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# Applications of Symmetric Quantum Calculus to the Class of Harmonic Functions

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**Abstract:** In the past few years, many scholars gave much attention to the use of $q$-calculus in geometric functions theory, and they defined new subclasses of analytic and harmonic functions. While using the symmetric $q$-calculus in geometric function theory, very little work has been published so far. In this research, with the help of fundamental concepts of symmetric $q$-calculus and the symmetric $q$-Salagean differential operator for harmonic functions, we define a new class of harmonic functions connected with Janowski functions $\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$. First, we illustrate the necessary and sufficient convolution condition for $\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$ and then prove that this sufficient condition is a sense preserving and univalent, and it is necessary for its subclass $\widetilde{\mathcal{TS}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$. Furthermore, by using this necessary and sufficient coefficient condition, we establish some novel results, particularly convexity, compactness, radii of $q$-starlike and $q$-convex functions of order $\alpha$, and extreme points for this newly defined class of harmonic functions. Our results are the generalizations of some previous known results.

**Keywords:** analytic functions; symmetric $q$-calculus; symmetric $q$-derivative operator; harmonic functions; Janowski functions; symmetric Salagean $q$-differential operator

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## 1. Introduction and Definitions

A continuous function $f = u + iv$ is harmonic in a domain $D \subseteq \mathbb{C}$ if $u$ and $v$ are real valued harmonic functions in $D$. In any simply connected subdomain of $D$, we can express $f = h + \overline{g}$, where $h$ is analytic and $g$ is co-analytic part of $f$ in $D$.

The Jacobian of $f = u + iv$ is given by

$$\mathcal{J}_f(z) = u_x v_y - v_x u_y,$$

and it can be written as:

$$\mathcal{J}_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = \left|h'(z)\right|^2 - \left|g'(z)\right|^2, \ \ z \in D.$$

If $f$ is analytic in $D$, then

$$f_{\overline{z}}(z) = 0 \text{ and } f_z(z) = f'(z).$$

The harmonic mapping $f$ is locally univalent (see [1]) at a point $z_0$ in domain $D$ if and only if

$$\mathcal{J}_f(z) \neq 0.$$

If $\mathcal{J}_f(z) > 0$ [2], then harmonic function $f = h + \overline{g}$ is sense preserving in $D$, or equivalently, $h^{'}(z) \neq 0$ and the dilatation

$$u(z) = \frac{g^{'}}{h^{'}},$$

are analytic and satisfy $|u(z)| < 1$, in $D$.

By demanding the harmonic function to be sense preserving, we can use some basic properties presented for analytic functions in [3].

The family of functions of the form $f = h + \overline{g}$ which are harmonic, normalized univalent for the conditions

$$h(0) = 0 = g(0) \text{ and } h^{'}(0) = 1$$

and also $f = h + \overline{g}$ sense preserving in

$$U = \{z : |z| < 1\}$$

is denoted by $\mathcal{H}$ and has a series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}, \ \ (z \in U), \tag{1}$$

where $h$ and $g$ are analytic functions in the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ g(z) = \sum_{n=1}^{\infty} b_n z^n, \ |b_1| < 1, \ \ (z \in U). \tag{2}$$

Let $\mathcal{H}^0$ denote the subclass of functions $f = h + \overline{g} \in \mathcal{H}$, if an analytic function $g(z)$ satisfies the additional condition $g^{'}(0) = 0$. The class of all univalent, sense-preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}^0$ is denoted by $\mathcal{S}_{\mathcal{H}}$. Moreover, if the co-analytic part of $g$ is zero, then the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of univalent functions. The class of functions $\mathcal{S}_{\mathcal{H}}$ defined by Clunie and Sheil-Small and investigated subfamilies of starlike and convex harmonic functions in $U$ (see [4,5]) is as follows:

$$\mathcal{S}^*_{\mathcal{H}} = \left\{ f \in \mathcal{S}_{\mathcal{H}} : \frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+z}{1-z}, \ \ (z \in U) \right\},$$

and

$$\mathcal{S}^c_{\mathcal{H}} = \{ f \in \mathcal{S}_{\mathcal{H}} : \mathcal{D}_{\mathcal{H}} f(z) \in \mathcal{S}^*_{\mathcal{H}} \ \ (z \in U) \},$$

where

$$\mathcal{D}_{\mathcal{H}} f(z) = z h^{'}(z) - \overline{z g^{'}(z)}. \tag{3}$$

Dziok [6] defined starlike harmonic functions $\mathcal{S}^*_{\mathcal{H}}(\mathcal{A}, \mathcal{B})$ in the domain of Janowski harmonic functions as follows:

$$\mathcal{S}^*_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) = \left\{ f \in \mathcal{S}_{\mathcal{H}} : \frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} \prec \frac{1+\mathcal{A}z}{1+\mathcal{B}z}, \ \ (z \in U) \right\},$$

where $\mathcal{D}_{\mathcal{H}} f(z)$ is given in (3). We can seen that

$$\mathcal{S}^*_{\mathcal{H}}(1, -1) = \mathcal{S}^*_{\mathcal{H}}.$$

Let $f = h + \overline{g}$ and $f_1 = h_1 + \overline{g_1}$ be the harmonic functions, and their convolution can be defined as:

$$(f * f_1)(z) = (h * h_1)(z) + \overline{(g * g_1)(z)}.$$

The subordination of two functions $h$ and $g$ is ( $(h(z) \prec g(z))$, $z \in U$, if there exists a complex-valued function $l$ which maps $U$ into itself such that $l(0) = 0$ and $h(z) = g(l(z))$. In particular, if $g$ is univalent in $U$, then we have

$$h(z) \prec g(z),\ z \in U \Leftrightarrow h(0) = g(0) \text{ and } h(U) \subset g(U).$$

