3}$$

An explicit representation is given by the Faà di Bruno's formula

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \ldots r_n!} f_r \left[\frac{g_1}{1!}\right]^{r_1} \left[\frac{g_2}{2!}\right]^{r_2} \cdots \left[\frac{g_n}{n!}\right]^{r_n}, \tag{4}$$

where the sum runs over all the partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ denotes the number of parts of the considered partition [5].

The $B_{n,k}$ coefficients satisfy the recursion $\forall\, n$

$$B_{n,1} = g_n\,, \quad B_{n,n} = g_1^n\,,$$
$$B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \ldots, g_{n-k-h+1})\, g_{h+1}\,. \tag{5}$$

3. LT of Composite Functions

Let $f(g(t))$ be a composite function that is analytic in a neighborhood of the origin, and whose Taylor's expansion is given by

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\,, \quad a_n = D_t^n[f(g(t))]_{t=0}\,. \tag{6}$$

According to the preceding equations, it results in

$$\begin{aligned} a_0 &= f(\overset{\circ}{g}_0)\,, \\ a_n &= D_t^n[f(g(t))]_{t=0} = \sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k\,, \quad (n \geq 1)\,, \end{aligned} \tag{7}$$

where

$$\overset{\circ}{f}_k := D_x^k f(x)|_{x=g(0)}, \qquad \overset{\circ}{g}_h := D_t^h g(t)|_{t=0}. \tag{8}$$

Then, the following result easily follows.

Theorem 1. *Consider a composite function $f(g(t))$ that is analytic in a neighborhood of the origin, and can be expressed by Taylor's expansion in (6). For its LT the following asymptotic representation holds*

$$\begin{aligned} \int_0^{+\infty} f(g(t))e^{-ts}dt &\simeq \frac{f(\overset{\circ}{g}_0)}{s} + \sum_{n=1}^{N} \int_0^{+\infty} \sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k\, \frac{t^n}{n!}\, e^{-ts}dt = \\ &= \frac{f(\overset{\circ}{g}_0)}{s} + \sum_{n=1}^{N} \left(\sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k \right) \int_0^{+\infty} \frac{t^n}{n!}\, e^{-ts}dt = \\ &= \frac{f(\overset{\circ}{g}_0)}{s} + \sum_{n=1}^{N} \left(\sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k \right) \frac{1}{s^{n+1}}\,, \end{aligned} \tag{9}$$

where N denotes a finite expansion order.

3.1. The Particular Case of the Exponential Function

In the particular case when $f(x) = e^x$, that is considering the function $e^{g(t)}$, and assuming $g(0) = 0$, we have the simple form

$$\sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1})\, \overset{\circ}{f}_k = \sum_{k=1}^{n} B_{n,k}(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_{n-k+1}) = B_n(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_n)\,, \tag{10}$$

where the B_n are the *complete Bell's polynomials*. It results $B_0(g_0) := f(g_0)$, and the first few values of B_n, for $n = 1, 2, \ldots, 5$, are given by

$$B_1 = g_1,$$

$$B_2 = g_1^2 + g_2,$$

$$B_3 = g_1^3 + 3g_1g_2 + g_3,$$

$$B_4 = g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4,$$

$$B_5 = g_1^5 + 10g_1^3g_2 + 15g_1g_2^2 + 10g_1^2g_3 + 10g_2g_3 + 5g_1g_4 + g_5,$$

$$B_6 = g_1^6 + 15g_1^4g_2 + 45g_1^2g_2^2 + 15g_2^3 + 20g_1^3g_3 + 60g_1g_2g_3 + 10g_3^2 + 15g_1^2g_4 + 15g_2g_4 + 6g_1g_5 + g_6,$$

$$\begin{aligned} B_7 = {} & g_1^7 + 21g_1^5g_2 + 105g_1^3g_2^2 + 105g_1g_2^3 + 35g_1^4g_3 + 210g_1^2g_2g_3 + 105g_2^2g_3 + 70g_1g_3^2 + 35g_1^3g_4 + \\ & 105g_1g_2g_4 + 35g_3g_4 + 21g_1^2g_5 + 21g_2g_5 + 7g_1g_6 + g_7, \end{aligned}$$

$$\begin{aligned} B_8 = {} & g_1^8 + 28g_1^6g_2 + 210g_1^4g_2^2 + 420g_1^2g_2^3 + 105g_2^4 + 56g_1^5g_3 + 560g_1^3g_2g_3 + 840g_1g_2^2g_3 + 280g_1^2g_3^2 + \\ & 280g_2g_3^2 + 70g_1^4g_4 + 420g_1^2g_2g_4 + 210g_2^2g_4 + 280g_1g_3g_4 + 35g_4^2 + 56g_1^3g_5 + 168g_1g_2g_5 + \\ & 56g_3g_5 + 28g_1^2g_6 + 28g_2g_6 + 8g_1g_7 + g_8, \end{aligned}$$

$$\begin{aligned} B_9 = {} & g_1^9 + 36g_1^7g_2 + 378g_1^5g_2^2 + 1260g_1^3g_2^3 + 945g_1g_2^4 + 84g_1^6g_3 + 1260g_1^4g_2g_3 + 3780g_1^2g_2^2g_3 + \\ & 1260g_2^3g_3 + 840g_1^3g_3^2 + 2520g_1g_2g_3^2 + 280g_3^3 + 126g_1^5g_4 + 1260g_1^3g_2g_4 + 1890g_1g_2^2g_4 + \\ & 1260g_1^2g_3g_4 + 1260g_2g_3g_4 + 315g_1g_4^2 + 126g_1^4g_5 + 756g_1^2g_2g_5 + 378g_2^2g_5 + 504g_1g_3g_5 + \\ & 126g_4g_5 + 84g_1^3g_6 + 252g_1g_2g_6 + 84g_3g_6 + 36g_1^2g_7 + 36g_2g_7 + 9g_1g_8 + g_9, \end{aligned}$$

$$\begin{aligned} B_{10} = {} & g_1^{10} + 45g_1^8g_2 + 630g_1^6g_2^2 + 3150g_1^4g_2^3 + 4725g_1^2g_2^4 + 945g_2^5 + 120g_1^7g_3 + 2520g_1^5g_2g_3 + \\ & 12600g_1^3g_2^2g_3 + 12600g_1g_2^3g_3 + 2100g_1^4g_3^2 + 12600g_1^2g_2g_3^2 + 6300g_2^2g_3^2 + 2800g_1g_3^3 + \\ & 210g_1^6g_4 + 3150g_1^4g_2g_4 + 9450g_1^2g_2^2g_4 + 3150g_2^3g_4 + 4200g_1^3g_3g_4 + 12600g_1g_2g_3g_4 + \\ & 2100g_3^2g_4 + 1575g_1^2g_4^2 + 1575g_2g_4^2 + 252g_1^5g_5 + 2520g_1^3g_2g_5 + 3780g_1g_2^2g_5 + 2520g_1^2g_3g_5 + \\ & 2520g_2g_3g_5 + 1260g_1g_4g_5 + 126g_5^2 + 210g_1^4g_6 + 1260g_1^2g_2g_6 + 630g_2^2g_6 + 840g_1g_3g_6 + \\ & 210g_4g_6 + 120g_1^3g_7 + 360g_1g_2g_7 + 120g_3g_7 + 45g_1^2g_8 + 45g_2g_8 + 10g_1g_9 + g_{10}\,. \end{aligned}$$

The values of the complete Bell's polynomials for particular choices of the relevant parameters can be found in [6].

The complete Bell's polynomials satisfy the identity (see, e.g., [4])

$$B_{n+1}(g_1, \ldots, g_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(g_1, \ldots, g_{n-k})\, g_{k+1}. \tag{11}$$

In this case Equation (9) reduces to

$$\int_0^{+\infty} \exp(g(t))\, e^{-ts} dt \simeq \frac{\exp(\overset{\circ}{g}_0)}{s} + \sum_{n=1}^{N} B_n(\overset{\circ}{g}_1, \overset{\circ}{g}_2, \ldots, \overset{\circ}{g}_n)\, \frac{1}{s^{n+1}}\,. \tag{12}$$

In what follows, we evaluate the approximation of the LT of nested functions. The reported results have been obtained using the computer algebra program Mathematica©.

Examples

We first recall the case of the LT of nested exponential functions, showing two particular examples.

- Consider the Bessel function $g(t) := J_1(t)$ and the LT of the corresponding exponential function. We find

$$\int_0^{+\infty} \exp(J_1(t))\, e^{-ts} dt = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{4s^3} - \frac{3}{4s^4} - \frac{11}{16s^5} - \frac{19}{32s^6} + \frac{91}{64s^7} + \\ + \frac{701}{128s^8} + \frac{953}{256s^9} - \frac{15245}{512s^{10}} + O\left(\frac{1}{s^{11}}\right), \tag{13}$$

for $s \to \infty$.

- Consider the function $g(t) := \arctan(t)$ and the LT of the corresponding exponential function. We find

$$\int_0^{+\infty} \exp(\arctan(t))\, e^{-ts} dt = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} - \frac{7}{s^5} - \frac{5}{s^6} + \frac{145}{s^7} + \\ + \frac{5}{s^8} - \frac{6095}{s^9} + \frac{5815}{s^{10}} + O\left(\frac{1}{s^{11}}\right), \tag{14}$$

for $s \to \infty$.

4. LT in Two Known Cases

We considered two cases concerning composite functions whose transform and anti-transform are known (see [15]). By using the computer algebra program Mathematica©, we have been able to prove the correctness of the methodology used.

4.1. Case #1

Consider the function $l(t) = \log[\cosh(t)]$. The LT of $l(t)$ is found to be [15]:

$$L(s) = \frac{1}{2s}\left[\psi\left(\frac{1}{2} + \frac{s}{4}\right) - \psi\left(\frac{s}{4}\right)\right] - \frac{1}{s^2}, \tag{15}$$

for $\Re s > 0$, and where $\psi(z)$ is the logarithmic derivative of the gamma function, given by

$$\psi(z) \equiv \tfrac{d}{dz} \ln \Gamma(z) = \tfrac{\Gamma'(z)}{\Gamma(z)}. \tag{16}$$

Using our methodology, we find that

$$L(s) \simeq \tilde{L}(s) = \frac{1}{s^3} - \frac{2}{s^5} + \frac{16}{s^7} - \frac{272}{s^9} + \frac{7936}{s^{11}}, \tag{17}$$

so that, the inverse Laplace transformation is given by

$$\tilde{l}(t) \simeq \left(\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{45} - \frac{17t^8}{2520} + \frac{31t^{10}}{14175}\right) H(t), \tag{18}$$

with $H(\cdot)$ denoting the Heaviside distribution which can be defined as follows:

$$H(x) = \int_{-\infty}^{x} \delta(u) du, \tag{19}$$

in terms of the Dirac delta distribution $\delta(\cdot)$.

