Article

A Note on Degenerate Catalan-Daehee Numbers and Polynomials

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Abstract: In this paper, we consider the degenerate forms of the Catalan–Daehee polynomials and numbers by the Volkenborn integrals and obtain diverse explicit expressions and formulas. Moreover, we show the expressions of the degenerate Catalan–Daehee numbers in terms of λ-Daehee numbers, Stirling numbers of the first kind and Bernoulli polynomials, and we also obtain a relation covering the Bernoulli numbers, the degenerate Catalan–Daehee numbers and Stirling numbers of the second kind. In addition, we prove an implicit summation formula and a symmetric identity, and we derive an explicit expression for the degenerate Catalan–Daehee polynomials including the Stirling numbers of the first kind and Bernoulli polynomials.

Keywords: Catalan-Daehee polynomials; degenerate Catalan-Daehee polynomials; Volkenborn integral; Bernoulli polynomials; degenerate exponential function

MSC: 11B68; 11B83; 11S80

1. Introduction

Special polynomials and numbers possess various significance in many areas of mathematics, engineering, physics and other related disciplines covering the topics such as differential equations, functional analysis, quantum mechanics, mathematical physics, mathematical analysis and so on (cf. [1–21], see also the references cited therein). These numbers and polynomials can be explored by utilizing multifarious tools like generating functions, p-adic analysis, differential equations, probability theory, combinatorial methods and umbral calculus techniques. Two of the significant families of polynomials and numbers are the Daehee (cf. [1,4,8,10,11,14,18–20]) and Catalan (cf. [1,4–7,9–11,13–15,19]) polynomials and numbers. Recently, Kim et al. [11] introduced the Catalan–Daehee polynomials by means of the bosonic p-adic integrals and provided several interesting properties and relations. Then, the Catalan–Daehee polynomials with their multifarious extensions have been studied and developed by several mathematicians, some of whom are Jang, Ma, Lee, Dolgy and Khan; see [1,4,10,14,19]. Diverse q-extensions of the Catalan–Daehee polynomials were considered and investigated by many authors in [14,19]. w-extensions of the Catalan–Daehee polynomials and numbers were considered and investigated by Kim et al. in [4]. By inspiring and motivating the studies of Catalan–Daehee polynomials and their degenerate versions, in this paper, we consider the degenerate forms of the Catalan–Daehee numbers and polynomials by using the bosonic p-adic integral on $\mathbb{Z}_p$ and obtain diverse explicit expressions and formulas. Moreover, we show the expressions of the degenerate Catalan–Daehee numbers in terms of λ-Daehee numbers, Bernoulli polynomials and Stirling numbers of the first kind, and we also obtain a relation covering the Bernoulli

numbers, the degenerate Catalan–Daehee numbers and Stirling numbers of the second kind. Furthermore, we derive an explicit expression for the degenerate Catalan–Daehee polynomials including the degenerate Bernoulli polynomials and Stirling numbers of the first kind.

Throughout this paper, we make use of the following notations: $\mathbb{Z}$ indicates the set of all integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}_p$ indicates the ring of the p-adic integers, $\mathbb{Q}_p$ denotes the field of the p-adic numbers, and $\mathbb{C}_p$ indicates the p-adic completion of the algebraic closure of $\mathbb{Q}_p$, where p be a fixed prime number. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N} = \{1, 2, 3, \cdots\}$. The notation $|.|_p$ indicates the p-adic norm on $\mathbb{C}_p$ normalized by $|p|_p = 1/p$. For any uniformly differentiable function $g : \mathbb{Z}_p \to \mathbb{C}_p$, the Volkenborn integral or bosonic p-adic integral on $\mathbb{Z}_p$ is provided, see [1,2,4,8,11,17,18,20,21], as follows:

$$\int_{\mathbb{Z}_p} g(\xi)d\mu(\xi) = \lim_{m \to \infty} \frac{1}{p^m} \sum_{\xi=0}^{p^m-1} g(\xi). \tag{1}$$

From (1), we observe that

$$\begin{aligned}
\int_{Z_p} (g(\xi+1) - g(x))d\mu(\xi) &= \lim_{m \to \infty} \frac{1}{p^m} \sum_{\xi=0}^{p^m-1} (g(\xi+1) - g(\xi)) \\
&= \lim_{m \to \infty} \frac{1}{p^m} (g(p^m) - g(0)) \\
&= \lim_{m \to \infty} \frac{g'(p^m)p^m \log m}{p^m \log m} = g'(0),
\end{aligned}$$

which gives the following expression

$$\int_{Z_p} g(\xi+1)d\mu(\xi) = \int_{Z_p} g(\xi)d\mu(\xi) + g'(0). \tag{2}$$

Similar to the above, we also obtain the following expression

$$\int_{Z_p} g(\xi+m)d\mu(x) = \int_{Z_p} g(\xi)d\mu(\xi) + \sum_{l=0}^{m-1} g'(l), \tag{3}$$

where $g'(l) = \frac{dg(\xi)}{d\xi}|_{\xi=l}$.