The calculus without limit is known as the quantum (or $q$-) calculus, and due to its important applications, it has been used in various areas of science, such as mathematics and physics. The significance of the $q$-derivative operator $(\partial_q)$ is moderately apparent due to its applications in the analysis of various subclasses of analytic functions. Firstly, Jackson [7] discussed the applications of the $q$-calculus by introducing $q$-derivative and $q$-integral operators. At the end of nineteen century, the $q$-deformation of the class of starlike functions was presented by Ismail et al. [8]. In 1989, Srivastava [9] used $q$-derivative $(\partial_q)$ systematically in the context of geometric function theory (GFT). After that, a number of researchers got motivation from the aforementioned works [7–9] and gave their findings to a GFT of complex analysis. For instance, Kanas and Raducanu [10] introduced the $q$-Ruscheweyh differential operator and discussed its important properties in GFT, and Srivastava and Bansal [11] defined a new class of close-to-convexity for certain Mittag–Leffer type functions. Zang et al. [12] provided the generalization of the conic domain with the help of the basic (or $q$-) calculus operator theory along with a definition of subordination, and then discussed some of its applications for a subclass of $q$-starlike functions. Furthermore, in [13], Mohammed and Darus examined the geometric properties of the $q$-operator to some subclasses of analytic functions in $U$. Raza et al. [14] published a paper in which they defined a new subclass of analytic functions associated with a $q$-derivative operator and investigated coefficient estimates. Recently, Khan et al. [15] evaluated inclusion relations of the $q$-Bessel functions, and in [16] they investigated the $q$-analogues of a Ruscheweyh-type operator and explored coefficient estimates, closure theorems, and extreme points for the functions belonging to this new class. Furthermore, the applications of the operators of the $q$-calculus and the fractional $q$-calculus in GFT were systematically given in a survey-cum-expository review article by Srivastava [17]. In addition, numerous authors have examined various applications of $q$-derivative operators upon the several new subclasses of $q$-starlike functions in open unit disks (see, for example, [18–22]).

The symmetric $q$-calculus has been indicated to be significant in various areas, such as fractional calculus and quantum mechanics [23,24]. In 2016, Sun et al. established the ideas of the fractional $q$-symmetric integrals and $q$-symmetric derivatives and then investigated some of their properties. Additionally, they used fractional difference operators and $q$-symmetric fractional integrals and studied boundary value problems with non-local boundary conditions. Kanas et al. [25] considered a symmetric $q$-derivative $(\widetilde{\partial}_q)$ operator and formulated a new subclass of analytic functions in open unit disk $U$, and examined some of its applications in the conic domain. Recently, Khan et al. [26] utilized the basic ideas of symmetric $q$-calculus and conic regions, and then defined a new version of the generalized symmetric conic domains; in addition, they used it to define a new subclass of $q$-starlike functions in the open unit disk $U$ and established some new results. It was Khan et al. [27] who utilized a $q$-symmetric operator and provided the generalization of the conic domain, and interpreted a subclasses of $q$-starlike and $q$-convex functions. More recently, Khan et al. [28] defined a symmetric $q$-difference operator for $m$-fold symmetric functions, and by considering this operator, they investigated some useful results for $m$-fold symmetric bi-univalent functions. In paper [29] Khan et al. expanded the idea of a $q$-symmetric derivative operator for multivalent functions and then established some new applications of this operator for multivalent $q$-starlike functions.

Now we mention some concept details and definitions of the symmetric $q$-difference calculus which will be used in this manuscript. We presume throughout this paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\}, \quad (\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}), \ -1 \leq \mathcal{A} < \mathcal{B} \leq 1.$$

The symmetric $q$-number for $n \in \mathbb{N}$ can be defined as:

$$\widetilde{[n]}_q = \frac{q^{-n} - q^n}{q^{-1} - q} \tag{4}$$

and for $n = 0$, then we have $\widetilde{[n]}_q = 0$.

The symmetric $q$-number shift factorial be defined by:

$$\widetilde{[n]}_q! = \widetilde{[n]}_q \widetilde{[n-1]}_q \widetilde{[n-2]}_q ... \widetilde{[2]}_q \widetilde{[1]}_q, \quad n \geq 1,$$

and for $n = 0$, then

$$\widetilde{[n]}_q! = 1$$

and for $q \to 1-,$ then

$$\widetilde{[n]}_q! = n!.$$

**Definition 1** ([30])**.** *The symmetric q-derivative (q-difference) operator $\widetilde{\partial}_q h(z)$ for the analytic function is defined by*

$$\begin{aligned} \widetilde{\partial}_q h(z) &= \frac{1}{z}\left(\frac{h(qz) - h(q^{-1}z)}{q - q^{-1}}\right), \ z \in U, \\ &= 1 + \sum_{n=1}^{\infty} \widetilde{[n]}_q a_n z^{n-1}, \ (z \neq 0, \ q \neq 1), \end{aligned} \tag{5}$$

*and*

$$\widetilde{\partial}_q z^n = \widetilde{[n]}_q z^{n-1}, \ \ \widetilde{\partial}_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} \widetilde{[n]}_q a_n z^{n-1}.$$

*We can observe that*

$$\lim_{q \to 1-} \widetilde{\partial}_q h(z) = h'(z).$$

The following applications of the symmetric $q$-derivative ($q$-difference) operator defined in (5) lead to symmetric Salagean $q$-differential operator, which is defined as:

**Definition 2** ([31])**.** *For the positive integer $m$, the symmetric Salagean $q$-differential operator for analytic function $h$ is defined by*

$$\begin{aligned} \widetilde{D}_q^0 h(z) &= h(z), \ \widetilde{D}_q^1 h(z) = z\widetilde{\partial}_q h(z) = \frac{h(qz) - h(q^{-1}z)}{q - q^{-1}}, ..., \\ \widetilde{D}_q^m h(z) &= \widetilde{D}_q\left(\widetilde{D}_q^{m-1} h(z)\right) \\ &= z + \sum_{n=2}^{\infty} \widetilde{[n]}_q^m a_n z^n. \end{aligned}$$

*We observe that*

$$\widetilde{D}_q^m h(z) = h(z) * \widetilde{D}_q\left(\frac{z}{1-z}\right) \tag{6}$$

*and*

$$\begin{aligned} \widetilde{D}_q\left(\frac{z}{1-z}\right) &= z + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^n \\ &= \frac{z}{(1 - q^{-1}z)(1 - qz)}. \end{aligned} \tag{7}$$

*It can be seen that*

$$\lim_{q\to 1-} \widetilde{D}_q^m h(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n,$$