4.2. Case #2

Let us consider the function $l(t) = J_0(t^2)$. The LT of $l(t)$ is found to be [15]:

$$L(s) = \frac{\pi s}{16}\left\{\left[J_{1/4}(s^2/8)\right]^2 + \left[Y_{1/4}(s^2/8)\right]^2\right\}, \tag{20}$$

for $\Re s > 0$.

Using our methodology, we find that

$$L(s) \simeq \tilde{L}(s) = \frac{1}{s} - \frac{6}{s^5} + \frac{630}{s^9} - \frac{207900}{s^{13}} + \frac{141891750}{s^{17}} - \frac{164991726900}{s^{21}}, \tag{21}$$

so that, the inverse Laplace transformation is given by

$$\tilde{I}(t) \simeq \left(1 - \frac{t^4}{4} + \frac{t^8}{64} - \frac{t^{12}}{2304} + \frac{t^{16}}{147456} - \frac{t^{20}}{14745600}\right) H(t) \,. \tag{22}$$

5. A First Extension of Bell's Polynomials

We consider the second-order Bell's polynomials, $Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \ldots; f_n, g_n, h_n)$, defined by the n-th derivative of the composite function $\Phi(t) := f(g(h(t)))$.

Consider the functions $x = h(t)$, $z = g(x)$, and $y = f(z)$, and suppose that $h(t)$, $g(x)$, and $f(z)$ are n times differentiable with respect to their variables, so that the composite function $\Phi(t) := f(g(h(t)))$ can be differentiated n times with respect to t, by using the chain rule.

We use, as before, the following notation:

$$\Phi_j := D_t^j \Phi(t), \quad f_h := D_y^h f(y)|_{y=g(x)}, \quad g_k := D_x^k g(x)|_{x=h(t)}, \quad h_r := D_t^r h(t).$$

Then, the n-th derivative can be represented by the compact symbol:

$$\Phi_n = Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \ldots; f_n, g_n, h_n) = Y_n^{[2]}([f, g, h]_n) \,, \tag{23}$$

where the $Y_n^{[2]}$ are defined as the second order Bell's polynomials.

The first few terms are as follows.

$$Y_1^{[2]}([f,g,h]_1) = f_1 g_1 h_1;$$

$$Y_2^{[2]}([f,g,h]_2) = f_1 g_1 h_2 + f_1 g_2 h_1^2 + f_2 g_1^2 h_1^2;$$

$$Y_3^{[2]}([f,g,h]_3) = f_1 g_1 h_3 + f_1 g_3 h_1^3 + 3 f_1 g_2 h_1 h_2 + 3 f_2 g_1 g_2 h_1^3 + f_3 g_1^3 h_1^3;$$

$$Y_4^{[2]}([f,g,h]_4) = f_4 g_1^4 h_1^4 + 6 f_3 g_1^2 g_2 h_1^4 + 3 f_2 g_2^2 h_1^4 + 4 f_2 g_1 g_3 h_1^4 + f_1 g_4 h_1^4 + 6 f_3 g_1^3 h_1^2 h_2 +$$

$$+ 18 f_2 g_1 g_2 h_1^2 h_2 + 6 f_1 g_3 h_1^2 h_2 + 3 f_2 g_1^2 h_2^2 + 3 f_1 g_2 h_2^2 + 4 f_2 g_1^2 h_1 h_3 + 4 f_1 g_2 h_1 h_3 + f_1 g_1 h_4;$$

$$Y_5^{[2]}([f,g,h]_5) = f_5 g_1^5 h_1^5 + 10 f_4 g_1^3 g_2 h_1^5 + 15 f_3 g_1 g_2^2 h_1^5 + 10 f_3 g_1^2 g_3 h_1^5 + 10 f_2 g_2 g_3 h_1^5 +$$

$$+ 5 f_2 g_1 g_4 h_1^5 + f_1 g_5 h_1^5 + 10 f_4 g_1^4 h_1^3 h_2 + 60 f_3 g_1^2 g_2 h_1^3 h_2 + 30 f_2 g_2^2 h_1^3 h_2 + 40 f_2 g_1 g_3 h_1^3 h_2 +$$

$$+ 10 f_1 g_4 h_1^3 h_2 + 15 f_3 g_1^3 h_1 h_2^2 + 45 f_2 g_1 g_2 h_1 h_2^2 + 15 f_1 g_3 h_1 h_2^2 + 10 f_3 g_1^3 h_1^2 h_3 + 30 f_2 g_1 g_2 h_1^2 h_3 +$$

$$+ 10 f_1 g_3 h_1^2 h_3 + 10 f_2 g_1^2 h_2 h_3 + 10 f_1 g_2 h_2 h_3 + 5 f_2 g_1^2 h_1 h_4 + 5 f_1 g_2 h_1 h_4 + f_1 g_1 h_5 \,.$$

A more extended table is given in Appendix A.

The connections to the ordinary Bell's polynomials are highlighted below.

Theorem 2. *For every integer n, the polynomials $Y_n^{[2]}$ are represented in terms of the ordinary Bell's polynomials by the following equation, where a compact notation similar to the one in (23) is used:*

$$\begin{aligned} &Y_n^{[2]}([f,g,h]_n) = \\ &= Y_n(f_1, Y_1([g,h]_1); f_2, Y_2([g,h]_2); \dots; f_n, Y_n([g,h]_n)) \end{aligned} \tag{24}$$

Proof. Using induction, we can conclude that (24) is true for $n = 1$, since

$$Y_1^{[2]}([f,g,h]_1) = f_1\, g_1\, h_1 = f_1\, Y_1([g,h]_1) = Y_1(f_1, Y_1([g,h]_1)).$$

Then, assuming that (24) is true for every n, it follows that

$$\begin{aligned} Y_{n+1}^{[2]}([f,g,h]_{n+1}) &= D_t\, Y_n^{[2]}([f,g,h]_n) = D_t\, Y_n(f_1, Y_1([g,h]_1); \dots; f_n, Y_n([g,h]_n)) = \\ &= Y_{n+1}(f_1, Y_1([g,h]_1); f_2, Y_2([g,h]_2); \dots; f_{n+1}, Y_{n+1}([g,h]_{n+1}))\,. \end{aligned} \tag{25}$$

□

Consequently, we have the theorem:

Theorem 3. *The second-order Bell's polynomials verify the recursion*

$$\begin{aligned} &Y_0^{[2]} = f_1; \\ &Y_{n+1}^{[2]}([f,g,h]_{n+1}) = \\ &= \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}^{[2]}(f_2, g_1, h_1; f_3, g_2, h_2; \dots; f_{n-k+1}, g_{n-k}, h_{n-k}) Y_{k+1}([g,h]_{k+1}). \end{aligned} \tag{26}$$

Proof. By means of (24) we express $Y_{n+1}^{[2]}([f,g,h]_{n+1})$ in terms of

$$Y_{n+1}(f_1, Y_1([g,h]_1); \dots; f_{n+1}, Y_{n+1}([g,h]_{n+1})).$$

Then, by using the recursion (9) and again Equation (24), the expression (26) follows. □

6. LT of Second-Order Nested Functions

Let be $f(g(h(t)))$ be a composite function that is analytic in a neighborhood of the origin and, therefore, can be expressed by the Taylor's expansion

$$f(g((h(t))) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\,, \quad a_n = D_t^n [f(g((h(t)))]_{t=0}\,. \tag{27}$$

According to the preceding equations, it results

$$\begin{aligned} &a_0 = \overset{\circ}{f}_0 = f(g(h(0))\,, \\ &a_n = D_t^n [f(g((h(t)))]_{t=0} = Y_n^{[2]}([\overset{\circ}{f}, \overset{\circ}{g}, \overset{\circ}{h}]_n)\,, \quad (n \geq 1)\,, \end{aligned} \tag{28}$$

where

$$\overset{\circ}{f}_h := D_x^h f(y)|_{y=g(0)}, \quad \overset{\circ}{g}_k := D_t^k g(x)|_{x=h(0)}, \quad \overset{\circ}{h}_r := D_t^r h(t)|_{t=0}\,. \tag{29}$$

This expansion can be used to evaluate the LT of analytic nested functions.

Theorem 4. *Consider a nested function $f(g((h(t)))$ that is analytic in a neighborhood of the origin, and whose Taylor's expansion is given by (27). For its LT, the following asymptotic representation holds*

$$\int_0^{+\infty} f(g((h(t))) e^{-ts} dt \simeq \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{N} Y_n^{[2]}([\overset{\circ}{f}, \overset{\circ}{g}, \overset{\circ}{h}]_n) \int_0^{+\infty} \frac{t^n}{n!} e^{-ts} dt = \\ = \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{N} Y_n^{[2]}([\overset{\circ}{f}, \overset{\circ}{g}, \overset{\circ}{h}]_n) \frac{1}{s^{n+1}}, \tag{30}$$

where N denotes a finite expansion order.

Proof. It is a straightforward application of the definition of the second-order Bell's polynomials. □

Example 1. • *Assuming $f(x) = e^{x-1}$, $g(y) = \cos(y)$, $h(t) = \sin(t)$, it results in (see Figure 1)*

$$\int_0^{+\infty} \exp[\cos(\sin(t)) - 1]\, e^{-ts} dt = \frac{1}{s} - \frac{1}{s^3} + \frac{8}{s^5} - \frac{127}{s^7} + \frac{3523}{s^9} - \frac{146964}{s^{11}} + O\left(\frac{1}{s^{13}}\right), \tag{31}$$

for $s \to \infty$. The corresponding inverse LT is approximated by (see Figure 2)

$$\tilde{l}(t) \simeq \left(1 - \frac{1}{2}\, t^2 + \frac{1}{3}\, t^4 - \frac{127}{720}\, t^6 + \frac{3523}{40320}\, t^8 - \frac{12247}{302400}\, t^{10}\right) H(t)\,. \tag{32}$$

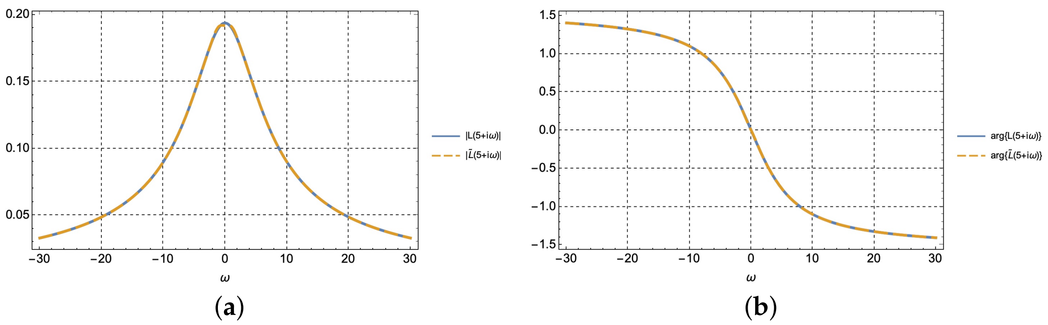


Figure 1. Magnitude (**a**) and argument (**b**) of the Laplace transform of $\exp[\cos(\sin(t)) - 1]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous integral expression $L(s)$ for $s = 5 + \mathrm{i}\,\omega$.

