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Generalizations of Topological Decomposition and Zeno Sequence in Fibered *n*-Spaces

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Abstract: The space-time geometry is rooted in the Minkowski 4-manifold. Minkowski and Euclidean topological 4-manifolds behave differently in view of compactness and local homogeneity. As a result, Zeno sequences are selectively admitted in such topological spaces. In this paper, the generalizations of topologically fibered n-spaces are proposed to formulate topological decomposition and the formation of projective fibered n-subspaces. The concept of quasi-compact fibering is introduced to analyze the formation of Zeno sequences in topological n-spaces (i.e., n-manifolds), where a quasi-compact fiber relaxes the Minkowski-type (algebraically) strict ordering relation under topological projections. The topological analyses of fibered Minkowski as well as Euclidean 4-spaces are presented under quasi-compact fibering and topological projections. The topological n-spaces endowed with quasi-compact fibers facilitated the detection of local as well as global compactness and the non-analytic behavior of a continuous function. It is illustrated that the 5-manifold with boundary embedding Minkowski 4-space transformed a quasi-compact fiber into a compact fiber maintaining generality.

Keywords: topological spaces; Zeno sequence; topological fibering; topological projection; manifolds

MSC: 54A10; 54A20; 54A35; 54F65

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1. Introduction

In general, Minkowski 4-space $(X_M \equiv M^4, \tau_E)$ is often considered a (pseudo) Euclidean topological 4-manifold, not necessarily with any boundary (i.e., the space $X_M \equiv M^4$ is equipped with local Euclidean topology). In the topological 4-manifold (X_M, τ_E), the Euclidean distance metric is computed as $\forall x_a, x_b \in X_M, d(x_a, x_b) = \left(\sum_{i \in [0,3]} (x_{a(i)} - x_{b(i)})^2 \right)^{1/2}$, where the coordinates of any point are represented as $x_a = \left\langle x_{a(0)}, x_{a(1)}, x_{a(2)}, x_{a(3)} \right\rangle$. As a result, one can formulate a metric topology on (X_M, τ_E) based on the neighborhoods of any point $x_a \in X_M$, which is given as $N_{x(a)} = \{x_b : d(x_a, x_b) < \varepsilon\}$. However, in 1967, Zeeman pointed out that the Euclidean topological 4-manifold (X_M, τ_E) is a locally homogeneous space, although Minkowski 4-space X_M is not a locally homogeneous space everywhere [1]. Moreover, Zeeman proposed a new topology (X_M, τ_{Zeeman}), called Zeeman topology, which is a finer topology on X_M. The Zeeman topology effectively decomposed the Minkowski 4-space X_M into two topological subspaces, while allowing the continuity of $f : I \to (X_M, \tau_{Zeeman})$ in the interval $I \equiv [0, 1]$. However, this invites the formation of Zeno sequences in (X_M, τ_{Zeeman}) with respect to (X_M, τ_E) mainly because of the differences in local homogeneities of the corresponding topological spaces [1]. Furthermore, the topological space (X_M, τ_{Zeeman}) is not equipped with a countable neighborhood basis [2]. As an alternative, a new topological space (X_M, τ_{path}) was proposed, which is now called path-topology in the Minkowski 4-space X_M. The topological space (X_M, τ_{path}) preserves causal structures and the homeomorphism of conformal groups. In other words, the topological space (X_M, τ_{path}) is considered to be a connected 4-manifold in R^4, where $R = [-\infty, +\infty]$ is the set of extended real numbers. However, the path-topological space (X_M, τ_{path}) does not retain the property of local compactness [2]. Furthermore, the topological space (X_M, τ_{path}) is not a regular and normal topological space. If an n-manifold is closed, then it is called as an *aspherical n-manifold* if the universal covering of the n-manifold is contractible generating a $K(\pi, 1) - space$ [3]. It helps in determining the homeomorphism between Euclidean space and the covering of the n-manifold through the formation of fundamental subgroups of Abelian variety. In case of $n = 4$, the resulting 4-manifold is the (aspherical) Minkowski space-time, which admits hyperbolic surfaces and the manifold can be fibered [4]. First, we briefly present the concept of the Zeno sequence and the comparative analysis of the diverse properties of two topological spaces under consideration, M^4 and the corresponding Euclidean 4-space E^4. Next, we detail the motivations as well as the contributions made in this paper.

1.1. Concept of Zeno Sequence

The origin of the mathematical concept of the Zeno sequence is rooted in the paradox of Zeno of Elea (sophist philosopher and mathematician, 490–430 BC), which has profound effects on our understandings of time in the space-time geometry of physics, existing quantum mechanical behaviors at sub-atomic scale, and the presence of Zeno effects in the modeling of hybrid systems. The Zeno paradox essentially forces us to rethink the relationship of mathematical continuity (i.e., axioms of real numbers) and the perceived reality in space-time geometry [5,6]. According to Zeno, the infinite space and time are either atomic or infinitely divisible, indicating that a sequence $(-\infty < L_l) < \lim_{x \to +\infty} f(x) < (L_h < +\infty)$ generated by $f : X \to R$ may not have a strong convergence property at its limiting values [6]. We can find the (approximate) resemblance of a similar concept in pure mathematics: that a convergent sequence is essentially a bounded Cauchy sequence, but a Cauchy sequence may not always converge. An alternate (simplistic) view of the Zeno sequence can be presented as an infinitely countable division of a finite interval, where the sequence may or may not converge depending upon the nature of local homogeneity.