The Bernoulli polynomials $B_m(\xi)$ are defined by the Volkenborn integral on $\mathbb{Z}_p$ as follows (see [2,8,17]):

$$\int_{\mathbb{Z}_p} e^{(\xi+\omega)z}d\mu(\omega) = \frac{z}{e^z - 1}e^{\xi z} = \sum_{m=0}^{\infty} B_m(\xi)\frac{z^m}{m!}. \tag{4}$$

Upon setting $\xi = 0$ in (4), we obtain $B_m(0) := B_m$ called the Bernoulli numbers. Also from (4), we have

$$\int_{\mathbb{Z}_p} (\xi+\omega)^m d\mu(\omega) = B_m(\xi), \ (m \geq 0). \tag{5}$$

These numbers and polynomials have a lot of applications in analytic number theory, such as evaluating the zeta function, estimating the harmonic series, summing powers of integers, as well as finding asymptotics of Stirling's formula, modular forms and Iwasawa theory.

For $z \in \mathbb{C}_p$ with $|z|_p < p^{-\frac{1}{p-1}}$, the familiar Daehee polynomials, which are closely related to the Bernoulli polynomials, (cf. [2,8]) are defined as follows:

$$\int_{\mathbb{Z}_p} (1+z)^{\xi+\varpi} d\mu(\varpi) = \frac{\log(1+z)}{z}(1+z)^{\xi} = \sum_{m=0}^{\infty} D_m(\xi)\frac{z^m}{m!}. \tag{6}$$

Taking $\xi = 0$ in (6), we obtain $D_m(0) := D_m$ termed the usual Daehee numbers. Several extensions and some applications of the Daehee polynomials and numbers have been studied by many mathematicians in [1–4,8,10,11,14,17,19–21].

The Catalan polynomials are defined by the following generating function (see [1,4–7,9–11,13–15,19]):

$$\frac{2}{1+\sqrt{1-4z}}(1-4z)^{\frac{\xi}{2}} = \sum_{m=0}^{\infty} C_m(\xi)z^m, \tag{7}$$

where $z \in \mathbb{C}_p$ with $|z|_p < p^{-\frac{1}{p-1}}$. Note that $C_m(0) := C_m$ are called the Catalan numbers, which are a sequence of natural numbers that seem in a lot of counting problems in the theory of combinatorics. These numbers count specific kinds of binary trees, permutations, lattice paths and many more combinatorial objects. Moreover, the mentioned numbers fulfill a basic recurrence relationship and possess a closed-form identity with regard to the binomial coefficients. Also, these numbers satisfy the following relation:

$$C_m = \frac{1}{m+1}\binom{2m}{m} \quad (m \in \mathbb{N}_0).$$

The degenerate Catalan polynomials are defined by the following generating function (see [15]):

$$2\frac{(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}}{1+\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}}} = \sum_{m=0}^{\infty} C_{m,\lambda}(\xi)z^m, \tag{8}$$

where $z \in \mathbb{C}_p$ with $|z|_p < p^{-\frac{1}{p-1}}$. Note that $C_{m,\lambda}(0) := C_{m,\lambda}$, which are called the degenerate Catalan numbers.

For $z \in \mathbb{C}_p$ with $|z|_p < p^{-\frac{1}{p-1}}$, Jeong et al. [3] considered the degenerate Daehee polynomials as follows:

$$\int_{\mathbb{Z}_p} (1+\log(1+\lambda z)^{\frac{1}{\lambda}})^{\xi+\varpi} d\mu(\varpi)$$

$$= \frac{\log(1+\log(1+\lambda z)^{\frac{1}{\lambda}})}{\log(1+\lambda z)^{\frac{1}{\lambda}}}(1+\log(1+\lambda z)^{\frac{1}{\lambda}})^{\xi} = \sum_{m=0}^{\infty} D_m(\xi|\lambda)\frac{z^m}{m!}. \tag{9}$$

In the case when $\xi = 0$, $D_m(0|\lambda) := D_m(\lambda)$ are termed the degenerate Daehee numbers. Note that

$$\lim_{\lambda \to 0} D_m(\xi|\lambda) = D_m(\xi), (m \geq 0).$$

By means of the Volkenborn integral on $\mathbb{Z}_p$, Kim et al. [10,11] considered the Catalan–Daehee polynomials as follows:

$$\int_{\mathbb{Z}_p} (1-4z)^{\frac{\xi+\varpi}{2}} d\mu(\varpi) = \frac{\frac{1}{2}\log(1-4z)}{\sqrt{1-4z}-1}(1-4z)^{\frac{\xi}{2}} = \sum_{m=0}^{\infty} d_m(\xi)z^m. \tag{10}$$

Setting $\xi = 0$, we obtain $d_m(0) := d_m$, termed the Catalan–Daehee numbers (cf. [1,4,10,11,14,19]).

For $m \geq 0$, the Stirling numbers of the first kind (cf. [1,4,10,11,14,19]) are defined by

$$\frac{1}{r!}(\log(1+z))^r = \sum_{m=r}^{\infty} S_1(m,r)\frac{z^m}{m!}, \quad (r \geq 0). \tag{11}$$

From (11), it is easy to see that

$$(\xi)_m = \sum_{l=0}^{m} S_1(m,l)\xi^l, \tag{12}$$

where $(\xi)_0 = 1$, and $(\xi)_m = \xi(\xi-1)\cdots(\xi-m+1), (m \geq 1)$.

For $m \geq 0$, the Stirling numbers of the second kind (cf. [1,4,10,11,14,16]) are defined by

$$\frac{1}{r!}(e^z - 1)^r = \sum_{m=r}^{\infty} S_2(m,r)\frac{z^m}{m!}. \tag{13}$$

From (13), we see that (see [8–15,19,20])

$$\xi^m = \sum_{l=0}^{m} S_2(m,l)(\xi)_l. \tag{14}$$

In combinatorics, the Stirling numbers of the first kind count the numbers of permutations in accordance with their number of permutations cycles. The Stirling partition number (or known as Stirling numbers of the second kind) arise in the theory of combinatorics, particularly in the research of partitions and these integers are the numbers of ways to partition a set of n elements into k non-empty subsets.