*which is the famous Salagean operator defined in [32].*

**Definition 3** ([31]). *For the positive integer $m$, the symmetric Salagean $q$-differential operator for harmonic function $f = h + \overline{g}$ can be defined as:*

$$\widetilde{D}_q^m f(z) = \widetilde{D}_q^m h(z) + (-1)^m \widetilde{D}_q^m \overline{g(z)}, \tag{8}$$

*where*

$$\begin{aligned} \widetilde{D}_q^m h(z) &= z + \sum_{n=2}^{\infty} \widetilde{[n]}_q^m a_n z^n, \\ \widetilde{D}_q^m g(z) &= \sum_{n=1}^{\infty} \widetilde{[n]}_q^m b_n z^n. \end{aligned}$$

**Remark 1.** *For $q \to 1-$, the operator $\widetilde{D}_q^m$ reduces to the operator $D^m$ which is the modified Salagean operator for the harmonic function $f = h + \overline{g}$ investigated in [33].*

In the article [34], Jahangiri first applied $q$-calculus operator theory and defined a Salagean $q$-differential operator for the harmonic function. Furthermore, Arif et al. [35] defined harmonic $q$-starlike functions associated with symmetrical points and Janowski functions. Srivastava et al. [36] used the fundamental concepts of $q$-calculus operator theory and defined a new class of $k$-symmetric harmonic functions. Recently, Zhang et al. [31] used symmetric $q$-calculus operator theory and defined a symmetric Salagean $q$-differential operator for analytic functions and for complex harmonic functions, and then investigated some useful properties of this operator.

In this paper we use the concepts of symmetric $q$-calculus theory and define a new subclass of harmonic functions and will establish some novel results, and these results are the generalizations of some existence results.

By taking the motivation from the recent published paper of Zhang et al. [31], we define a new subclass $\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m, q, \mathcal{A}, \mathcal{B})$ of harmonic functions $f \in \mathcal{H}^0$ in the domain of Janowski functions, along with a symmetric $q$-Salagean differential operator $\widetilde{D}_q^m$.

**Definition 4.** *Let $\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m, q, \mathcal{A}, \mathcal{B})$ be the class of harmonic functions $f \in \mathcal{H}^0$ which satisfy the condition*

$$\frac{\widetilde{D}_q^{m+1} f(z)}{\widetilde{D}_q^m f(z)} \prec \frac{1 + \mathcal{A}z}{1 + \mathcal{B}z}, \ (q \in (0,1), \ -1 \leq \mathcal{A} < \mathcal{B} \leq 1, \ z \in U). \tag{9}$$

*Inequality (9) is equivalent to the condition*

$$\left| \frac{\widetilde{D}_q^{m+1} f(z) - \widetilde{D}_q^m f(z)}{\mathcal{B}\widetilde{D}_q^{m+1} f(z) - \mathcal{A}\widetilde{D}_q^m f(z)} \right| < 1. \tag{10}$$

**Definition 5.** *We denote by $\widetilde{\mathcal{TS}_{\mathcal{H}}^0}(m, q, \mathcal{A}, \mathcal{B})$ a subclass of harmonic functions $f = h + \overline{g} \in \widetilde{\mathcal{S}_{\mathcal{H}}^0}(m, q, \mathcal{A}, \mathcal{B})$, where*

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \ \ g(z) = (-1)^m \sum_{n=2}^{\infty} |b_n| z^n, \ z \in U. \tag{11}$$

Clearly, the function $f = h + \overline{g} \in \widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$ satisfies the condition

$$\left| \frac{\widetilde{D}_q^{m+1} f(z)}{\widetilde{D}_q^{m} f(z)} - \frac{1 - \mathcal{A}\mathcal{B}}{1 - \mathcal{B}^2} \right| < \frac{\mathcal{B} - \mathcal{A}}{1 - \mathcal{B}^2}, \text{ if } \mathcal{B} \neq 1,$$

and

$$Re\left( \frac{\widetilde{D}_q^{m+1} f(z)}{\widetilde{D}_q^{m} f(z)} \right) > \frac{1 + \mathcal{A}}{2}, \text{ if } \mathcal{B} = 1.$$

In particular, if we take $\mathcal{B} = q$, then for the same $q$, the class $\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$ may equivalently be defined by

$$\left| \frac{\widetilde{D}_q^{m+1} f(z)}{\widetilde{D}_q^{m} f(z)} - \frac{1 - \mathcal{A}q}{1 - q^2} \right| < \frac{q - \mathcal{A}}{1 - q^2}, \ (-q \leq \mathcal{A} < q, \ z \in U).$$

**Remark 2.** *For $q \to 1-$, then $\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B}) = \mathcal{H}^{\lambda}(A, \mathcal{B})$ as defined by Dziok et al. in [37].*

**Remark 3.** *For $\lambda = 0$, then this class $\mathcal{H}^{\lambda}(\mathcal{A}, \mathcal{B})$, as studied in [6], and for $\lambda = 1$, then this class $\mathcal{H}^{\lambda}(\mathcal{A}, \mathcal{B})$, as studied in [38,39].*

**Remark 4.** *The class $\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, (1+q)\alpha - 1, q)$, $(0 \leq \alpha < 1)$ is denoted by $\widetilde{\mathcal{H}}_q^{m}(\alpha)$, and for $m = 0$ and $m = 1$; then $\widetilde{\mathcal{H}}_q^{m}(\alpha) = \widetilde{\mathcal{H}}_q^{0}(\alpha)$ and $\widetilde{\mathcal{H}}_q^{m}(\alpha) = \widetilde{\mathcal{H}}_q^{1}(\alpha)$ are the $q$-analogues of harmonic starlike and harmonic convex functions of order $\alpha$, respectively.*

**Remark 5.** *Further, as $q \to 1-$, then $\widetilde{\mathcal{H}}_q^{0}(\alpha) = \mathcal{S}_{\mathcal{H}}^{*}(\alpha)$ and $\widetilde{\mathcal{H}}_q^{1}(\alpha) = \mathcal{S}_{\mathcal{H}}^{C}(\alpha)$ are the well-known harmonic starlike and harmonic convex functions of order $\alpha$, which was examined by Jahangiri [40].*

**Definition 6.** *For the functions $f \in \mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$ such that*

$$\frac{f(rz)}{r} \in \widetilde{\mathcal{H}}_q^{*}(\alpha), \ r \in (0, 1)$$

*is called the radius of $q$-starlikeness of order $\alpha$ and is denoted by*

$$r\widetilde{\mathcal{H}}_q^{*}(\alpha)\left(\mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})\right).$$

In this study, we define a new class $\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$ of harmonic functions $f \in \mathcal{H}^0$, related with symmetric Salagean $q$-differential operator. First of all, in Theorem 1, we prove the necessary and sufficient convolution condition. In Theorem 2, we obtain that this sufficient coefficient condition for $f \in \mathcal{H}^0$ is sense preserving and univalent in the same class. Next, in Theorem 3, we prove that this coefficient condition is necessary for the functions in its subclass $\widetilde{\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$. Furthermore, by using this necessary and sufficient coefficient condition, we also investigate some novel results, particularly, convexity, compactness, radii of $q$-starlike and $q$-convex functions of order $\alpha$, and extreme points for the functions in the class $\widetilde{\mathcal{T}\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$.