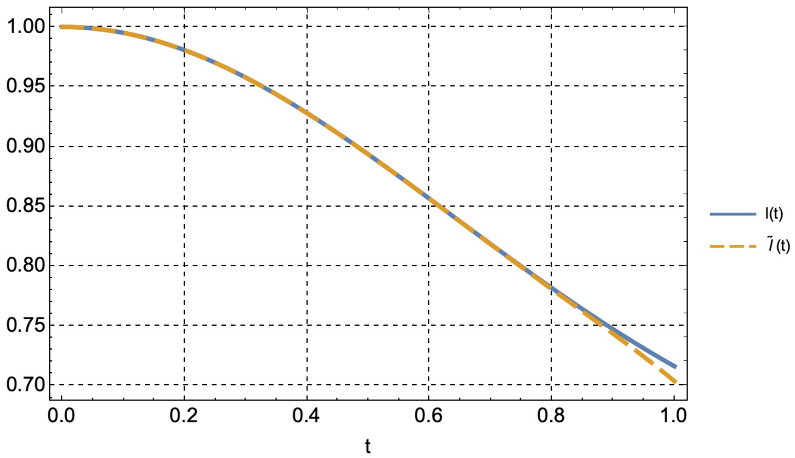


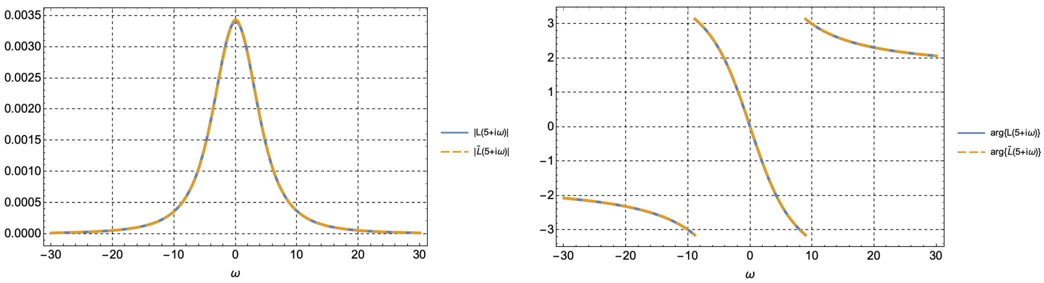
Figure 2. Distribution of $l(t) = \exp[\cos(\sin(t)) - 1]$ and the relevant approximant $\tilde{l}(t)$.

Example 2. • *Upon assuming* $f(x) = \log(1 + \frac{x}{2})$, $g(y) = \cosh(y) - 1$, $h(t) = \sin(t)$, *it results in (see Figure 3)*

$$\int_0^{+\infty} \log\left[1 + \frac{\cosh(\sin(t)) - 1}{2}\right] e^{-ts} dt = \frac{1}{2s^3} - \frac{9}{4s^5} - \frac{27}{2s^7} + \frac{1169}{8s^9} - \frac{5869}{2s^{11}} + O\left(\frac{1}{s^{13}}\right), \qquad (33)$$

for $s \to \infty$. *The corresponding inverse LT can be approximated as (see Figure 4)*

$$\tilde{l}(t) \simeq \left(\frac{1}{4} t^2 - \frac{3}{32} t^4 + \frac{3}{160} t^6 - \frac{167}{46080} t^8 + \frac{5869}{7257600} t^{10}\right) H(t)\,. \qquad (34)$$



(**a**) (**b**)

Figure 3. Magnitude (**a**) and argument (**b**) of the Laplace transform of $\log\left[1 + \frac{\cosh(\sin(t))-1}{2}\right]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous integral expression $L(s)$ for $s = 5 + \mathrm{i}\,\omega$.

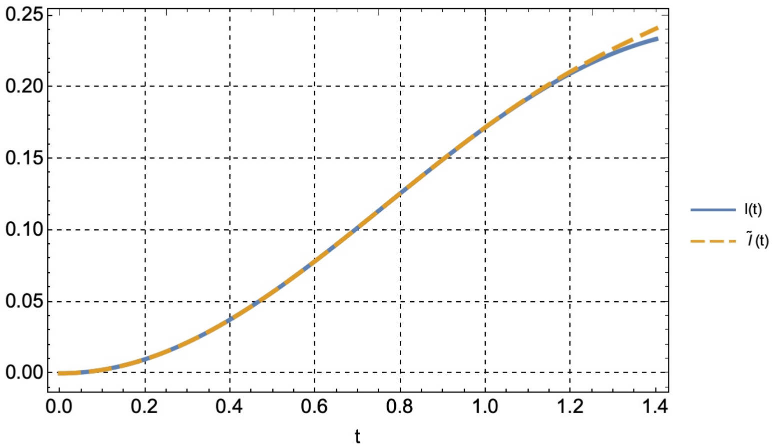


Figure 4. Distribution of $l(t) = \log\left[1 + \frac{\cosh(\sin(t))-1}{2}\right]$ and the relevant approximant $\tilde{l}(t)$.

7. Higher Order Bell's Polynomials

Consider the nested function $\Phi(t) := f_{(1)}(f_{(2)}(\cdots(f_{(M)}(t))))$, i.e., the composition of the functions $x_{M-1} = f_{(M)}(t), \ldots, x_1 = f_{(2)}(x_2), y = f_{(1)}(x_1)$, and suppose that $f_{(M)}, \ldots, f_{(2)}, f_{(1)}$ are n times differentiable with respect to their independent variables. Then, $\Phi(t)$ can be differentiated n times with respect to t using the chain rule. By definition we put $x_M := t$, so that $y = \Phi(t)$.

We use the following notation:

$$\begin{aligned}
&\Phi_h := D_t^h \Phi(t), \\
&f_{(1),h} := D_{x_1}^h f_{(1)}|_{x_1 = f_{(2)}(f_{(3)}(\cdots(f_{(M)}(t))))}, \\
&f_{(2),k} := D_{x_2}^k f_{(2)}|_{x_2 = f_{(3)}(f_{(4)}(\cdots(f_{(M)}(t))))}, \\
&\ldots\ldots\ldots\ldots\ldots \\
&f_{(M),j} := D_{x_M}^j f_{(M)}|_{x_M = t}.
\end{aligned} \qquad (35)$$

Then, the n-th derivative can be represented as

$$\Phi_n = Y_n^{[M-1]}(f_{(1),1}, \dots, f_{(M),1}; f_{(1),2}, \dots, f_{(M),2}; \dots; f_{(1),n}, \dots, f_{(M),n}),$$

where the $Y_n^{[M-1]}$ are, by definition, Bell's polynomials of order $M-1$.

The above Theorems 2 and 3 can be generalized as follows.

Theorem 5. *For every integer n, the polynomials $Y_n^{[M-1]}$ are expressed in terms of Bell's polynomials of a lower order, through the following equation:*

$$\begin{aligned} &Y_n^{[M-1]}\Big(f_{(1),1}, \dots, f_{(M),1}; \dots; f_{(1),n}, \dots, f_{(M),n}\Big) = \\ &= Y_n\Big(f_{(1),1}, Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}); \\ &\qquad f_{(1),2}, Y_2^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; f_{(2),2}, \dots, f_{(M),2}); \dots \\ &\qquad\quad \dots; f_{(1),n}, Y_n^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n})\Big). \end{aligned} \tag{36}$$

Theorem 6. *The following recurrence relation for the Bell's polynomials $Y_n^{[M-1]}$ of order $M-1$ holds true:*

$$\begin{aligned} &Y_0^{[M-1]} = f_{(1),1}; \\ &Y_{n+1}^{[M-1]}\Big(f_{(1),1}, \dots, f_{(M),1}; \dots; f_{(1),n+1}, \dots, f_{(M),n+1}\Big) = \\ &= \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}^{[M-1]}\Big(f_{(1),2}, f_{(2),1}, \dots, f_{(M),1}; f_{(1),3}, f_{(2),2}, \dots, f_{(M),2}; \dots \\ &\qquad\qquad \dots; f_{(1),n-k+1}, f_{(2),n-k}, \dots, f_{(M),n-k}\Big) \times \\ &\qquad \times Y_{k+1}^{[M-2]}\Big(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),k+1}, \dots, f_{(M),k+1}\Big). \end{aligned} \tag{37}$$

Example 3. *We apply the above results to the case of the LT of nested* sine *functions, assuming $M = 4$ and $M = 7$.*

- *Let be $M = 4$. We have (see Figure 5):*

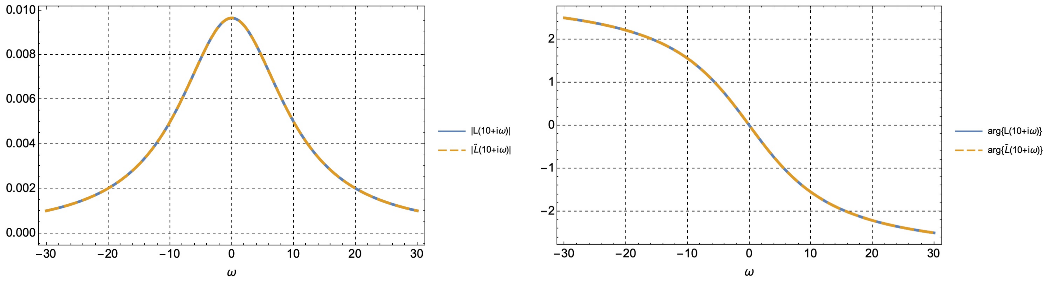
$$f_4(t) = f_3(t) = f_2(t) = f_1(t) = \sin(t), \quad f(t) = \sin(\sin(\sin(\sin(t)))),$$

$$\int_0^\infty \exp(-s\,t) \sin(\sin(\sin(\sin(t))))\, dt = \frac{1}{s^2} - \frac{4}{s^4} + \frac{64}{s^6} - \frac{2160}{s^8} + \frac{121600}{s^{10}} + O\left(\frac{1}{s^{12}}\right),$$

for $s \to \infty$.

The corresponding inverse LT is approximated by (see Figure 6).

$$\tilde{I}(t) \simeq \left(t - \frac{2}{3}\,t^3 + \frac{8}{15}\,t^5 - \frac{3}{7}\,t^7 + \frac{190}{567}\,t^9\right) H(t)\,. \tag{38}$$



(**a**) (**b**)

Figure 5. Magnitude (**a**) and argument (**b**) of the Laplace transform of $\sin(\sin(\sin(\sin(t))))$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous integral expression $L(s)$ for $s = 10 + \mathrm{i}\,\omega$.