2. Explicit Expressions for Degenerate Catalan–Daehee Numbers and Polynomials

In this part, we consider and investigate degenerate forms of the Catalan–Daehee polynomials and numbers derived from the bosonic p-adic integral on $\mathbb{Z}_p$. We start with the following definition.

Definition 1. *Let $\lambda, z \in \mathbb{C}_p$ with $|\lambda z| < p^{-\frac{1}{p-1}}$. Now, we define the degenerate Catalan–Daehee numbers by means of the following Volkenborn integral*

$$\int_{\mathbb{Z}_p} (1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} d\mu(\xi) = \frac{\frac{1}{2}\log(1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}})}{\sqrt{1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}} - 1}} = \sum_{n=0}^{\infty} d_{n,\lambda} z^n. \tag{15}$$

Note that

$$\lim_{\lambda \to 0} d_{m,\lambda} = d_m, (m \geq 0).$$

Now we provide some formulas and relations for the numbers $d_{m,\lambda}$ by the following theorems.

Theorem 1. *Let $0 \leq n$. The following relation is valid:*

$$d_{n,\lambda} = \begin{cases} 1, & \text{if } n = 0 \\ \frac{(-4)^n}{n!}D_n(\lambda/4) - \sum_{m=0}^{n-1}\frac{(-4)^{n-m-1}}{(n-m-1)!}D_{n-m-1}(\lambda/4)C_{m,\lambda}, & \text{if } n \geq 1. \end{cases}$$

Proof. From (6), (8) and (15), we have

$$
\begin{aligned}
\sum_{m=0}^{\infty} d_{m,\lambda} z^m &= \frac{1}{2}\left(\frac{\log(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})}{(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})-1}\right)\left(\sqrt{(1-4\log(1+\lambda z)^{\frac{1}{\lambda}}+1)}\right)\\
&= \left(\sum_{l=0}^{\infty}(-4)^l D_l(\lambda/4)\frac{z^l}{l!}\right)\left(1-\sum_{r=0}^{\infty}C_{r,\lambda}z^{r+1}\right)\\
&= \sum_{m=0}^{\infty}(-4)^m\frac{D_m(\lambda/4)}{m!}z^m - \sum_{m=1}^{\infty}\left(\sum_{r=0}^{m-1}\frac{(-4)^{m-r-1}}{(m-r-1)!}D_{m-r-1}(\lambda/4)C_{r,\lambda}\right)z^m\\
&= 1 + \sum_{m=1}^{\infty}\left(\frac{(-4)^m}{m!}D_m(\lambda/4)-\sum_{r=0}^{m-1}\frac{(-4)^{m-r-1}}{(m-r-1)!}D_{m-r-1}(\lambda/4)C_{r,\lambda}\right)z^m.
\end{aligned}
\tag{16}
$$

Therefore, comparing the coefficients on both sides of (16) gives the asserted result. □

Theorem 2. *Let* $0 \leq n$. *The following relation is valid:*

$$
d_{n,\lambda} = \sum_{r=0}^{m} d_r r!\lambda^{m-r}S_1(m,r).
$$

Proof. By (10), (11) and (15), we see that

$$
\begin{aligned}
\int_{\mathbb{Z}_p}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}d\mu(\xi) &= \sum_{r=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{\xi}{2}}{r}d\mu(\xi)r!\frac{(-4\log(1+\lambda z)^{\frac{1}{\lambda}})^r}{r!}\\
&= \sum_{r=0}^{\infty}d_r r!\sum_{m=r}^{\infty}\lambda^{m-r}S_1(m,r)\frac{z^m}{m!}\\
&= \sum_{m=0}^{\infty}\left(\sum_{r=0}^{m}d_r r!\lambda^{m-r}S_1(m,r)\right)\frac{z^m}{m!}.
\end{aligned}
\tag{17}
$$

Hence, by (15) and (17), we obtain the claimed result. □

We provide the following theorem:

Theorem 3. *Let* $0 \leq n$. *The following relation is valid:*

$$
d_n = \sum_{m=0}^{n} d_{m,\lambda}\lambda^{n-m}S_2(n,m)\frac{m!}{n!}.
$$

Proof. Using (10) and (13) and by replacing z by $\frac{1}{\lambda}(e^{\lambda z}-1)$ in (15), we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p}(1-4z)^{\frac{\xi}{2}}d\mu(\xi) &= \sum_{r=0}^{\infty}d_{r,\lambda}r!\frac{(\frac{1}{\lambda}[e^{\lambda z}-1])^r}{r!}\\
&= \sum_{r=0}^{\infty}d_{r,\lambda}\lambda^{-r}r!\sum_{m=r}^{\infty}S_2(m,r)\lambda^m\frac{z^m}{m!}\\
&= \sum_{m=0}^{\infty}\left(\sum_{r=0}^{m}d_{r,\lambda}\lambda^{m-r}r!S_2(m,r)\right)\frac{z^m}{m!}.
\end{aligned}
\tag{18}
$$