## 2. Main Results

**Theorem 1.** *Let $f \in \mathcal{H}^0$. Then, the function $f \in \widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$ if and only if*

$$\widetilde{D}_q^{m} f(z) * \phi(z, \gamma) \neq 0, \ \ (\zeta \in \mathbb{C}, |\zeta| = 1, \ z \in U \backslash \{0\}),$$

*where*

$$\phi(z,\zeta) = \frac{(\mathcal{B}-\mathcal{A})\zeta z + (1+\mathcal{A}\zeta)qz^2}{(1-q^{-1}z)(1-qz)} - \overline{\left(\frac{2z + (\mathcal{A}+\mathcal{B})\overline{\zeta}z - (1+\mathcal{A}\overline{\zeta})qz^2}{(1-q^{-1}z)(1-qz)}\right)}. \tag{12}$$

**Proof.** Let $f = h + \overline{g} \in \mathcal{H}^0$ of the form (1). Then, the function $f \in \widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$ if and only if inequality (9) holds, or equivalently

$$\frac{\widetilde{D}_q^{m+1} f(z)}{\widetilde{D}_q^m f(z)} \neq \frac{1+\mathcal{A}\zeta}{1+\mathcal{B}\zeta}, \quad (\zeta \in \mathbb{C},\ |\zeta| = 1,\ z \in U \backslash \{0\})$$

which by (8) is given by

$$\begin{aligned} & (1+\mathcal{B}\zeta)\left[\widetilde{D}_q^m\left(\widetilde{D}_q h(z)\right) + (-1)^{m+1}\overline{\widetilde{D}_q^m\left(\widetilde{D}_q g(z)\right)}\right] \\ & -(1+\mathcal{A}\zeta)\left[\widetilde{D}_q^m h(z) + (-1)^m \overline{\widetilde{D}_q^m g(z)}\right] \\ \neq\ & 0. \end{aligned} \tag{13}$$

We use (6) and (7), so condition (13) can be given as:

$$\begin{aligned} & \widetilde{D}_q^m h(z) * \left[(1+\mathcal{B}\zeta)\frac{z}{(1-q^{-1}z)(1-qz)} - (1+\mathcal{A}\zeta)\frac{z}{1-z}\right] \\ & -(-1)^m \overline{\widetilde{D}_q^m g(z)} * \left[(1+\mathcal{B}\zeta)\frac{\overline{z}}{(1-q^{-1}\overline{z})(1-q\overline{z})} + (1+\mathcal{A}\zeta)\frac{\overline{z}}{(1-\overline{z})}\right] \\ \neq\ & 0. \end{aligned}$$

By using the convolution between two harmonic functions, we obtain

$$\widetilde{D}_q^m f(z) * \phi(z,\zeta) \neq 0,$$

where the harmonic function $\phi(z,\zeta)$ is given by (12). □

If we consider $q \to 1-$ in Theorem 1, we get the following result involving the Salagean operator $\widetilde{D}^m$.

**Corollary 1.** *Let $f \in \mathcal{H}^0$ and function $f \in \widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,\mathcal{A},\mathcal{B})$ if and only if*

$$\widetilde{D}^m f(z) * \phi(z,\gamma) \neq 0, \quad (\zeta \in \mathbb{C}, |\zeta| = 1,\ z \in U \backslash \{0\}),$$

*where*

$$\phi(z,\zeta) = \frac{(\mathcal{B}-\mathcal{A})\zeta z + (1+\mathcal{A}\zeta)z^2}{(1-z)^2} - \overline{\left(\frac{2z + (\mathcal{A}+\mathcal{B})\overline{\zeta}z - (1+\mathcal{A}\overline{\zeta})z^2}{(1-z)^2}\right)}.$$

**Remark 6.** *The result of Corollary 1 with $\phi(z,\gamma)$ given by (12) improves the results of (Dziok et al. [37], Theorem 1, p. 3).*

**Theorem 2.** *Let $f = h + \overline{g} \in \mathcal{H}^0$ of the form (1) and $q \in (0,1)$, $-1 \leq \mathcal{A} < \mathcal{B} \leq 1$. If*

$$\sum_{n=2}^{\infty} L_n|a_n| + M_n|b_n| \leq \mathcal{B} - \mathcal{A}, \tag{14}$$

*where*

$$L_n = \widetilde{[n]}_q^m \left\{ \widetilde{[n]}_q (1+\mathcal{B}) - (1+\mathcal{A}) \right\}, \tag{15}$$

$$M_n = \widetilde{[n]}_q^m \left\{ \widetilde{[n]}_q (1+\mathcal{B}) + (1+\mathcal{A}) \right\}, \tag{16}$$

*and* $\widetilde{[n]}_q$ *is given by (4), then*

*(i) for* $q \to 1-$, *the function* $f$ *is locally univalent and sense-preserving in* $U$.

*(ii) and* $f \in \widetilde{\mathcal{S}_{\mathcal{H}}^0}(m, q, \mathcal{A}, \mathcal{B})$.