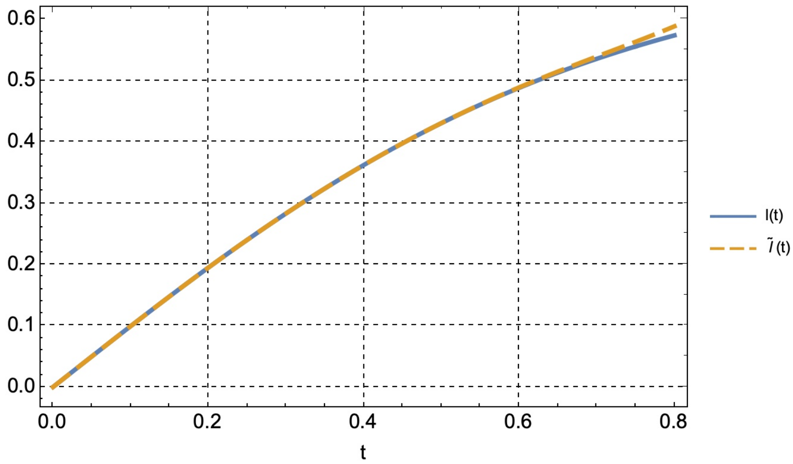


Figure 6. Distribution of $l(t) = \sin(\sin(\sin(\sin(t))))$ and the relevant approximant $\tilde{l}(t)$.

• *Let be* $M = 7$*. We have (see Figure 7):*

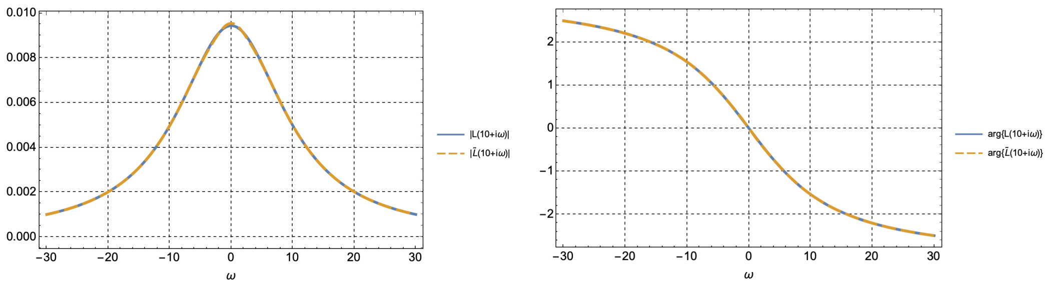
$$f_7(t) = f_6(t) = \cdots = f_1(t) = \sin(t)\,, \quad f(t) = \sin(\sin(\cdots \sin(\sin(t))))\,,$$

$$\int_0^\infty \exp(-s\,t) f(t)\, dt = \frac{1}{s^2} - \frac{7}{s^4} + \frac{217}{s^6} - \frac{14903}{s^8} + \frac{1776817}{s^{10}} + O\left(\frac{1}{s^{12}}\right),$$

for $s \to \infty$*.*

The corresponding inverse LT is approximated by (see Figure 8).

$$\tilde{l}(t) \simeq \left(t - \frac{7}{6}\,t^3 + \frac{217}{120}\,t^5 - \frac{2129}{720}\,t^7 + \frac{253831}{51840}\,t^9\right) H(t)\,. \tag{39}$$



(**a**) (**b**)

Figure 7. Magnitude (**a**) and argument (**b**) of the Laplace transform of $\sin(\sin(\ldots(\sin(t))))$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous integral expression $L(s)$ for $s = 10 + \mathrm{i}\,\omega$.

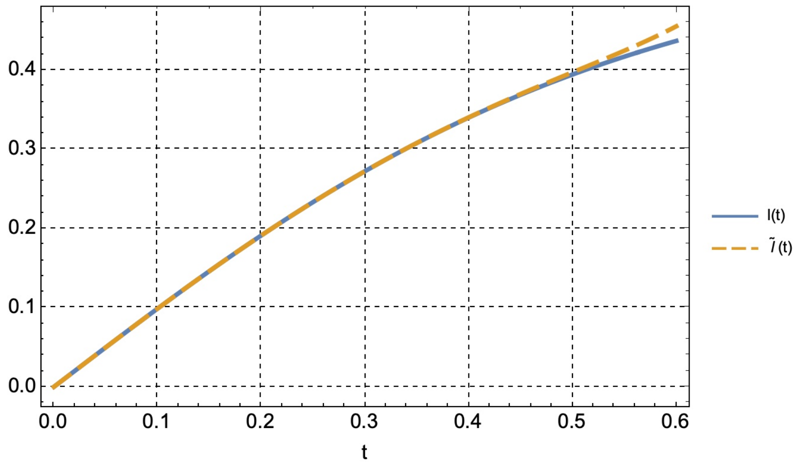


Figure 8. Distribution of $l(t) = \sin(\sin(\ldots(\sin(t))))$ and the relevant approximant $\tilde{l}(t)$.

8. Conclusions

We have presented a method for approximating the integral of analytic composite functions. We started from the Taylor expansion of the considered function in a neighborhood of the origin. Since the coefficients can be expressed in terms of Bell's polynomials, the integral is reduced to the computation of an approximating series, which obviously converges if the integral is convergent. Then, this methodology has been applied to the case of the LT of an analytic composite function, starting from the case of analytic nested exponential functions. Furthermore, the evaluation of the LT of analytic nested functions is discussed, and the first few second-order Bell's polynomials used in the framework of the presented methodology are reported in Appendix A, whereas those of order 4 are given in Appendix B. A graphical verification of the proposed technique, performed in the case when the analytical forms of both the transform and anti-transform are known, proved the correctness of our results. In future studies, attention will be devoted to the evaluation of more complex functions, such as the basic class of symmetric orthogonal polynomials (BCSOP) introduced in [18].

Author Contributions: The authors have contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

In[∘]:= $Y[n\_] := \sum_{k=1}^{n} (\mathrm{BellY}[n, k, \mathrm{Table}[h_m, \{m, 1, n-k+1\}]]\, g_k)$

In[∘]:= $Y2[n\_] := \sum_{k=1}^{n} (\mathrm{BellY}[n, k, \mathrm{Table}[Y[m], \{m, 1, n-k+1\}]]\, f_k)$

In[∘]:= `Y2[1] // FullSimplify // Expand`

Out[∘]= $f_1 g_1 h_1$

In[∘]:= `Y2[2] // FullSimplify // Expand`

Out[∘]= $f_2 g_1^2 h_1^2 + f_1 g_2 h_1^2 + f_1 g_1 h_2$

In[∘]:= `Y2[3] // FullSimplify // Expand`

Out[∘]= $f_3 g_1^3 h_1^3 + 3 f_2 g_1 g_2 h_1^3 + f_1 g_3 h_1^3 + 3 f_2 g_1^2 h_1 h_2 + 3 f_1 g_2 h_1 h_2 + f_1 g_1 h_3$

In[∘]:= `Y2[4] // FullSimplify // Expand`

Out[∘]=
$$f_4 g_1^4 h_1^4 + 6 f_3 g_1^2 g_2 h_1^4 + 3 f_2 g_2^2 h_1^4 + 4 f_2 g_1 g_3 h_1^4 + f_1 g_4 h_1^4 + 6 f_3 g_1^3 h_1^2 h_2 + 18 f_2 g_1 g_2 h_1^2 h_2 +$$
$$6 f_1 g_3 h_1^2 h_2 + 3 f_2 g_1^2 h_2^2 + 3 f_1 g_2 h_2^2 + 4 f_2 g_1^2 h_1 h_3 + 4 f_1 g_2 h_1 h_3 + f_1 g_1 h_4$$

In[∘]:= `Y2[5] // FullSimplify // Expand`

Out[∘]=
$$f_5 g_1^5 h_1^5 + 10 f_4 g_1^3 g_2 h_1^5 + 15 f_3 g_1 g_2^2 h_1^5 + 10 f_3 g_1^2 g_3 h_1^5 + 10 f_2 g_2 g_3 h_1^5 + 5 f_2 g_1 g_4 h_1^5 + f_1 g_5 h_1^5 +$$
$$10 f_4 g_1^4 h_1^3 h_2 + 60 f_3 g_1^2 g_2 h_1^3 h_2 + 30 f_2 g_2^2 h_1^3 h_2 + 40 f_2 g_1 g_3 h_1^3 h_2 + 10 f_1 g_4 h_1^3 h_2 +$$
$$15 f_3 g_1^3 h_1 h_2^2 + 45 f_2 g_1 g_2 h_1 h_2^2 + 15 f_1 g_3 h_1 h_2^2 + 10 f_3 g_1^3 h_1^2 h_3 + 30 f_2 g_1 g_2 h_1^2 h_3 +$$
$$10 f_1 g_3 h_1^2 h_3 + 10 f_2 g_1^2 h_2 h_3 + 10 f_1 g_2 h_2 h_3 + 5 f_2 g_1^2 h_1 h_4 + 5 f_1 g_2 h_1 h_4 + f_1 g_1 h_5$$

In[∘]:= `Y2[6] // FullSimplify // Expand`

Out[∘]=
$$f_6 g_1^6 h_1^6 + 15 f_5 g_1^4 g_2 h_1^6 + 45 f_4 g_1^2 g_2^2 h_1^6 + 15 f_3 g_2^3 h_1^6 + 20 f_4 g_1^3 g_3 h_1^6 + 60 f_3 g_1 g_2 g_3 h_1^6 +$$
$$10 f_2 g_3^2 h_1^6 + 15 f_3 g_1^2 g_4 h_1^6 + 15 f_2 g_2 g_4 h_1^6 + 6 f_2 g_1 g_5 h_1^6 + f_1 g_6 h_1^6 + 15 f_5 g_1^5 h_1^4 h_2 +$$
$$150 f_4 g_1^3 g_2 h_1^4 h_2 + 225 f_3 g_1 g_2^2 h_1^4 h_2 + 150 f_3 g_1^2 g_3 h_1^4 h_2 + 150 f_2 g_2 g_3 h_1^4 h_2 + 75 f_2 g_1 g_4 h_1^4 h_2 +$$
$$15 f_1 g_5 h_1^4 h_2 + 45 f_4 g_1^4 h_1^2 h_2^2 + 270 f_3 g_1^2 g_2 h_1^2 h_2^2 + 135 f_2 g_2^2 h_1^2 h_2^2 + 180 f_2 g_1 g_3 h_1^2 h_2^2 +$$
$$45 f_1 g_4 h_1^2 h_2^2 + 15 f_3 g_1^3 h_2^3 + 45 f_2 g_1 g_2 h_2^3 + 15 f_1 g_3 h_2^3 + 20 f_4 g_1^4 h_1^3 h_3 + 120 f_3 g_1^2 g_2 h_1^3 h_3 +$$
$$60 f_2 g_2^2 h_1^3 h_3 + 80 f_2 g_1 g_3 h_1^3 h_3 + 20 f_1 g_4 h_1^3 h_3 + 60 f_3 g_1^3 h_1 h_2 h_3 + 180 f_2 g_1 g_2 h_1 h_2 h_3 +$$
$$60 f_1 g_3 h_1 h_2 h_3 + 10 f_2 g_1^2 h_3^2 + 10 f_1 g_2 h_3^2 + 15 f_3 g_1^3 h_1^2 h_4 + 45 f_2 g_1 g_2 h_1^2 h_4 +$$
$$15 f_1 g_3 h_1^2 h_4 + 15 f_2 g_1^2 h_2 h_4 + 15 f_1 g_2 h_2 h_4 + 6 f_2 g_1^2 h_1 h_5 + 6 f_1 g_2 h_1 h_5 + f_1 g_1 h_6$$