Moreover,

$$
\int_{\mathbb{Z}_p}(1-4z)^{\frac{\xi}{2}}d\mu(\xi) = \frac{\frac{1}{2}\log(1-4z)}{\sqrt{1-4z}-1} = \sum_{m=0}^{\infty}d_m z^m.
\tag{19}
$$

Hence, by (18) and (19), we obtain the desired result. □

Theorem 4. *Let $0 \leq n$. The following relation is valid:*

$$d_{n,\lambda} = \sum_{m=0}^{n} d_m (-1)^m 4^m \lambda^{n-m} S_1(n,m) \frac{m!}{n!}.$$

Proof. From (10), (11) and (15), we observe that

$$
\begin{aligned}
\int_{\mathbb{Z}_p} (1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} d\mu(\xi) &= \sum_{r=0}^{\infty} (-1)^r 4^r \int_{\mathbb{Z}_p} \binom{\frac{\xi}{2}}{r} d\mu(\xi) r! \frac{[\log(1+\lambda z)^{\frac{1}{\lambda}}]^r}{r!} \\
&= \sum_{r=0}^{\infty} d_r (-1)^r 4^r \lambda^{m-r} r! \sum_{m=r}^{\infty} S_1(m,r) \frac{z^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} d_r (-1)^r 4^r \lambda^{m-r} r! S_1(m,r) \right) \frac{z^m}{m!}.
\end{aligned}
\tag{20}
$$

Hence, by (15) and (20), we obtain the claimed equality. □

For $\lambda \in \mathbb{Z}_p$, $|z| < p^{-\frac{1}{p-1}}$, λ-Daehee polynomials $D_{m,\lambda}(\xi)$ are defined as follows (see [3]):

$$\int_{\mathbb{Z}_p} (1+z)^{\lambda \varpi + \xi} d\mu(\varpi) = \frac{\lambda \log(1+z)}{(1+z)^{\lambda} - 1} (1+z)^{\xi} = \sum_{m=0}^{\infty} D_{m,\lambda}(\xi) \frac{z^m}{m!}. \tag{21}$$

When $\xi = 0$, $D_{m,\lambda}(0) := D_{m,\lambda}$ are called the λ-Daehee numbers.

Theorem 5. *Let $0 \leq n$. The following relation is valid:*

$$d_{n,\lambda} = \sum_{m=0}^{n} D_{m,\frac{1}{2}} \lambda^{n-m} S_1(n,m) \frac{(-4)^m}{n!}.$$

Proof. Using (11), (15) and (21) and also upon setting $\lambda = \frac{1}{2}$ and $z \to -4\log(1+\lambda z)^{\frac{1}{\lambda}}$ in (21), we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p} (1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} d\mu(\xi) &= \sum_{r=0}^{\infty} D_{r,\frac{1}{2}} \frac{(-4\log(1+\lambda z)^{\frac{1}{\lambda}})^r}{r!} \\
&= \sum_{r=0}^{\infty} D_{r,\frac{1}{2}} (-4)^r \lambda^{-r} \sum_{m=r}^{\infty} S_1(m,r) \frac{\lambda^m z^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} D_{r,\frac{1}{2}} \lambda^{m-r} S_1(m,r) \frac{(-4)^r}{m!} \right) z^m.
\end{aligned}
\tag{22}
$$

Thus, by (15) and (22), we obtain the assertion in the theorem. □

Theorem 6. *Let $0 \leq n$. The following relation is valid:*

$$d_{n,\lambda} = \sum_{m=0}^{n} 2^m (-1)^m B_m \lambda^{n-m} S_1(n,m) \frac{1}{n!}.$$

Proof. Using (4), (11) and (15), we have

$$\frac{\frac{1}{2}\log(1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})}{\sqrt{1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}} - 1}} = \int_{\mathbb{Z}_p} (1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})^{\frac{\varpi}{2}} d\mu(\varpi)$$

$$= \sum_{r=0}^{\infty} 2^{-r} \frac{1}{r!} (-4\log(1 + \lambda z)^{\frac{1}{\lambda}})^r \int_{\mathbb{Z}_p} (\varpi)^r d\mu(\varpi)$$

$$= \sum_{r=0}^{\infty} 2^r (-1)^r B_r \lambda^{-r} \sum_{m=r}^{\infty} S_1(m,r) \frac{\lambda^m z^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} 2^r (-1)^r B_r \lambda^{m-r} S_1(m,r) \frac{1}{m!} \right) z^m.$$

$\square$

A corollary of Theorems 5 and 6 is given by

$$\sum_{m=0}^{n} (-1)^m B_m 2^m \lambda^{n-m} S_1(n,m) = \sum_{m=0}^{n} D_{m,\frac{1}{2}} \lambda^{n-m} S_1(n,m)(-4)^m.$$

Theorem 7. *Let* $0 \leq n$. *The following relation is valid:*

$$d_{n,\lambda} = \sum_{m=0}^{n} d_m \lambda^{n-m} S_1(n,m) \frac{m!}{n!}.$$

Proof. By means of (10), (11) and (15) and also by substituting z by $\log(1 + \lambda z)^{\frac{1}{\lambda}}$ in (21), we obtain

$$\frac{\frac{1}{2}\log(1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})}{\sqrt{1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}} - 1}} = \sum_{r=0}^{\infty} d_r r! \frac{[\log(1 + \lambda z)^{\frac{1}{\lambda}}]^r}{r!}$$

$$= \sum_{r=0}^{\infty} d_r \lambda^{-r} r! \frac{(\log(1 + \lambda z))^r}{r!} \qquad (23)$$

$$= \sum_{r=0}^{\infty} d_r \lambda^{-r} r! \sum_{m=r}^{\infty} S_1(m,r) \frac{\lambda^m z^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} d_r \lambda^{m-r} r! S_1(m,r) \right) \frac{z^m}{m!}.$$