*Equality occurs for the function*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\mathcal{B}-\mathcal{A}}{L_n} \gamma_n z^n + \sum_{n=2}^{\infty} \frac{\mathcal{B}-\mathcal{A}}{M_n} \overline{\beta_n z^n}$$

*and*

$$\sum_{n=2}^{\infty} (|\gamma_n| + |\beta_n|) = 1.$$

**Proof.** It is obvious that for part (i), theorem is true for

$$f(z) = z.$$

Let $f = h + \overline{g}$ and

$$a_n \neq 0 \text{ or } b_n \neq 0 \text{ for } n \geq 2.$$

Since $\widetilde{[n]}_q > 1$, we identify from (15) and (16) that

$$L_n \geq M_n > \widetilde{[n]}_q (\mathcal{B} - \mathcal{A}),$$

by which Condition (14) indicates the condition

$$\sum_{n=2}^{\infty} \widetilde{[n]}_q (|a_n| + |b_n|) < 1, \tag{17}$$

and

$$\begin{aligned} \left| \partial_q h(z) - \partial_q g(z) \right| &\geq 1 - \sum_{n=2}^{\infty} \widetilde{[n]}_q |a_n| |z|^{n-1} - \sum_{n=2}^{\infty} \widetilde{[n]}_q |b_n| |z|^{n-1} \\ &> 1 - |z| \sum_{n=2}^{\infty} \widetilde{[n]}_q (|a_n| + |a_n|) \geq 1 - |z| > 0 \end{aligned}$$

in $U$, and thus as $q \to 1-$, $\left| h'(z) \right| > \left| g'(z) \right|$ in $U$. Hence, part (i) is complete. Moreover, if $z_1, z_2 \in U$ and for some $q$ $(0 < q < 1)$, $q^{-1} z_1 \neq q z_2$. Then, for that $q$,

$$\begin{aligned} \left| \frac{(q^{-1} z_1)^n - (q z_2)^n}{q^{-1} z_1 - q z_2} \right| &= \left| \sum_{t=1}^{n} \left( q^{-1} z_1 \right)^{t-1} (q z_2)^{n-t} \right| \\ &\leq \sum_{t=1}^{n} \left| q^{-1} \right|^{t-1} |z_1|^{t-1} q^{n-t} |z_2|^{n-t} \\ &< \widetilde{[n]}_q, \text{ for } (n = 2, 3, \ldots). \end{aligned}$$

Hence, for that value of $q$, from (17), we have

$$\begin{aligned}
&\left|f(qz_1)-f(q^{-1}z_2)\right| \\
\geq\ &\left|qz_1-q^{-1}z_2-\sum_{n=2}^{\infty}a_n\left((qz_1)^n-\left(q^{-1}z_2\right)^n\right)\right| \\
&-\left|\sum_{n=2}^{\infty}b_n\overline{\left((qz_1)^n-(q^{-1}z_2)^n\right)}\right| \\
\geq\ &\left|qz_1-q^{-1}z_2\right|\left(1-\sum_{n=2}^{\infty}|a_n|\left|\frac{(qz_1)^n-(q^{-1}z_2)^n}{qz_1-q^{-1}z_2}\right|-\sum_{n=2}^{\infty}|b_n|\left|\frac{(qz_1)^n-(q^{-1}z_2)^n}{qz_1-q^{-1}z_2}\right|\right) \\
>\ &\left|qz_1-q^{-1}z_2\right|\left(1-\sum_{n=2}^{\infty}\widetilde{[n]}_q|a_n|-\sum_{n=2}^{\infty}\widetilde{[n]}_q|b_n|\right)>0,
\end{aligned}$$

which illustrates that $f$ is univalent in $U$. This confirms the result (i).

To prove that $f\in\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$, we only need to show that $f$ satisfies the condition (10). Consider $|z|=r$, $(0<r<1)$; we can write (10) as:

$$\begin{aligned}
&\left|\widetilde{D}_q^{m+1}f(z)-\widetilde{D}_q^m f(z)\right|-\left|\mathcal{B}\left(\widetilde{D}_q^{m+1}f(z)\right)-\mathcal{A}\widetilde{D}_q^m f(z)\right| \\
=\ &\left|\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-1\right)a_nz^n-(-1)^m\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+1\right)\overline{b_nz^n}\right| \\
&-\left|\begin{array}{c}(\mathcal{B}-\mathcal{A})z+\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\mathcal{B}\left(\widetilde{[n]}_q\right)-\mathcal{A}\right)a_nz^n \\ -(-1)^m\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\mathcal{B}\left(\widetilde{[n]}_q\right)+\mathcal{A}\right)\overline{b_nz^n}\end{array}\right| \\
\leq\ &\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-1\right)a_nr^n+\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+1\right)\overline{b_nr^n} \\
&-(\mathcal{B}-\mathcal{A})r+\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\mathcal{B}\widetilde{[n]}_q-\mathcal{A}\right)a_nr^n \\
&+\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\mathcal{B}\widetilde{[n]}_q+\mathcal{A}\right)\overline{b_nr^n} \\
<\ &\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|)r^n-(\mathcal{B}-\mathcal{A}) \\
\leq\ &\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|)r^n-(\mathcal{B}-\mathcal{A})\leq 0.
\end{aligned}$$

This is the case if condition (14) holds. Hence, condition (10) is proved. □

**Example 1.** *The function* $f=h+\overline{g}$ *given by*

$$f(z)=z+\sum_{n=2}^{\infty}\mathcal{T}_nz^n+\sum_{n=1}^{\infty}\mathcal{R}_n\overline{z}^n,$$

*where*

$$\begin{aligned}
\mathcal{T}_n &= \frac{(2+\delta)(\mathcal{B}-\mathcal{A})\mu_n}{2(n+\delta)(n+1+\delta)\widetilde{[n]}_q^m\left\{\widetilde{[n]}_q(1+\mathcal{B})-(1+\mathcal{A})\right\}},\\
\mathcal{R}_n &= \frac{(1+\delta)(\mathcal{B}-\mathcal{A})\mu_n}{2(n+\delta)(n+1+\delta)\widetilde{[n]}_q^m\left\{\widetilde{[n]}_q(1+\mathcal{B})+(1+\mathcal{A})\right\}},
\end{aligned}$$