In[∘]:= `Y2[7] // FullSimplify // Expand`

Out[∘]=
$$f_7 g_1^7 h_1^7 + 21 f_6 g_1^5 g_2 h_1^7 + 105 f_5 g_1^3 g_2^2 h_1^7 + 105 f_4 g_1 g_2^3 h_1^7 + 35 f_5 g_1^4 g_3 h_1^7 + 210 f_4 g_1^2 g_2 g_3 h_1^7 +$$
$$105 f_3 g_2^2 g_3 h_1^7 + 70 f_3 g_1 g_3^2 h_1^7 + 35 f_4 g_1^3 g_4 h_1^7 + 105 f_3 g_1 g_2 g_4 h_1^7 + 35 f_2 g_3 g_4 h_1^7 +$$
$$21 f_3 g_1^2 g_5 h_1^7 + 21 f_2 g_2 g_5 h_1^7 + 7 f_2 g_1 g_6 h_1^7 + f_1 g_7 h_1^7 + 21 f_6 g_1^6 h_1^5 h_2 + 315 f_5 g_1^4 g_2 h_1^5 h_2 +$$
$$945 f_4 g_1^2 g_2^2 h_1^5 h_2 + 315 f_3 g_2^3 h_1^5 h_2 + 420 f_4 g_1^3 g_3 h_1^5 h_2 + 1260 f_3 g_1 g_2 g_3 h_1^5 h_2 +$$
$$210 f_2 g_3^2 h_1^5 h_2 + 315 f_3 g_1^2 g_4 h_1^5 h_2 + 315 f_2 g_2 g_4 h_1^5 h_2 + 126 f_2 g_1 g_5 h_1^5 h_2 + 21 f_1 g_6 h_1^5 h_2 +$$
$$105 f_5 g_1^5 h_1^3 h_2^2 + 1050 f_4 g_1^3 g_2 h_1^3 h_2^2 + 1575 f_3 g_1 g_2^2 h_1^3 h_2^2 + 1050 f_3 g_1^2 g_3 h_1^3 h_2^2 +$$
$$1050 f_2 g_2 g_3 h_1^3 h_2^2 + 525 f_2 g_1 g_4 h_1^3 h_2^2 + 105 f_1 g_5 h_1^3 h_2^2 + 105 f_4 g_1^4 h_1 h_2^3 + 630 f_3 g_1^2 g_2 h_1 h_2^3 +$$
$$315 f_2 g_2^2 h_1 h_2^3 + 420 f_2 g_1 g_3 h_1 h_2^3 + 105 f_1 g_4 h_1 h_2^3 + 35 f_5 g_1^5 h_1^4 h_3 + 350 f_4 g_1^3 g_2 h_1^4 h_3 +$$
$$525 f_3 g_1 g_2^2 h_1^4 h_3 + 350 f_3 g_1^2 g_3 h_1^4 h_3 + 350 f_2 g_2 g_3 h_1^4 h_3 + 175 f_2 g_1 g_4 h_1^4 h_3 + 35 f_1 g_5 h_1^4 h_3 +$$
$$210 f_4 g_1^4 h_1^2 h_2 h_3 + 1260 f_3 g_1^2 g_2 h_1^2 h_2 h_3 + 630 f_2 g_2^2 h_1^2 h_2 h_3 + 840 f_2 g_1 g_3 h_1^2 h_2 h_3 +$$
$$210 f_1 g_4 h_1^2 h_2 h_3 + 105 f_3 g_1^3 h_2^2 h_3 + 315 f_2 g_1 g_2 h_2^2 h_3 + 105 f_1 g_3 h_2^2 h_3 + 70 f_3 g_1^3 h_1 h_3^2 +$$
$$210 f_2 g_1 g_2 h_1 h_3^2 + 70 f_1 g_3 h_1 h_3^2 + 35 f_4 g_1^4 h_1^3 h_4 + 210 f_3 g_1^2 g_2 h_1^3 h_4 + 105 f_2 g_2^2 h_1^3 h_4 +$$
$$140 f_2 g_1 g_3 h_1^3 h_4 + 35 f_1 g_4 h_1^3 h_4 + 105 f_3 g_1^3 h_1 h_2 h_4 + 315 f_2 g_1 g_2 h_1 h_2 h_4 +$$
$$105 f_1 g_3 h_1 h_2 h_4 + 35 f_2 g_1^2 h_3 h_4 + 35 f_1 g_2 h_3 h_4 + 21 f_3 g_1^3 h_1^2 h_5 + 63 f_2 g_1 g_2 h_1^2 h_5 +$$
$$21 f_1 g_3 h_1^2 h_5 + 21 f_2 g_1^2 h_2 h_5 + 21 f_1 g_2 h_2 h_5 + 7 f_2 g_1^2 h_1 h_6 + 7 f_1 g_2 h_1 h_6 + f_1 g_1 h_7$$

In[∘]:= `Y2[8] // FullSimplify // Expand`

Out[∘]=
$$f_8 g_1^8 h_1^8 + 28 f_7 g_1^6 g_2 h_1^8 + 210 f_6 g_1^4 g_2^2 h_1^8 + 420 f_5 g_1^2 g_2^3 h_1^8 + 105 f_4 g_2^4 h_1^8 + 56 f_6 g_1^5 g_3 h_1^8 +$$
$$560 f_5 g_1^3 g_2 g_3 h_1^8 + 840 f_4 g_1 g_2^2 g_3 h_1^8 + 280 f_4 g_1^2 g_3^2 h_1^8 + 280 f_3 g_2 g_3^2 h_1^8 + 70 f_5 g_1^4 g_4 h_1^8 +$$
$$420 f_4 g_1^2 g_2 g_4 h_1^8 + 210 f_3 g_2^2 g_4 h_1^8 + 280 f_3 g_1 g_3 g_4 h_1^8 + 35 f_2 g_4^2 h_1^8 + 56 f_4 g_1^3 g_5 h_1^8 +$$
$$168 f_3 g_1 g_2 g_5 h_1^8 + 56 f_2 g_3 g_5 h_1^8 + 28 f_3 g_1^2 g_6 h_1^8 + 28 f_2 g_2 g_6 h_1^8 + 8 f_2 g_1 g_7 h_1^8 + f_1 g_8 h_1^8 +$$
$$28 f_7 g_1^7 h_1^6 h_2 + 588 f_6 g_1^5 g_2 h_1^6 h_2 + 2940 f_5 g_1^3 g_2^2 h_1^6 h_2 + 2940 f_4 g_1 g_2^3 h_1^6 h_2 + 980 f_5 g_1^4 g_3 h_1^6 h_2 +$$
$$5880 f_4 g_1^2 g_2 g_3 h_1^6 h_2 + 2940 f_3 g_2^2 g_3 h_1^6 h_2 + 1960 f_3 g_1 g_3^2 h_1^6 h_2 + 980 f_4 g_1^3 g_4 h_1^6 h_2 +$$
$$2940 f_3 g_1 g_2 g_4 h_1^6 h_2 + 980 f_2 g_3 g_4 h_1^6 h_2 + 588 f_3 g_1^2 g_5 h_1^6 h_2 + 588 f_2 g_2 g_5 h_1^6 h_2 +$$
$$196 f_2 g_1 g_6 h_1^6 h_2 + 28 f_1 g_7 h_1^6 h_2 + 210 f_6 g_1^6 h_1^4 h_2^2 + 3150 f_5 g_1^4 g_2 h_1^4 h_2^2 + 9450 f_4 g_1^2 g_2^2 h_1^4 h_2^2 +$$
$$3150 f_3 g_2^3 h_1^4 h_2^2 + 4200 f_4 g_1^3 g_3 h_1^4 h_2^2 + 12600 f_3 g_1 g_2 g_3 h_1^4 h_2^2 + 2100 f_2 g_3^2 h_1^4 h_2^2 +$$
$$3150 f_3 g_1^2 g_4 h_1^4 h_2^2 + 3150 f_2 g_2 g_4 h_1^4 h_2^2 + 1260 f_2 g_1 g_5 h_1^4 h_2^2 + 210 f_1 g_6 h_1^4 h_2^2 + 420 f_5 g_1^5 h_1^2 h_2^3 +$$
$$4200 f_4 g_1^3 g_2 h_1^2 h_2^3 + 6300 f_3 g_1 g_2^2 h_1^2 h_2^3 + 4200 f_3 g_1^2 g_3 h_1^2 h_2^3 + 4200 f_2 g_2 g_3 h_1^2 h_2^3 +$$
$$2100 f_2 g_1 g_4 h_1^2 h_2^3 + 420 f_1 g_5 h_1^2 h_2^3 + 105 f_4 g_1^4 h_2^4 + 630 f_3 g_1^2 g_2 h_2^4 + 315 f_2 g_2^2 h_2^4 +$$
$$420 f_2 g_1 g_3 h_2^4 + 105 f_1 g_4 h_2^4 + 56 f_6 g_1^6 h_1^5 h_3 + 840 f_5 g_1^4 g_2 h_1^5 h_3 + 2520 f_4 g_1^2 g_2^2 h_1^5 h_3 +$$
$$840 f_3 g_2^3 h_1^5 h_3 + 1120 f_4 g_1^3 g_3 h_1^5 h_3 + 3360 f_3 g_1 g_2 g_3 h_1^5 h_3 + 560 f_2 g_3^2 h_1^5 h_3 +$$
$$840 f_3 g_1^2 g_4 h_1^5 h_3 + 840 f_2 g_2 g_4 h_1^5 h_3 + 336 f_2 g_1 g_5 h_1^5 h_3 + 56 f_1 g_6 h_1^5 h_3 + 560 f_5 g_1^5 h_1^3 h_2 h_3 +$$
$$5600 f_4 g_1^3 g_2 h_1^3 h_2 h_3 + 8400 f_3 g_1 g_2^2 h_1^3 h_2 h_3 + 5600 f_3 g_1^2 g_3 h_1^3 h_2 h_3 + 5600 f_2 g_2 g_3 h_1^3 h_2 h_3 +$$
$$2800 f_2 g_1 g_4 h_1^3 h_2 h_3 + 560 f_1 g_5 h_1^3 h_2 h_3 + 840 f_4 g_1^4 h_1 h_2^2 h_3 + 5040 f_3 g_1^2 g_2 h_1 h_2^2 h_3 +$$
$$2520 f_2 g_2^2 h_1 h_2^2 h_3 + 3360 f_2 g_1 g_3 h_1 h_2^2 h_3 + 840 f_1 g_4 h_1 h_2^2 h_3 + 280 f_4 g_1^4 h_1^2 h_3^2 +$$
$$1680 f_3 g_1^2 g_2 h_1^2 h_3^2 + 840 f_2 g_2^2 h_1^2 h_3^2 + 1120 f_2 g_1 g_3 h_1^2 h_3^2 + 280 f_1 g_4 h_1^2 h_3^2 + 280 f_3 g_1^3 h_2 h_3^2 +$$
$$840 f_2 g_1 g_2 h_2 h_3^2 + 280 f_1 g_3 h_2 h_3^2 + 70 f_5 g_1^5 h_1^4 h_4 + 700 f_4 g_1^3 g_2 h_1^4 h_4 + 1050 f_3 g_1 g_2^2 h_1^4 h_4 +$$
$$700 f_3 g_1^2 g_3 h_1^4 h_4 + 700 f_2 g_2 g_3 h_1^4 h_4 + 350 f_2 g_1 g_4 h_1^4 h_4 + 70 f_1 g_5 h_1^4 h_4 + 420 f_4 g_1^4 h_1^2 h_2 h_4 +$$
$$2520 f_3 g_1^2 g_2 h_1^2 h_2 h_4 + 1260 f_2 g_2^2 h_1^2 h_2 h_4 + 1680 f_2 g_1 g_3 h_1^2 h_2 h_4 + 420 f_1 g_4 h_1^2 h_2 h_4 +$$
$$210 f_3 g_1^3 h_2^2 h_4 + 630 f_2 g_1 g_2 h_2^2 h_4 + 210 f_1 g_3 h_2^2 h_4 + 280 f_3 g_1^3 h_1 h_3 h_4 + 840 f_2 g_1 g_2 h_1 h_3 h_4 +$$
$$280 f_1 g_3 h_1 h_3 h_4 + 35 f_2 g_1^2 h_4^2 + 35 f_1 g_2 h_4^2 + 56 f_4 g_1^4 h_1^3 h_5 + 336 f_3 g_1^2 g_2 h_1^3 h_5 +$$
$$168 f_2 g_2^2 h_1^3 h_5 + 224 f_2 g_1 g_3 h_1^3 h_5 + 56 f_1 g_4 h_1^3 h_5 + 168 f_3 g_1^3 h_1 h_2 h_5 + 504 f_2 g_1 g_2 h_1 h_2 h_5 +$$
$$168 f_1 g_3 h_1 h_2 h_5 + 56 f_2 g_1^2 h_3 h_5 + 56 f_1 g_2 h_3 h_5 + 28 f_3 g_1^3 h_1^2 h_6 + 84 f_2 g_1 g_2 h_1^2 h_6 +$$
$$28 f_1 g_3 h_1^2 h_6 + 28 f_2 g_1^2 h_2 h_6 + 28 f_1 g_2 h_2 h_6 + 8 f_2 g_1^2 h_1 h_7 + 8 f_1 g_2 h_1 h_7 + f_1 g_1 h_8$$

```
In[•]:= Y2[9] // FullSimplify // Expand
```