Thus, by (15) and (23), we obtain the claimed equality in the theorem. $\square$

Now, from (11) and (15), we observe that

$$(1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} = \sum_{r=0}^{\infty} \binom{\frac{\xi}{2}}{r} r! (-4)^r \frac{[\log(1 + \lambda z)^{\frac{1}{\lambda}}]^r}{r!}$$

$$= \sum_{r=0}^{\infty} \binom{\frac{\xi}{2}}{r} r! \lambda^{-r} (-4)^r \sum_{m=r}^{\infty} S_1(m,r) \frac{\lambda^m z^m}{m!} \qquad (24)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} \binom{\frac{\xi}{2}}{r} r! \lambda^{m-r} (-4)^r S_1(m,r) \right) \frac{z^m}{m!}.$$

Definition 2. *We define a degenerate form of the Catalan–Daehee polynomials as follows:*

$$\int_{\mathbb{Z}_p} (1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})^{\frac{\xi+\varpi}{2}} d\mu(\varpi) = \frac{\frac{1}{2}\log(1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})}{\sqrt{1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}} - 1}} (1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$$

$$= \sum_{m=0}^{\infty} d_{m,\lambda}(\xi) z^m. \tag{25}$$

Upon setting $\xi = 0$ in (25), we have $d_{m,\lambda}(0) := d_{m,\lambda}$ called the degenerate Catalan–Daehee numbers in (15).

Theorem 8. *Let $0 \le n$. The following relation is valid:*

$$d_{n,\lambda}(\xi) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{\frac{\xi}{2}}{m} (-1)^m 2^{2m} \lambda^{l-m} S_1(l,m) d_{n-l,\lambda} \frac{1}{l!}.$$

Proof. From (11), (15) and (25), we note that

$$\frac{\frac{1}{2}\log(1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})}{\sqrt{1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}} - 1}} (1 - 4\log(1 + \lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$$

$$= \sum_{m=0}^{\infty} d_{m,\lambda} z^m \sum_{r=0}^{\infty} \binom{\frac{\xi}{2}}{r} (-1)^r 2^{2r} \frac{(\log(1 + \lambda z)^{\frac{1}{\lambda}})^r}{r!}$$

$$= \sum_{m=0}^{\infty} d_{m,\lambda} z^m \sum_{r=0}^{\infty} \binom{\frac{\xi}{2}}{r} (-1)^r 2^{2r} \lambda^{-r} \sum_{l=r}^{\infty} S_1(l,r) \frac{\lambda^l z^l}{l!} \tag{26}$$

$$= \sum_{m=0}^{\infty} d_{m,\lambda} z^m \sum_{l=0}^{\infty} \sum_{r=0}^{l} \binom{\frac{\xi}{2}}{r} (-1)^r 2^{2r} \lambda^{l-r} S_1(l,r) \frac{z^l}{l!}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} \sum_{r=0}^{l} \binom{\frac{\xi}{2}}{r} (-1)^r 2^{2r} \lambda^{l-r} S_1(l,r) d_{m-l,\lambda} \frac{1}{l!} \right) z^m.$$

By (25) and (26), we obtain the claimed relation in the theorem. $\quad\square$

Theorem 9. *Let $0 \le n$. The following relation is valid:*

$$d_n(\xi) = \sum_{m=0}^{n} d_{m,\lambda}(\xi) \lambda^{n-m} S_1(n,m) \frac{m!}{n!}.$$

Proof. By means of (10), (11) and also substituting z by $\frac{1}{\lambda}(e^{\lambda z} - 1)$ in (25), we obtain that

$$\sum_{r=0}^{\infty} d_{r,\lambda}(\xi) r! \frac{(\frac{1}{\lambda}e^{\lambda z} - 1)^r}{r!} = \sum_{r=0}^{\infty} d_{r,\lambda}(\xi) r! \lambda^{-r} \sum_{m=r}^{\infty} S_1(m,r) \frac{\lambda^m z^m}{m!}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{m} d_{r,\lambda}(\xi) \lambda^{m-r} S_1(m,r) r! \right) \frac{z^m}{m!},$$

which refers to the asserted result in the theorem. $\quad\square$

We give the following theorem:

Theorem 10. *Let $0 \le n$. The following relation is valid:*

$$B_n(\xi) = (-1)^n \sum_{m=0}^{n} d_{m,\lambda}(\xi) m! \lambda^{n-m} m! 2^n S_2(n,m).$$

Proof. From (4) and (13), and also substituting z by $\frac{e^{-\frac{\lambda z}{4}}-1}{\lambda}$ in (25), we obtain

$$\int_{\mathbb{Z}_p}(1+z)^{\frac{\xi+\varpi}{2}}d\mu(\varpi) = \sum_{m=0}^{\infty}2^{-m}\int_{\mathbb{Z}_p}(\xi+\varpi)^m d\mu(\varpi)\frac{z^m}{m!}. \tag{27}$$