*belonging to the class* $\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$*, for* $\delta > -2$*,* $\mu_n \in \mathbb{C}$*,* $|\mu_n| = 1$*. This is the case, because know that*

$$\begin{aligned}
&\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left\{\widetilde{[n]}_q(1+\mathcal{B})-(1+\mathcal{A})\right\}|\mathcal{T}_n| \\
&+\sum_{n=1}^{\infty}\widetilde{[n]}_q^m\left\{\widetilde{[n]}_q(1+\mathcal{B})+(1+\mathcal{A})\right\}|\mathcal{R}_n| \\
\leq\ & \sum_{n=2}^{\infty}\frac{(2+\delta)(\mathcal{B}-\mathcal{A})}{2(n+\delta)(n+1+\delta)}+\sum_{n=1}^{\infty}\frac{(1+\delta)(\mathcal{B}-\mathcal{A})}{2(n+\delta)(n+1+\delta)} \\
=\ & \frac{(2+\delta)(\mathcal{B}-\mathcal{A})}{2}\sum_{n=2}^{\infty}\frac{1}{(n+\delta)(n+1+\delta)} \\
&+\frac{(1+\delta)(\mathcal{B}-\mathcal{A})}{2}\sum_{n=1}^{\infty}\frac{1}{(n+\delta)(n+1+\delta)}, \\
=\ & \frac{(2+\delta)(\mathcal{B}-\mathcal{A})}{2}\sum_{n=2}^{\infty}\left(\frac{1}{(n+\delta)}-\frac{1}{(n+1+\delta)}\right) \\
&+\frac{(1+\delta)(\mathcal{B}-\mathcal{A})}{2}\sum_{n=1}^{\infty}\left(\frac{1}{(n+\delta)}-\frac{1}{(n+1+\delta)}\right), \\
=\ & \mathcal{B}-\mathcal{A}.
\end{aligned}$$

**Theorem 3.** *Let* $f = h + \overline{g} \in \mathcal{H}^0$ *and* $f \in \mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$ *if and only if Condition (14) holds that is*

$$\sum_{n=2}^{\infty} L_n|a_n| + M_n|b_n| \leq \mathcal{B}-\mathcal{A}, \tag{18}$$

*where* $L_n$ *and* $M_n$ *are defined by (15) and (16).*

**Proof.** If part of Theorem 2 is proved, and only if, we let $f \in \mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$. Then by condition (9), we have from (10) that for any $z \in U$.

$$\left|\frac{\Upsilon(\mathcal{B},\mathcal{A})a_n z^n+\Phi(\mathcal{B},\mathcal{A})\overline{b_n z^n}}{(\mathcal{B}-\mathcal{A})z-C(\mathcal{B},\mathcal{A})|a_n|z^n-D(\mathcal{B},\mathcal{A})\overline{|b_n|z^n}}\right|<1,$$

where

$$\begin{aligned}
\Upsilon(\mathcal{B},\mathcal{A}) &= \sum_{n=2}^{\infty}\left(\Psi_n^q\right)^m\left(\Psi_n^q-1\right),\\
\Phi(\mathcal{B},\mathcal{A}) &= \sum_{n=2}^{\infty}\left(\Psi_n^q\right)^m\left(\Psi_n^q+1\right),
\end{aligned}$$

$$\begin{aligned} C(\mathcal{B},\mathcal{A}) &= \sum_{n=2}^{\infty}\left(\Psi_n^q\right)^m\left(\mathcal{B}\left(\Psi_n^q\right)-\mathcal{A}\right), \\ D(\mathcal{B},\mathcal{A}) &= \sum_{n=2}^{\infty}\left(\Psi_n^q\right)^m\left(\mathcal{B}\left(\Psi_n^q\right)+\mathcal{A}\right). \end{aligned}$$

For $(0 \le r < 1)$, and $z = r$, we obtain

$$\frac{\Upsilon(\mathcal{B},\mathcal{A})|a_n|r^{n-1}+\Phi(\mathcal{B},\mathcal{A})\overline{|b_n|r^{n-1}}}{(\mathcal{B}-\mathcal{A})-C(\mathcal{B},\mathcal{A})|a_n|r^{n-1}-D(\mathcal{B},\mathcal{A})\overline{|b_n|r^{n-1}}} < 1,$$

which illustrate that

$$\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|)r^{n-1} < \mathcal{B}-\mathcal{A}. \tag{19}$$

Let $\sigma_n$ represent the sequence of partial sums of the series

$$\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|).$$

Then, $\sigma_n$ is a non-decreasing sequence, and by (19) it is bounded above. Thus, it is convergent for $r \to 1^-$ and

$$\sum_{n=2}^{\infty}(L_n|a_n|+M_n|b_n|) = \lim_{n\to\infty}\sigma_n \le \mathcal{B}-\mathcal{A}.$$

This gives condition (14). □

**Remark 7.** *For $q \to 1-$, the result of Theorem 3 coincides with the result given in [37].*

Taking $\mathcal{B} = q$ and $\mathcal{A} = (1+q)\alpha - 1$ $(0 \le \alpha < 1)$ in Theorem 3, we attain Corollary 2.

**Corollary 2.** *Let $f = h + \overline{g} \in \mathcal{H}^0$ and $f \in \mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$ if and only if condition (14) holds; that is,*

$$\sum_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\left(\widetilde{[n]}_q-\alpha\right)|a_n|+\left(\widetilde{[n]}_q+\alpha\right)|b_n|\right) \le (1-\alpha). \tag{20}$$

**Remark 8.** *If we take $m = 0$ and $m = 1$ in (20), then Corollary 2 provides a necessary and sufficient condition for $f = h + \overline{g} \in \mathcal{H}^0$, and it is given by*

$$\sum_{n=2}^{\infty}\left\{\left(\widetilde{[n]}_q-\alpha\right)|a_n|+\left(\widetilde{[n]}_q+\alpha\right)\right\}|b_n| \le 1-\alpha, \tag{21}$$

$$\sum_{n=2}^{\infty}\widetilde{[n]}_q\left\{\left(\widetilde{[n]}_q-\alpha\right)|a_n|+\left(\widetilde{[n]}_q+\alpha\right)\right\}|b_n| \le 1-\alpha. \tag{22}$$

**Theorem 4.** *The class $\mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$ is a convex and compact subclass of $f = h + \overline{g} \in \mathcal{H}^0$, where $h$ and $g$ are given by (11).*