Out[•]=

$$
\begin{aligned}
& f_9\, g_1^9\, h_1^9 + 36\, f_8\, g_1^7\, g_2\, h_1^9 + 378\, f_7\, g_1^5\, g_2^2\, h_1^9 + 1260\, f_6\, g_1^3\, g_2^3\, h_1^9 + 945\, f_5\, g_1\, g_2^4\, h_1^9 + \\
& 84\, f_7\, g_1^6\, g_3\, h_1^9 + 1260\, f_6\, g_1^4\, g_2\, g_3\, h_1^9 + 3780\, f_5\, g_1^2\, g_2^2\, g_3\, h_1^9 + 1260\, f_4\, g_2^3\, g_3\, h_1^9 + \\
& 840\, f_5\, g_1^3\, g_3^2\, h_1^9 + 2520\, f_4\, g_1\, g_2\, g_3^2\, h_1^9 + 280\, f_3\, g_3^3\, h_1^9 + 126\, f_6\, g_1^5\, g_4\, h_1^9 + 1260\, f_5\, g_1^3\, g_2\, g_4\, h_1^9 + \\
& 1890\, f_4\, g_1\, g_2^2\, g_4\, h_1^9 + 1260\, f_4\, g_1^2\, g_3\, g_4\, h_1^9 + 1260\, f_3\, g_2\, g_3\, g_4\, h_1^9 + 315\, f_3\, g_1\, g_4^2\, h_1^9 + \\
& 126\, f_5\, g_1^4\, g_5\, h_1^9 + 756\, f_4\, g_1^2\, g_2\, g_5\, h_1^9 + 378\, f_3\, g_2^2\, g_5\, h_1^9 + 504\, f_3\, g_1\, g_3\, g_5\, h_1^9 + 126\, f_2\, g_4\, g_5\, h_1^9 + \\
& 84\, f_4\, g_1^3\, g_6\, h_1^9 + 252\, f_3\, g_1\, g_2\, g_6\, h_1^9 + 84\, f_2\, g_3\, g_6\, h_1^9 + 36\, f_3\, g_1^2\, g_7\, h_1^9 + 36\, f_2\, g_2\, g_7\, h_1^9 + \\
& 9\, f_2\, g_1\, g_8\, h_1^9 + f_1\, g_9\, h_1^9 + 36\, f_8\, g_1^8\, h_1^7\, h_2 + 1008\, f_7\, g_1^6\, g_2\, h_1^7\, h_2 + 7560\, f_6\, g_1^4\, g_2^2\, h_1^7\, h_2 + \\
& 15\,120\, f_5\, g_1^2\, g_2^3\, h_1^7\, h_2 + 3780\, f_4\, g_2^4\, h_1^7\, h_2 + 2016\, f_6\, g_1^5\, g_3\, h_1^7\, h_2 + 20\,160\, f_5\, g_1^3\, g_2\, g_3\, h_1^7\, h_2 + \\
& 30\,240\, f_4\, g_1\, g_2^2\, g_3\, h_1^7\, h_2 + 10\,080\, f_4\, g_1^2\, g_3^2\, h_1^7\, h_2 + 10\,080\, f_3\, g_2\, g_3^2\, h_1^7\, h_2 + 2520\, f_5\, g_1^4\, g_4\, h_1^7\, h_2 + \\
& 15\,120\, f_4\, g_1^2\, g_2\, g_4\, h_1^7\, h_2 + 7560\, f_3\, g_2^2\, g_4\, h_1^7\, h_2 + 10\,080\, f_3\, g_1\, g_3\, g_4\, h_1^7\, h_2 + 1260\, f_2\, g_4^2\, h_1^7\, h_2 + \\
& 2016\, f_4\, g_1^3\, g_5\, h_1^7\, h_2 + 6048\, f_3\, g_1\, g_2\, g_5\, h_1^7\, h_2 + 2016\, f_2\, g_3\, g_5\, h_1^7\, h_2 + 1008\, f_3\, g_1^2\, g_6\, h_1^7\, h_2 + \\
& 1008\, f_2\, g_2\, g_6\, h_1^7\, h_2 + 288\, f_2\, g_1\, g_7\, h_1^7\, h_2 + 36\, f_1\, g_8\, h_1^7\, h_2 + 378\, f_7\, g_1^7\, h_1^5\, h_2^2 + 7938\, f_6\, g_1^5\, g_2\, h_1^5\, h_2^2 + \\
& 39\,690\, f_5\, g_1^3\, g_2^2\, h_1^5\, h_2^2 + 39\,690\, f_4\, g_1\, g_2^3\, h_1^5\, h_2^2 + 13\,230\, f_5\, g_1^4\, g_3\, h_1^5\, h_2^2 + 79\,380\, f_4\, g_1^2\, g_2\, g_3\, h_1^5\, h_2^2 + \\
& 39\,690\, f_3\, g_2^2\, g_3\, h_1^5\, h_2^2 + 26\,460\, f_3\, g_1\, g_3^2\, h_1^5\, h_2^2 + 13\,230\, f_4\, g_1^3\, g_4\, h_1^5\, h_2^2 + 39\,690\, f_3\, g_1\, g_2\, g_4\, h_1^5\, h_2^2 + \\
& 13\,230\, f_2\, g_3\, g_4\, h_1^5\, h_2^2 + 7938\, f_3\, g_1^2\, g_5\, h_1^5\, h_2^2 + 7938\, f_2\, g_2\, g_5\, h_1^5\, h_2^2 + 2646\, f_2\, g_1\, g_6\, h_1^5\, h_2^2 + \\
& 378\, f_1\, g_7\, h_1^5\, h_2^2 + 1260\, f_6\, g_1^6\, h_1^3\, h_2^3 + 18\,900\, f_5\, g_1^4\, g_2\, h_1^3\, h_2^3 + 56\,700\, f_4\, g_1^2\, g_2^2\, h_1^3\, h_2^3 + \\
& 18\,900\, f_3\, g_2^3\, h_1^3\, h_2^3 + 25\,200\, f_4\, g_1^3\, g_3\, h_1^3\, h_2^3 + 75\,600\, f_3\, g_1\, g_2\, g_3\, h_1^3\, h_2^3 + 12\,600\, f_2\, g_3^2\, h_1^3\, h_2^3 + \\
& 18\,900\, f_3\, g_1^2\, g_4\, h_1^3\, h_2^3 + 18\,900\, f_2\, g_2\, g_4\, h_1^3\, h_2^3 + 7560\, f_2\, g_1\, g_5\, h_1^3\, h_2^3 + 1260\, f_1\, g_6\, h_1^3\, h_2^3 + \\
& 945\, f_5\, g_1^5\, h_1\, h_2^4 + 9450\, f_4\, g_1^3\, g_2\, h_1\, h_2^4 + 14\,175\, f_3\, g_1\, g_2^2\, h_1\, h_2^4 + 9450\, f_3\, g_1^2\, g_3\, h_1\, h_2^4 + \\
& 9450\, f_2\, g_2\, g_3\, h_1\, h_2^4 + 4725\, f_2\, g_1\, g_4\, h_1\, h_2^4 + 945\, f_1\, g_5\, h_1\, h_2^4 + 84\, f_7\, g_1^7\, h_1^6\, h_3 + 1764\, f_6\, g_1^5\, g_2\, h_1^6\, h_3 + \\
& 8820\, f_5\, g_1^3\, g_2^2\, h_1^6\, h_3 + 8820\, f_4\, g_1\, g_2^3\, h_1^6\, h_3 + 2940\, f_5\, g_1^4\, g_3\, h_1^6\, h_3 + 17\,640\, f_4\, g_1^2\, g_2\, g_3\, h_1^6\, h_3 + \\
& 8820\, f_3\, g_2^2\, g_3\, h_1^6\, h_3 + 5880\, f_3\, g_1\, g_3^2\, h_1^6\, h_3 + 2940\, f_4\, g_1^3\, g_4\, h_1^6\, h_3 + 8820\, f_3\, g_1\, g_2\, g_4\, h_1^6\, h_3 + \\
& 2940\, f_2\, g_3\, g_4\, h_1^6\, h_3 + 1764\, f_3\, g_1^2\, g_5\, h_1^6\, h_3 + 1764\, f_2\, g_2\, g_5\, h_1^6\, h_3 + 588\, f_2\, g_1\, g_6\, h_1^6\, h_3 + \\
& 84\, f_1\, g_7\, h_1^6\, h_3 + 1260\, f_6\, g_1^6\, h_1^4\, h_2\, h_3 + 18\,900\, f_5\, g_1^4\, g_2\, h_1^4\, h_2\, h_3 + 56\,700\, f_4\, g_1^2\, g_2^2\, h_1^4\, h_2\, h_3 + \\
& 18\,900\, f_3\, g_2^3\, h_1^4\, h_2\, h_3 + 25\,200\, f_4\, g_1^3\, g_3\, h_1^4\, h_2\, h_3 + 75\,600\, f_3\, g_1\, g_2\, g_3\, h_1^4\, h_2\, h_3 + \\
& 12\,600\, f_2\, g_3^2\, h_1^4\, h_2\, h_3 + 18\,900\, f_3\, g_1^2\, g_4\, h_1^4\, h_2\, h_3 + 18\,900\, f_2\, g_2\, g_4\, h_1^4\, h_2\, h_3 + 7560\, f_2\, g_1\, g_5\, h_1^4\, h_2\, h_3 + \\
& 1260\, f_1\, g_6\, h_1^4\, h_2\, h_3 + 3780\, f_5\, g_1^5\, h_1^2\, h_2^2\, h_3 + 37\,800\, f_4\, g_1^3\, g_2\, h_1^2\, h_2^2\, h_3 + 56\,700\, f_3\, g_1\, g_2^2\, h_1^2\, h_2^2\, h_3 + \\