On the other hand, we have

$$\sum_{r=0}^{\infty}d_{r,\lambda}(\xi)r!\frac{(\frac{e^{-\frac{\lambda z}{4}}-1}{\lambda})^r}{r!} = \sum_{r=0}^{\infty}d_{r,\lambda}(\xi)r!\lambda^{-r}\sum_{m=r}^{\infty}S_2(m,r)\frac{(-4)^m\lambda^m z^m}{m!}$$

$$= \sum_{m=0}^{\infty}\sum_{r=0}^{m}d_{r,\lambda}(\xi)r!\lambda^{m-r}(-1)^m 4^m S_2(m,r)\frac{z^m}{m!}. \tag{28}$$

Hence, by (27) and (28), we obtain the desired result in the theorem. $\square$

Theorem 11. *Let $0 \leq n$. The following relation is valid:*

$$d_{n,\lambda}(\xi) = \sum_{m=0}^{n}2^m(-1)^m B_m(\xi)\lambda^{n-m}S_1(n,m)\frac{1}{n!}.$$

Proof. From (4), (11) and (25), we have

$$\frac{\frac{1}{2}\log(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})}{\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}-1}}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}} = \int_{\mathbb{Z}_p}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi+\varpi}{2}}d\mu(\varpi)$$

$$= \sum_{r=0}^{\infty}2^{-r}\frac{1}{r!}(-4\log(1+\lambda z)^{\frac{1}{\lambda}})^r\int_{\mathbb{Z}_p}(\xi+\varpi)^r d\mu(\varpi) \tag{29}$$

$$= \sum_{r=0}^{\infty}2^r(-1)^r B_r(\xi)\lambda^{-r}\sum_{m=r}^{\infty}S_1(m,r)\frac{\lambda^m z^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left(\sum_{r=0}^{m}2^r(-1)^r B_r(\xi)\lambda^{m-r}S_1(m,r)\frac{1}{m!}\right)z^m.$$

Thus, by (25) and (29), we prove the claimed equality in the theorem. $\square$

Theorem 12. *Let $0 \leq n$. The following relation is valid:*

$$d_{n,\lambda}(\xi) = \sum_{l=0}^{n}D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{n-l}S_1(n,l)\frac{1}{n!}$$

$$-\sum_{l=0}^{n-m-1}\sum_{m=1}^{n}D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{n-m-1-l}S_1(n-m-1,l)C_{m,\lambda}\frac{1}{(n-m-1)!}.$$

Proof. From (6), (8), (11) and (25), we have

$$\sum_{m=0}^{\infty} d_{m,\lambda}(\xi) z^m = \frac{\frac{1}{2}\log(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})}{\sqrt{1-4\log(1+\lambda z)^{\frac{1}{\lambda}}-1}}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}$$

$$= \frac{\log(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})}{-4\log(1+\lambda z)^{\frac{1}{\lambda}}}(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})^{\frac{\xi}{2}}\frac{1}{2}\left(\sqrt{(1-4\log(1+\lambda z)^{\frac{1}{\lambda}})}+1\right)$$

$$= \left(\sum_{m=0}^{\infty} D_m\left(\frac{\xi}{2}\right)\frac{(-4\log(1+\lambda z)^{\frac{1}{\lambda}})^m}{m!}\right)\left(1-\sum_{r=0}^{\infty} C_{r,\lambda} z^{r+1}\right)$$

$$= \left(\sum_{l=0}^{\infty} D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{-l}\sum_{m=l}^{\infty} S_1(m,l)\frac{\lambda^m z^m}{m!}\right)\left(1-\sum_{r=0}^{\infty} C_{r,\lambda} z^{r+1}\right)$$

$$= \left(\sum_{m=0}^{\infty}\sum_{l=0}^{m} D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{m-l}S_1(m,l)\frac{z^m}{m!}\right)\left(1-\sum_{r=0}^{\infty} C_{r,\lambda} z^{r+1}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{m} D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{m-l}S_1(m,l)\frac{z^m}{m!}$$

$$-\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m} D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{m-l}S_1(m,l)\frac{z^m}{m!}\right)\left(\sum_{r=0}^{\infty} C_{r,\lambda} z^{r+1}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{m} D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{m-l}S_1(m,l)\frac{z^m}{m!}$$

$$-\sum_{m=0}^{\infty}\sum_{l=0}^{m-r-1}\sum_{r=1}^{m} D_l\left(\frac{\xi}{2}\right)(-4)^l\lambda^{m-r-1-l}S_1(m-r-1,l)C_{r,\lambda}\frac{z^m}{(m-r-1)!}. \tag{30}$$

Therefore, by (30), we obtain the claimed equality. □

Now, we investigate some summation and symmetric formulas for the degenerate Catalan–Daehee polynomials.

We note that the following series manipulation formula hold (cf. [20]):

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{31}$$

We give the following theorem.

Theorem 13. *The following summation formula*

$$d_{k+l,\lambda}(\xi) = \sum_{n,m=0}^{k,l}\sum_{r=0}^{n+m}\binom{k}{n}\binom{l}{m}\binom{(\xi-\omega)/2}{r}$$

$$x\frac{(k+l-n-m)!}{(k+l)!}(-4)^r r!\lambda^{n+m-r}S_1(n+m,r)\,d_{k+l-n-m,\lambda}(\omega)$$

holds.