**Proof.** Let for $j = 1,2$, $f_j \in \mathcal{T}\widetilde{\mathcal{S}_{\mathcal{H}}^0}(m,q,\mathcal{A},\mathcal{B})$, and let for this $m$ it be of the form

$$f_j(z) = z - \sum_{n=2}^{\infty}|a_{j,n}|z^n + (-1)^m\sum_{n=2}^{\infty}|b_{j,n}|\overline{z}^n,\ z \in U. \tag{23}$$

Then, for $0 \leq \rho \leq 1$,

$$\begin{aligned} F(z) &= \rho f_1(z) + (1-\rho) f_2(z) \\ &= z - \sum_{n=2}^{\infty} (\rho |a_{1,n}| + (1-\rho)|a_{2,n}|) z^n \\ &\quad + (-1)^m \sum_{n=2}^{\infty} (\rho |b_{1,n}| + (1-\rho)|b_{2,n}|) \bar{z}^n. \end{aligned}$$

By Theorem 3, we attain

$$\begin{aligned} & \sum_{n=2}^{\infty} \{ L_n (\rho |a_{1,n}| + (1-\rho)|a_{2,n}|) + M_n (\rho |b_{1,n}| + (1-\rho)|b_{2,n}|) \} \\ = \; & \rho \sum_{n=2}^{\infty} \{ L_n |a_{1,n}| + M_n |b_{1,n}| \} + (1-\rho) \sum_{n=2}^{\infty} \{ L_n |a_{2,n}| + M_n |b_{2,n}| \} \\ \leq \; & \rho(\mathcal{B} - \mathcal{A}) + (1-\rho)(\mathcal{B} - \mathcal{A}) = \mathcal{B} - \mathcal{A}. \end{aligned}$$

Therefore, $F \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$. Hence, $\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$ is convex.

On the other hand, if we assume $f_j \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B}), j \in N = \{1, 2, 3 \ldots\}$, then by Theorem 3, we obtain

$$\sum_{n=2}^{\infty} (L_n |a_{j,n}| + M_n |b_{j,n}|) \leq \mathcal{B} - \mathcal{A}. \tag{24}$$

Hence for $|z| \leq r$ $(0 < r < 1)$

$$\begin{aligned} |f_j(z)| &\leq r + \sum_{n=2}^{\infty} (|a_{j,n}| + |b_{j,n}|) r^n \leq \mathcal{B} - \mathcal{A} \\ &\leq r + \frac{\sum_{n=2}^{\infty} (L_n |a_{j,n}| + M_n |b_{j,n}|) r^n}{\left(\widetilde{[2]}_q\right)^m \left\{ \widetilde{[2]}_q (1 + \mathcal{B}) - (1 + \mathcal{A}) \right\}} \\ &< r + \frac{\mathcal{B} - \mathcal{A}}{\left(\widetilde{[2]}_q\right)^m \left\{ \widetilde{[2]}_q (1 + \mathcal{B}) - (1 + \mathcal{A}) \right\}} r^2. \end{aligned}$$

Similarly, we get for $|z| \leq r$, and $(0 < r < 1)$,

$$|f_j(z)| > r - \frac{\mathcal{B} - \mathcal{A}}{\left(\widetilde{[2]}_q\right)^m \left\{ \widetilde{[2]}_q (1 + \mathcal{B}) - (1 + \mathcal{A}) \right\}} r^2.$$

Therefore, class $\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$ is locally uniformly bounded.

If we assume that $f_j \to f$, then we conclude that $|a_{j,n}| \to |a_n|$ and $|b_{j,n}| \to |b_n|$ as $j \to \infty$ for any $n = 2, 3 \ldots$. Hence, from (24), we get

$$\sum_{n=2}^{\infty} (L_n |a_n| + M_n |b_n|) \leq \mathcal{B} - \mathcal{A},$$

which illustrates that $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$. Thus, the class $\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$ is closed. This proves that class $\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$ is compact.

**Corollary 3.** *Let $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$. Then, for $|z| = r$ $(r < 1)$,*

$$r - \frac{\mathcal{B} - \mathcal{A}}{\Theta_1(m, q, \mathcal{A}, \mathcal{B})} r^2 < |f(z)| < r + \frac{\mathcal{B} - \mathcal{A}}{\Theta_1(m, q, \mathcal{A}, \mathcal{B})} r^2.$$

*Furthermore,*

$$\left\{w \in \mathbb{C} : |w| < 1 - \frac{\mathcal{B} - \mathcal{A}}{\Theta_1(m, q, \mathcal{A}, \mathcal{B})}\right\} \subset f(U),$$

*where*

$$\Theta_1(m, q, \mathcal{A}, \mathcal{B}) = \left(\widetilde{[2]}_q\right)^m \left\{\widetilde{[2]}_q(1 + \mathcal{B}) - (1 + \mathcal{A})\right\}.$$

□

In Theorem 5, we find the radius of the $q$-starlikeness of order $\alpha$ for $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$.

**Theorem 5.** *Let $0 \leq \alpha < 1$ and $L_n$, $M_n$ be defined by (15), and (16). Then,*

$$r\widetilde{\mathcal{H}}^*_q(\alpha)\left(\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})\right) = \inf_{n \geq 2}\left[\left(\frac{1 - \alpha}{\mathcal{B} - \mathcal{A}}\right)\left(\min\left\{\frac{L_n}{\widetilde{[n]}_q - \alpha}, \frac{M_n}{\widetilde{[n]}_q + \alpha}\right\}\right)\right]^{\frac{1}{n-1}}, \quad (25)$$

*where $\widetilde{[n]}_q$ defined by (4).*

**Proof.** Let $f = h + \overline{g} \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$; then, by Theorem 3, we have

$$\sum_{n=2}^{\infty} L_n|a_n| + M_n|b_n| \leq \mathcal{B} - \mathcal{A},$$

where $L_n$ and $M_n$ are defined in (15) and (16). Let $r_0$ be the radius of $q$-starlikeness of order $\alpha$. Then, $\frac{f(r_0 z)}{r_0} \in \widetilde{\mathcal{H}}^*_q(\alpha)$ if and only if from (21),

$$\sum_{n=2}^{\infty}\left\{\left\{\left(\widetilde{[n]} - \alpha\right)|a_n| + \left(\widetilde{[n]} + \alpha\right)\right\}|b_n|\right\} r_0^{k-1} \leq 1 - \alpha,$$

which is true if

$$\frac{\widetilde{[n]} - \alpha}{1 - \alpha} r_0^{k-1} \leq \frac{L_n}{\mathcal{B} - \mathcal{A}}, \quad n = 2, 3 \ldots,$$

and

$$\frac{\widetilde{[n]} + \alpha}{1 - \alpha} r_0^{k-1} \leq \frac{M_n}{\mathcal{B} - \mathcal{A}}, \quad n = 2, 3 \ldots,$$

or if

$$r_0 \leq \left[\frac{1 - \alpha}{\mathcal{B} - \mathcal{A}} \min\left\{\frac{L_n}{\widetilde{[n]} - \alpha}, \frac{M_n}{\widetilde{[n]} + \alpha}\right\}\right]^{\frac{1}{n-1}}.$$