& 37\,800\, f_3\, g_1^2\, g_3\, h_1^2\, h_2^2\, h_3 + 37\,800\, f_2\, g_2\, g_3\, h_1^2\, h_2^2\, h_3 + 18\,900\, f_2\, g_1\, g_4\, h_1^2\, h_2^2\, h_3 + 3780\, f_1\, g_5\, h_1^2\, h_2^2\, h_3 + \\
& 1260\, f_4\, g_1^4\, h_2^3\, h_3 + 7560\, f_3\, g_1^2\, g_2\, h_2^3\, h_3 + 3780\, f_2\, g_2^2\, h_2^3\, h_3 + 5040\, f_2\, g_1\, g_3\, h_2^3\, h_3 + 1260\, f_1\, g_4\, h_2^3\, h_3 + \\
& 840\, f_5\, g_1^5\, h_1^3\, h_3^2 + 8400\, f_4\, g_1^3\, g_2\, h_1^3\, h_3^2 + 12\,600\, f_3\, g_1\, g_2^2\, h_1^3\, h_3^2 + 8400\, f_3\, g_1^2\, g_3\, h_1^3\, h_3^2 + \\
& 8400\, f_2\, g_2\, g_3\, h_1^3\, h_3^2 + 4200\, f_2\, g_1\, g_4\, h_1^3\, h_3^2 + 840\, f_1\, g_5\, h_1^3\, h_3^2 + 2520\, f_4\, g_1^4\, h_1\, h_2\, h_3^2 + \\
& 15\,120\, f_3\, g_1^2\, g_2\, h_1\, h_2\, h_3^2 + 7560\, f_2\, g_2^2\, h_1\, h_2\, h_3^2 + 10\,080\, f_2\, g_1\, g_3\, h_1\, h_2\, h_3^2 + 2520\, f_1\, g_4\, h_1\, h_2\, h_3^2 + \\
& 280\, f_3\, g_1^3\, h_3^3 + 840\, f_2\, g_1\, g_2\, h_3^3 + 280\, f_1\, g_3\, h_3^3 + 126\, f_6\, g_1^6\, h_1^5\, h_4 + 1890\, f_5\, g_1^4\, g_2\, h_1^5\, h_4 + \\
& 5670\, f_4\, g_1^2\, g_2^2\, h_1^5\, h_4 + 1890\, f_3\, g_2^3\, h_1^5\, h_4 + 2520\, f_4\, g_1^3\, g_3\, h_1^5\, h_4 + 7560\, f_3\, g_1\, g_2\, g_3\, h_1^5\, h_4 + \\
& 1260\, f_2\, g_3^2\, h_1^5\, h_4 + 1890\, f_3\, g_1^2\, g_4\, h_1^5\, h_4 + 1890\, f_2\, g_2\, g_4\, h_1^5\, h_4 + 756\, f_2\, g_1\, g_5\, h_1^5\, h_4 + \\
& 126\, f_1\, g_6\, h_1^5\, h_4 + 1260\, f_5\, g_1^5\, h_1^3\, h_2\, h_4 + 12\,600\, f_4\, g_1^3\, g_2\, h_1^3\, h_2\, h_4 + 18\,900\, f_3\, g_1\, g_2^2\, h_1^3\, h_2\, h_4 + \\
& 12\,600\, f_3\, g_1^2\, g_3\, h_1^3\, h_2\, h_4 + 12\,600\, f_2\, g_2\, g_3\, h_1^3\, h_2\, h_4 + 6300\, f_2\, g_1\, g_4\, h_1^3\, h_2\, h_4 + 1260\, f_1\, g_5\, h_1^3\, h_2\, h_4 + \\
& 1890\, f_4\, g_1^4\, h_1\, h_2^2\, h_4 + 11\,340\, f_3\, g_1^2\, g_2\, h_1\, h_2^2\, h_4 + 5670\, f_2\, g_2^2\, h_1\, h_2^2\, h_4 + 7560\, f_2\, g_1\, g_3\, h_1\, h_2^2\, h_4 + \\
& 1890\, f_1\, g_4\, h_1\, h_2^2\, h_4 + 1260\, f_4\, g_1^4\, h_1^2\, h_3\, h_4 + 7560\, f_3\, g_1^2\, g_2\, h_1^2\, h_3\, h_4 + 3780\, f_2\, g_2^2\, h_1^2\, h_3\, h_4 + \\
& 5040\, f_2\, g_1\, g_3\, h_1^2\, h_3\, h_4 + 1260\, f_1\, g_4\, h_1^2\, h_3\, h_4 + 1260\, f_3\, g_1^3\, h_2\, h_3\, h_4 + 3780\, f_2\, g_1\, g_2\, h_2\, h_3\, h_4 + \\
& 1260\, f_1\, g_3\, h_2\, h_3\, h_4 + 315\, f_3\, g_1^3\, h_1\, h_4^2 + 945\, f_2\, g_1\, g_2\, h_1\, h_4^2 + 315\, f_1\, g_3\, h_1\, h_4^2 + 126\, f_5\, g_1^5\, h_1^4\, h_5 + \\
& 1260\, f_4\, g_1^3\, g_2\, h_1^4\, h_5 + 1890\, f_3\, g_1\, g_2^2\, h_1^4\, h_5 + 1260\, f_3\, g_1^2\, g_3\, h_1^4\, h_5 + 1260\, f_2\, g_2\, g_3\, h_1^4\, h_5 + \\
& 630\, f_2\, g_1\, g_4\, h_1^4\, h_5 + 126\, f_1\, g_5\, h_1^4\, h_5 + 756\, f_4\, g_1^4\, h_1^2\, h_2\, h_5 + 4536\, f_3\, g_1^2\, g_2\, h_1^2\, h_2\, h_5 + \\
& 2268\, f_2\, g_2^2\, h_1^2\, h_2\, h_5 + 3024\, f_2\, g_1\, g_3\, h_1^2\, h_2\, h_5 + 756\, f_1\, g_4\, h_1^2\, h_2\, h_5 + 378\, f_3\, g_1^3\, h_2^2\, h_5 + \\
& 1134\, f_2\, g_1\, g_2\, h_2^2\, h_5 + 378\, f_1\, g_3\, h_2^2\, h_5 + 504\, f_3\, g_1^3\, h_1\, h_3\, h_5 + 1512\, f_2\, g_1\, g_2\, h_1\, h_3\, h_5 + \\
& 504\, f_1\, g_3\, h_1\, h_3\, h_5 + 126\, f_2\, g_1^2\, h_4\, h_5 + 126\, f_1\, g_2\, h_4\, h_5 + 84\, f_4\, g_1^4\, h_1^3\, h_6 + 504\, f_3\, g_1^2\, g_2\, h_1^3\, h_6 + \\
& 252\, f_2\, g_2^2\, h_1^3\, h_6 + 336\, f_2\, g_1\, g_3\, h_1^3\, h_6 + 84\, f_1\, g_4\, h_1^3\, h_6 + 252\, f_3\, g_1^3\, h_1\, h_2\, h_6 + 756\, f_2\, g_1\, g_2\, h_1\, h_2\, h_6 + \\
& 252\, f_1\, g_3\, h_1\, h_2\, h_6 + 84\, f_2\, g_1^2\, h_3\, h_6 + 84\, f_1\, g_2\, h_3\, h_6 + 36\, f_3\, g_1^3\, h_1^2\, h_7 + 108\, f_2\, g_1\, g_2\, h_1^2\, h_7 + \\
& 36\, f_1\, g_3\, h_1^2\, h_7 + 36\, f_2\, g_1^2\, h_2\, h_7 + 36\, f_1\, g_2\, h_2\, h_7 + 9\, f_2\, g_1^2\, h_1\, h_8 + 9\, f_1\, g_2\, h_1\, h_8 + f_1\, g_1\, h_9
\end{aligned}
$$

Appendix B

```
In[•]:= M = 4;
In[•]:= N = 5;
```

$$Y_1 = \text{Table}\left[\sum_{k=1}^{n} \left(\text{BellY}\left[n,\, k,\, \text{Table}\left[f_{M,m},\, \{m,\, 1,\, n-k+1\}\right]\right] f_{M-1,k}\right),\, \{n,\, N\}\right];$$

$$\text{For}\Big[i = 2,\, i < M,\, {+}{+}i,$$

$$Y_i = \text{Table}\left[\sum_{k=1}^{n} \left(\text{BellY}[n,\, k,\, \text{Table}[Y_{i-1}[[m]],\, \{m,\, 1,\, n-k+1\}]]\, f_{M-i,k}\right),\, \{n,\, N\}\right];\ \Big]$$

In[•]:= $Y_{M-1}[[1]]$ // FullSimplify // Expand

Out[•]= $f_{1,1}\, f_{2,1}\, f_{3,1}\, f_{4,1}$

In[•]:= $Y_{M-1}[[2]]$ // FullSimplify // Expand

Out[•]= $f_{1,2}\, f_{2,1}^2\, f_{3,1}^2\, f_{4,1}^2 + f_{1,1}\, f_{2,2}\, f_{3,1}^2\, f_{4,1}^2 + f_{1,1}\, f_{2,1}\, f_{3,2}\, f_{4,1}^2 + f_{1,1}\, f_{2,1}\, f_{3,1}\, f_{4,2}$