Proof. Upon setting z by $z+u$ in (25), we investigate

$$\frac{\frac{1}{2}\log(1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})}{\sqrt{(1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})-1}} = (1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})^{-\frac{\xi}{2}}\sum_{k,l=0}^{\infty} d_{k+l,\lambda}(\xi)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!}$$

Again replacing ξ by ω in the last equation, and utilizing (11) and (31), we obtain

$$\frac{\frac{1}{2}\log(1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})}{\sqrt{(1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})-1}} = (1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})^{-\frac{\omega}{2}}\sum_{k,l=0}^{\infty}d_{k+l,\lambda}(\omega)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!}$$

By the last two equations, we obtain

$$\sum_{k,l=0}^{\infty}d_{k+l,\lambda}(\xi)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!} = (1-4\log(1+\lambda(z+u))^{\frac{1}{\lambda}})^{\frac{\xi-\omega}{2}}\sum_{k,l=0}^{\infty}d_{k+l,\lambda}(\omega)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!}$$

which means

$$\sum_{k,l=0}^{\infty}d_{k+l,\lambda}(\xi)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!} = \sum_{n,m=0}^{\infty}\sum_{r=0}^{n+m}\binom{(\xi-\omega)/2}{r}$$

$$\text{x}(-4)^r r!\lambda^{n+m-r}S_1(n+m,r)\frac{z^n}{n!}\frac{u^m}{m!}\sum_{k,l=0}^{\infty}d_{k+l,\lambda}(\omega)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!}.$$

Hence, we obtain

$$\sum_{k,l=0}^{\infty}d_{k+l,\lambda}(\xi)(k+l)!\frac{z^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l}\sum_{r=0}^{n+m}\binom{(\xi-\omega)/2}{r}$$

$$\text{x}(-4)^r r!\lambda^{n+m-r}S_1(n+m,r)\frac{d_{k+l-n-m,\lambda}(\omega)(k+l-n-m)!}{n!m!(k-l)!(l-m)!}z^k u^l,$$

which implies the asserted result. □

Corollary 1. *Letting $k=0$ in the results of Theorem 13, the following implicit summation formula holds:*

$$d_{l,\lambda}(\xi) = \sum_{m=0}^{l}\sum_{r=0}^{m}\binom{l}{m}\binom{(\xi-\omega)/2}{r}\frac{(l-m)!}{(l)!}(-4)^r r!\lambda^{m-r}S_1(m,r)\,d_{l-m,\lambda}(\omega).$$

Corollary 2. *Upon setting $k=0$ and replacing ξ by $\xi+\omega$ in (33) in the results of Theorem 13, we obtain*

$$d_{l,\lambda}(\xi+\omega) = \sum_{m=0}^{l}\sum_{r=0}^{m}\binom{l}{m}\binom{\xi/2}{r}\frac{(l-m)!}{(l)!}(-4)^r r!\lambda^{m-r}S_1(m,r)\,d_{l-m,\lambda}(\omega).$$

Now, we give the following theorem.

Theorem 14. *The following symmetric identity*

$$\sum_{k=0}^{n}d_{n-k,\lambda}(ab\xi)d_{k,\lambda}(ba\xi)a^{n-k}b^k = \sum_{k=0}^{n}d_{n-k,\lambda}(ba\xi)d_{k,\lambda}(ab\xi)b^{n-k}a^k \tag{32}$$

holds for $a,b\in\mathbb{R}$ and $n\geq0$.

Proof. Let

$$Y = \frac{1}{4}\frac{\log(1-4\log(1+\lambda az)^{\frac{1}{\lambda}})\log(1-4\log(1+\lambda bz)^{\frac{1}{\lambda}})}{\left(\sqrt{(1-4\log(1+\lambda az)^{\frac{1}{\lambda}})-1}\right)\left(\sqrt{(1-4\log(1+\lambda bz)^{\frac{1}{\lambda}})-1}\right)}$$

$$\times(1-4\log(1+\lambda az)^{\frac{1}{\lambda}})^{\frac{ab\xi}{2}}(1-4\log(1+\lambda bz)^{\frac{1}{\lambda}})^{\frac{ab\xi}{2}}.$$

Then, by (25), the expression for Y is symmetric in a and b, and we derive the following two expansions of Y:

$$
\begin{aligned}
Y &= \sum_{m=0}^{\infty} d_{m,\lambda}(ab\xi) a^m z^m \sum_{m=0}^{\infty} d_{m,\lambda}(ab\xi) b^m z^m \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} d_{n-k,\lambda}(ab\xi) d_{k,\lambda}(ab\xi) a^{n-k} b^k z^n
\end{aligned}
$$

and similarly

$$
Y = \sum_{n=0}^{\infty} \sum_{k=0}^{n} d_{n-k,\lambda}(ba\xi) d_{k,\lambda}(ba\xi) b^{n-k} a^k z^n,
$$

which gives the desired result (32). $\square$

3. Further Remarks

Before concluding this study, let us add more comments to better frame the derived results. With the help of the idea of Kim et al. [10] on the Catalan–Daehee polynomials, the degenerate Catalan–Daehee numbers and polynomials can be extended as follows:

$$
\begin{aligned}
&\frac{\frac{1}{2}\log(1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}})}{\sqrt{1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}} - 1}} \left(1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}}\right)^{\frac{\xi}{2}} \\
&= \sum_{r=0}^{\infty} B_r(\xi) \left(\frac{1}{2}\right)^r \frac{1}{r!} \left(\log(1 - 4\log(1+\lambda z)^{\frac{1}{\lambda}})\right)^r \\
&= \sum_{r=0}^{\infty} B_r(\xi) \left(\frac{1}{2}\right)^r \sum_{u=r}^{\infty} S_1(u,r)(-4)^u \frac{1}{u!} \left(\log(1+\lambda z)^{\frac{1}{\lambda}}\right)^u \\
&= \sum_{r=0}^{\infty} B_r(\xi) \left(\frac{1}{2}\right)^r \sum_{u=r}^{\infty} S_1(u,r)(-4)^u \lambda^{m-u} \sum_{m=u}^{\infty} S_1(m,u) \frac{z^m}{m!} \\
&= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{u} \sum_{u=0}^{m} B_r(\xi)(-1)^u 2^{2u-r} S_1(u,r) S_1(m,u) \right) \frac{z^m}{m!}.
\end{aligned}
\tag{33}
$$