It follows that the radius $r\widetilde{\mathcal{H}}^*_q(\alpha)\left(\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})\right)$ is given in (25). □

Similarly, we can find the radius of $q$-convexity of order $\alpha$ for $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})$.

**Theorem 6.** *Let $0 \leq \alpha < 1$, and $L_n$, $M_n$ be defined by (15) and (16). Then,*

$$r\widetilde{\mathcal{H}}^c_q(\alpha)\left(\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m, q, \mathcal{A}, \mathcal{B})\right)$$
$$= \inf_{n \geq 2}\left[\left(\frac{1 - \alpha}{(\mathcal{B} - \mathcal{A})\widetilde{[n]}_q}\right) \min\left\{\frac{L_n}{\widetilde{[n]}_q - \alpha}, \frac{M_n}{\widetilde{[n]}_q + \alpha}\right\}\right]^{\frac{1}{n-1}},$$

*where $\widetilde{[n]}_q$ is given by (4).*

**Theorem 7.** *Let* $f = h + \overline{g} \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$ *be of the form (11) if and only if*

$$f(z) = \sum_{n=1}^{\infty}\{y_n h_n(z) + x_n g_n(z)\}, \tag{26}$$

*where*

$$\begin{aligned} h_1(z) &= z,\ h_n(z) = z - \frac{\mathcal{B}-\mathcal{A}}{L_n}z^n,\ g_1(z) = z, \\ g_n(z) &= z - \frac{\mathcal{B}-\mathcal{A}}{M_n}\overline{z}^n,\ \text{for } n = 2,3,\ldots, \\ \text{and } x_n, y_n &\geq 0,\ y_1 = 1 - \sum_{n=2}^{\infty} y_n - \sum_{n=2}^{\infty} x_n. \end{aligned} \tag{27}$$

**Proof.** Let $f$ be given in (26); then from (27), and of the form

$$f(z) = z - \sum_{n=2}^{\infty} y_n\left(\frac{\mathcal{B}-\mathcal{A}}{L_n}\right)z^n + (-1)^m \sum_{n=2}^{\infty} x_n\left(\frac{\mathcal{B}-\mathcal{A}}{M_n}\right)\overline{z}^n,$$

which by Theorem 3, we prove that $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$. Since for function $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}$ $(m,q,\mathcal{A},\mathcal{B})$, we have

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(L_n y_n\left(\frac{\mathcal{B}-\mathcal{A}}{L_n}\right) + M_n x_n\left(\frac{\mathcal{B}-\mathcal{A}}{M_n}\right)\right) \\ =\ & (\mathcal{B}-\mathcal{A})\sum_{n=2}^{\infty}\{y_n + x_n\} \\ =\ & (\mathcal{B}-\mathcal{A})(1 - y_1 - x_1) \\ \leq\ & \mathcal{B}-\mathcal{A}. \end{aligned}$$

Conversely, let $f = h + \overline{g} \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$ and set

$$y_n = \frac{L_n}{\mathcal{B}-\mathcal{A}}|a_n|,\ x_n = \frac{M_n}{\mathcal{B}-\mathcal{A}}|b_n|.$$

Then, using (27), we obtain

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty}|a_n|z^n + (-1)^m\sum_{n=2}^{\infty}|b_n|\overline{z}^n \\ &= z - \sum_{n=2}^{\infty} y_n\left(\frac{\mathcal{B}-\mathcal{A}}{L_n}\right)z^n + (-1)^m\sum_{n=2}^{\infty} x_n\left(\frac{\mathcal{B}-\mathcal{A}}{M_n}\right)\overline{z}^n \\ &= z - \sum_{n=2}^{\infty} y_n(z - h_n(z)) + \sum_{n=2}^{\infty} x_n(g_n(z) - z) \\ &= \left\{1 - \sum_{n=2}^{\infty}(y_n + x_n)\right\}z + \sum_{n=2}^{\infty}\{y_n h_n(z) + x_n g_n(z)\}, \end{aligned}$$

which is of the form (26). This confirm the Theorem 7. □

**Remark 9.** *The points* $h_n$ *and* $g_n$ *are the extreme points of* $\mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$.

**Corollary 4.** *Let* $f \in \mathcal{T}\widetilde{\mathcal{S}^0_{\mathcal{H}}}(m,q,\mathcal{A},\mathcal{B})$ *be of the form (11). Then*

$$|a_n| \leq \frac{\mathcal{B}-\mathcal{A}}{L_n} \text{ and } |b_n| \leq \frac{\mathcal{B}-\mathcal{A}}{M_n},\ n = 2,3,4\ldots, \tag{28}$$

*where $L_n$ and $M_n$ are defined by (15) and (16) and the extremal functions $h_n(z)$ and $g_n(z)$ given in (27).*

## 3. Conclusions

Recently, many scholars have used $q$-calculus in geometric functions theory and defined new subclasses of $q$-starlike and convex functions and harmonic functions; see [11,12,14–17,34,35]. In this paper, we used the concept of a symmetric $q$-Salagean differential operator for harmonic functions, and we defined a new class of harmonic functions associated with Janowski functions, $\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$. For this newly defined class, we proved necessary and sufficient condition and established some novel results, such as convexity, compactness of the class $\widetilde{\mathcal{S}_{\mathcal{H}}^{0}}(m, q, \mathcal{A}, \mathcal{B})$, and radii of $q$-starlike and $q$-convex functions of order $\alpha$, along with extreme points. This research will motivate future research in the area of symmetric $q$-calculus operators together with harmonic functions.

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