In[•]:= $Y_{M-1}[[3]]$ // FullSimplify // Expand

Out[•]=

$$
\begin{aligned}
& f_{1,3}\, f_{2,1}^3\, f_{3,1}^3\, f_{4,1}^3 + 3\, f_{1,2}\, f_{2,1}\, f_{2,2}\, f_{3,1}^3\, f_{4,1}^3 + f_{1,1}\, f_{2,3}\, f_{3,1}^3\, f_{4,1}^3 + 3\, f_{1,2}\, f_{2,1}^2\, f_{3,1}\, f_{3,2}\, f_{4,1}^3 + \\
& 3\, f_{1,1}\, f_{2,2}\, f_{3,1}\, f_{3,2}\, f_{4,1}^3 + f_{1,1}\, f_{2,1}\, f_{3,3}\, f_{4,1}^3 + 3\, f_{1,2}\, f_{2,1}^2\, f_{3,1}^2\, f_{4,1}\, f_{4,2} + \\
& 3\, f_{1,1}\, f_{2,2}\, f_{3,1}^2\, f_{4,1}\, f_{4,2} + 3\, f_{1,1}\, f_{2,1}\, f_{3,2}\, f_{4,1}\, f_{4,2} + f_{1,1}\, f_{2,1}\, f_{3,1}\, f_{4,3}
\end{aligned}
$$

In[•]:= $Y_{M-1}[[4]]$ // FullSimplify // Expand

Out[•]=

$$
\begin{aligned}
& f_{1,4}\, f_{2,1}^4\, f_{3,1}^4\, f_{4,1}^4 + 6\, f_{1,3}\, f_{2,1}^2\, f_{2,2}\, f_{3,1}^4\, f_{4,1}^4 + 3\, f_{1,2}\, f_{2,2}^2\, f_{3,1}^4\, f_{4,1}^4 + \\
& 4\, f_{1,2}\, f_{2,1}\, f_{2,3}\, f_{3,1}^4\, f_{4,1}^4 + f_{1,1}\, f_{2,4}\, f_{3,1}^4\, f_{4,1}^4 + 6\, f_{1,3}\, f_{2,1}^3\, f_{3,1}^2\, f_{3,2}\, f_{4,1}^4 + \\
& 18\, f_{1,2}\, f_{2,1}\, f_{2,2}\, f_{3,1}^2\, f_{3,2}\, f_{4,1}^4 + 6\, f_{1,1}\, f_{2,3}\, f_{3,1}^2\, f_{3,2}\, f_{4,1}^4 + 3\, f_{1,2}\, f_{2,1}^2\, f_{3,2}^2\, f_{4,1}^4 + \\
& 3\, f_{1,1}\, f_{2,2}\, f_{3,2}^2\, f_{4,1}^4 + 4\, f_{1,2}\, f_{2,1}^2\, f_{3,1}\, f_{3,3}\, f_{4,1}^4 + 4\, f_{1,1}\, f_{2,2}\, f_{3,1}\, f_{3,3}\, f_{4,1}^4 + f_{1,1}\, f_{2,1}\, f_{3,4}\, f_{4,1}^4 + \\
& 6\, f_{1,3}\, f_{2,1}^3\, f_{3,1}^3\, f_{4,1}^2\, f_{4,2} + 18\, f_{1,2}\, f_{2,1}\, f_{2,2}\, f_{3,1}^3\, f_{4,1}^2\, f_{4,2} + 6\, f_{1,1}\, f_{2,3}\, f_{3,1}^3\, f_{4,1}^2\, f_{4,2} + \\
& 18\, f_{1,2}\, f_{2,1}^2\, f_{3,1}\, f_{3,2}\, f_{4,1}^2\, f_{4,2} + 18\, f_{1,1}\, f_{2,2}\, f_{3,1}\, f_{3,2}\, f_{4,1}^2\, f_{4,2} + 6\, f_{1,1}\, f_{2,1}\, f_{3,3}\, f_{4,1}^2\, f_{4,2} + \\
& 3\, f_{1,2}\, f_{2,1}^2\, f_{3,1}^2\, f_{4,2}^2 + 3\, f_{1,1}\, f_{2,2}\, f_{3,1}^2\, f_{4,2}^2 + 3\, f_{1,1}\, f_{2,1}\, f_{3,2}\, f_{4,2}^2 + 4\, f_{1,2}\, f_{2,1}^2\, f_{3,1}^2\, f_{4,1}\, f_{4,3} + \\
& 4\, f_{1,1}\, f_{2,2}\, f_{3,1}^2\, f_{4,1}\, f_{4,3} + 4\, f_{1,1}\, f_{2,1}\, f_{3,2}\, f_{4,1}\, f_{4,3} + f_{1,1}\, f_{2,1}\, f_{3,1}\, f_{4,4}
\end{aligned}
$$

In[∘]:= Y_{M-1}[[5]] // FullSimplify // Expand

Out[∘]=

$$\begin{aligned}
&f_{1,5} f_{2,1}^5 f_{3,1}^5 f_{4,1}^5 + 10 f_{1,4} f_{2,1}^3 f_{2,2} f_{3,1}^5 f_{4,1}^5 + 15 f_{1,3} f_{2,1} f_{2,2}^2 f_{3,1}^5 f_{4,1}^5 + \\
&10 f_{1,3} f_{2,1}^2 f_{2,3} f_{3,1}^5 f_{4,1}^5 + 10 f_{1,2} f_{2,2} f_{2,3} f_{3,1}^5 f_{4,1}^5 + 5 f_{1,2} f_{2,1} f_{2,4} f_{3,1}^5 f_{4,1}^5 + \\
&f_{1,1} f_{2,5} f_{3,1}^5 f_{4,1}^5 + 10 f_{1,4} f_{2,1}^4 f_{3,1}^3 f_{3,2} f_{4,1}^5 + 60 f_{1,3} f_{2,1}^2 f_{2,2} f_{3,1}^3 f_{3,2} f_{4,1}^5 + \\
&30 f_{1,2} f_{2,2}^2 f_{3,1}^3 f_{3,2} f_{4,1}^5 + 40 f_{1,2} f_{2,1} f_{2,3} f_{3,1}^3 f_{3,2} f_{4,1}^5 + 10 f_{1,1} f_{2,4} f_{3,1}^3 f_{3,2} f_{4,1}^5 + \\
&15 f_{1,3} f_{2,1}^3 f_{3,1} f_{3,2}^2 f_{4,1}^5 + 45 f_{1,2} f_{2,1} f_{2,2} f_{3,1} f_{3,2}^2 f_{4,1}^5 + 15 f_{1,1} f_{2,3} f_{3,1} f_{3,2}^2 f_{4,1}^5 + \\
&10 f_{1,3} f_{2,1}^3 f_{3,1}^2 f_{3,3} f_{4,1}^5 + 30 f_{1,2} f_{2,1} f_{2,2} f_{3,1}^2 f_{3,3} f_{4,1}^5 + 10 f_{1,1} f_{2,3} f_{3,1}^2 f_{3,3} f_{4,1}^5 + \\
&10 f_{1,2} f_{2,1}^2 f_{3,2} f_{3,3} f_{4,1}^5 + 10 f_{1,1} f_{2,2} f_{3,2} f_{3,3} f_{4,1}^5 + 5 f_{1,2} f_{2,1}^2 f_{3,1} f_{3,4} f_{4,1}^5 + \\
&5 f_{1,1} f_{2,2} f_{3,1} f_{3,4} f_{4,1}^5 + f_{1,1} f_{2,1} f_{3,5} f_{4,1}^5 + 10 f_{1,4} f_{2,1}^4 f_{3,1}^4 f_{4,1}^3 f_{4,2} + \\
&60 f_{1,3} f_{2,1}^2 f_{2,2} f_{3,1}^4 f_{4,1}^3 f_{4,2} + 30 f_{1,2} f_{2,2}^2 f_{3,1}^4 f_{4,1}^3 f_{4,2} + 40 f_{1,2} f_{2,1} f_{2,3} f_{3,1}^4 f_{4,1}^3 f_{4,2} + \\
&10 f_{1,1} f_{2,4} f_{3,1}^4 f_{4,1}^3 f_{4,2} + 60 f_{1,3} f_{2,1}^3 f_{3,1}^2 f_{3,2} f_{4,1}^3 f_{4,2} + 180 f_{1,2} f_{2,1} f_{2,2} f_{3,1}^2 f_{3,2} f_{4,1}^3 f_{4,2} + \\
&60 f_{1,1} f_{2,3} f_{3,1}^2 f_{3,2} f_{4,1}^3 f_{4,2} + 30 f_{1,2} f_{2,1}^2 f_{3,2}^2 f_{4,1}^3 f_{4,2} + 30 f_{1,1} f_{2,2} f_{3,2}^2 f_{4,1}^3 f_{4,2} + \\
&40 f_{1,2} f_{2,1}^2 f_{3,1} f_{3,3} f_{4,1}^3 f_{4,2} + 40 f_{1,1} f_{2,2} f_{3,1} f_{3,3} f_{4,1}^3 f_{4,2} + 10 f_{1,1} f_{2,1} f_{3,4} f_{4,1}^3 f_{4,2} + \\
&15 f_{1,3} f_{2,1}^3 f_{3,1}^3 f_{4,1} f_{4,2}^2 + 45 f_{1,2} f_{2,1} f_{2,2} f_{3,1}^3 f_{4,1} f_{4,2}^2 + 15 f_{1,1} f_{2,3} f_{3,1}^3 f_{4,1} f_{4,2}^2 + \\
&45 f_{1,2} f_{2,1}^2 f_{3,1} f_{3,2} f_{4,1} f_{4,2}^2 + 45 f_{1,1} f_{2,2} f_{3,1} f_{3,2} f_{4,1} f_{4,2}^2 + 15 f_{1,1} f_{2,1} f_{3,3} f_{4,1} f_{4,2}^2 + \\
&10 f_{1,3} f_{2,1}^3 f_{3,1}^3 f_{4,1}^2 f_{4,3} + 30 f_{1,2} f_{2,1} f_{2,2} f_{3,1}^3 f_{4,1}^2 f_{4,3} + 10 f_{1,1} f_{2,3} f_{3,1}^3 f_{4,1}^2 f_{4,3} + \\
&30 f_{1,2} f_{2,1}^2 f_{3,1} f_{3,2} f_{4,1}^2 f_{4,3} + 30 f_{1,1} f_{2,2} f_{3,1} f_{3,2} f_{4,1}^2 f_{4,3} + 10 f_{1,1} f_{2,1} f_{3,3} f_{4,1}^2 f_{4,3} + \\
&10 f_{1,2} f_{2,1}^2 f_{3,1}^2 f_{4,2} f_{4,3} + 10 f_{1,1} f_{2,2} f_{3,1}^2 f_{4,2} f_{4,3} + 10 f_{1,1} f_{2,1} f_{3,2} f_{4,2} f_{4,3} + \\
&5 f_{1,2} f_{2,1}^2 f_{3,1}^2 f_{4,1} f_{4,4} + 5 f_{1,1} f_{2,2} f_{3,1}^2 f_{4,1} f_{4,4} + 5 f_{1,1} f_{2,1} f_{3,2} f_{4,1} f_{4,4} + f_{1,1} f_{2,1} f_{3,1} f_{4,5}
\end{aligned}$$

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