By (25) and (33), we obtain the following relations

$$
d_{m,\lambda}(\xi) = \sum_{r=0}^{u} \sum_{u=0}^{m} B_{r,\lambda}(\xi)(-1)^u 2^{2u-r} S_1(u,r) S_1(m,u) \frac{(-1)^u}{m!}, \ (m \geq 0)
$$

and

$$
d_{m,\lambda} = \sum_{r=0}^{u} \sum_{u=0}^{m} B_r(-1)^u 2^{2u-r} S_1(u,r) S_1(m,u) \frac{(-1)^u}{m!}.
$$

From (25), we note that

$$
\sum_{r=0}^{\infty} d_{r,\lambda}(\xi) r! \frac{\left(\frac{1}{4\lambda}\left(e^{\lambda(1-e^{2z})} - 1\right)\right)^r}{r!} = \sum_{m=0}^{\infty} B_m(\xi) \frac{z^m}{m!}
$$

and also

$$
\sum_{r=0}^{\infty} d_{r,\lambda}(\xi) r! \frac{\left(\frac{1}{\lambda}\left(e^{\frac{\lambda}{4}(1-e^{2z})} - 1\right)\right)^r}{r!} = \sum_{m=0}^{\infty} B_m(\xi) \frac{z^m}{m!}.
\tag{34}
$$

The left-hand side of (34) is given by

$$\sum_{r=0}^{\infty} d_{r,\lambda}(\xi)r! \frac{\left(\frac{1}{\lambda}\left(e^{\frac{\lambda}{4}(1-e^{2z})}-1\right)\right)^{r}}{r!} = \sum_{r=0}^{\infty} d_{r,\lambda}(\xi)r! \sum_{u=r}^{\infty} S_{2}(u,r)\frac{(e^{2z}-1)^{u}}{u!}2^{-2u}\lambda^{u-r}(-1)^{u}$$

$$= \sum_{r=0}^{\infty} d_{r,\lambda}(\xi)r! \sum_{u=r}^{\infty} S_{2}(u,r) \sum_{m=u}^{\infty} S_{2}(m,u)\frac{2^{m-2u}z^{m}}{m!}\lambda^{u-r}(-1)^{u} \tag{35}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{r=0}^{u}\sum_{u=0}^{m} 2^{m-2u}\lambda^{u-r}d_{r,\lambda}(\xi)r!(-1)^{u}S_{2}(m,u)S_{2}(u,r)\right)\frac{z^{m}}{m!}.$$

Therefore, by (34) and (35), we obtain the following relations

$$B_{m}(\xi) = \sum_{r=0}^{u}\sum_{u=0}^{m} 2^{m-2u}\lambda^{u-r}d_{r,\lambda}(\xi)r!(-1)^{u}S_{2}(m,u)S_{2}(u,r), (m \geq 0)$$

and

$$B_{m} = \sum_{r=0}^{u}\sum_{u=0}^{m} 2^{m-2u}\lambda^{u-r}d_{r,\lambda}r!(-1)^{u}S_{2}(m,u)S_{2}(u,r).$$

The last results provide the relations including the classical Bernoulli polynomials and numbers, Stirling numbers of the second kind and the degenerate Catalan–Daehee polynomials and numbers.

4. Conclusions

In the present work, we first introduced the degenerate forms of the Catalan–Daehee numbers and polynomials by using the bosonic p-adic integral on $\mathbb{Z}_p$ and have obtained diverse explicit expressions and formulas. From Theorem 1 to Theorem 7, we investigated many relations for the degenerate Catalan–Daehee numbers associated with the Daehee polynomials, Catalan numbers, Stirling numbers of the first and second kinds, Bernoulli numbers, Catalan–Daehee numbers and λ-Daehee numbers. From Theorem 8 to Theorem 12, we derived some relations for the degenerate Catalan–Daehee polynomials associated with the degenerate Catalan–Daehee polynomials, the usual Catalan–Daehee numbers, Bernoulli polynomials, Stirling numbers of the first and second kinds, Daehee polynomials and Catalan numbers. Theorem 13 includes an implicit summation formula for the degenerate Catalan–Daehee polynomials. Moreover, Theorem 14 covers a symmetric relation for the degenerate Catalan–Daehee polynomials. Furthermore, we provided some relations including the classical Bernoulli polynomials and numbers, Stirling numbers of the second kind and the degenerate Catalan–Daehee polynomials and numbers in the Further Remarks.

It can be considered that it is not only the idea of this paper that can utilize similar polynomials, but these polynomials also have possible applications in other scientific areas besides the investigations at the end of the paper. In addition, by improving the aim of this paper, we proceed with this idea within our next research studies in several directions.

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