1.2. Topological Properties of Minkowski and Euclidean 4-Spaces

Let us consider a Minkowski topological 4-space (i.e., manifold M^4) and the corresponding Euclidean topological space, which is denoted as E^4. The topological behaviors of M^4 and E^4 are very different in view of the convergence of sequences, the compactness of subspaces, and the topological decomposition of respective spaces [1]. Interestingly, the topologically anomalous behaviors of M^4 with respect to E^4 can be observed even if we consider that both spaces are R^4 retaining dimensional-homogeneity, where R is the set of real numbers. We summarize the distinguishing analytical properties of two topologies as follows.

(**Prop. 1**): If (M^4, τ_M) is a Minkowski topological space, then it admits the Zeno sequence. On the contrary, the corresponding Euclidean topological 4-space (E^4, τ_E) does not necessarily admit the Zeno sequence. In other words, if $f : Z \to (E^4, \tau_E)$ generates a sequence (Z is the set of integers) such that $f(Z) \subset E^4$, then it eventually converges within (E^4, τ_E). However, the continuous function $f : Z \to (M^4, \tau_M)$ may not always converge in (M^4, τ_M).

(**Prop. 2**): The topological space (M^4, τ_M) is decomposable into two subspaces, such that $M^4 = (X_t \equiv R) \times (X_s \equiv R^3)$, which is in line with product topology. As a consequence, (M^4, τ_M) becomes a finer topological space compared to (E^4, τ_E), and it results in the formation of Zeeman topological space [1]. Note that (E^4, τ_E) is a locally homogeneous topological space, whereas (M^4, τ_M) is not a completely homogeneous topological space.

(**Prop. 3**): Let us consider that $I = [0, 1]$ is a unit interval in real line and $\pi_t : (M^4, \tau_M) \to X_t$ is a topological projection. If we consider a continuous function $f : I \to (M^4, \tau_M)$ preserving the partial ordering relation in X_t such that $\forall a, b \in I, [a < b] \Rightarrow [(\pi_t \circ f)(a) < (\pi_t \circ f)(b)]$, then $f(I)$ is well-behaved in (M^4, τ_M) by exhibiting analytic behavior and by retaining the compactness of $f(I)$. On the contrary, the continuous function $f : I \rightarrow (E^4, \tau_E)$ may be non-analytic almost everywhere in (E^4, τ_E).

These observations motivate us to investigate further the generalizations of topological decompositions in a fibered n-space and the possibilities of existence of the Zeno sequence in such a space. The motivations and contributions made in this paper are presented in the following sections (Sections 1.3 and 1.4, respectively).

1.3. Motivations

It is evident from the aforesaid observations that (M^4, τ_M) admits some of the topological pathogeneses compared to the topological space (E^4, τ_E). Interestingly, if X is a complete and affine manifold with metric-signature pair $\langle g, (+, +, +, -) \rangle$, then X has 0-curvature and it can cover manifold M^4 [7]. Note that the topological space (M^4, τ_M) retains the Hausdorff separation property admitting Zeno sequences, and it is *not* a T_4 topological space [1]. On the other hand, the topological space (E^4, τ_E) is locally homogeneous, and, as a result, it does not allow the formation of any Zeno sequence. Let us consider a Hausdorff topological n-space given as $(X \equiv R^n, \tau_X)$, which is decomposable as $X = X_m \times X_k$ in view of product topology such that $\dim(X_m) \geq 1$ and $\dim(X_k) \geq 1$. If we fix $\dim(X_m) = 1$ and $\dim(X_k) = 3$, then it is a Minkowski space, which can admit bilinear forms of rank-4, and it generates *space-fiber bundle* over the *time-fiber bundle* [8]. However, there is restriction on the fibrations over n-manifolds. In case of an n-manifold, if it is $n - even$ (i.e., Minkowski 4-space), then it cannot be fibered over S^1 due to the Chern-Gauss-Bonnet theorem [9]. Furthermore, it is known that even in the case of a Euclidean 4-manifold E^4, the manifold E^4 cannot be fibered by a 2-dimensional fiber [10].

Evidently, we cannot easily extend, analyze, or guarantee the aforesaid diverse observations or properties in an n-space without the required topological generalizations. Thus, a motivating question is: Is it possible to generalize the above mentioned diverse topological properties in a decomposable n-space, and how can we analyze anomalous as well as non-anomalous topological behaviors in such n-space considering topologies τ_M and τ_E in the corresponding 4-spaces (i.e., through the reduction of dimensions)? Moreover, some interesting questions are: How can we introduce fibering in such topological n-spaces, and are there any commonalities in topological properties in such a fibered n-space? This paper addresses these questions in view of general as well as geometric topology.

1.4. Contributions

The contributions made in this paper can be summarized as follows. In this paper, we present the generalizations of topological decomposition in a fibered (Hausdorff) n-space, and we analyzed the presence of the Zeno sequence in such a space under topological projections. The strict partial ordering of a continuous function $f : I \rightarrow (M^4, \tau_M)$ is relaxed, allowing the possibility of the formation of a Zeno sequence irrespective of the specific nature of topological space (i.e., either 4-Minkowski or 4-Euclidean under reduced dimensions). This results in the concept of *quasi-compactness* of a topological subspace and the formation of a *quasi-compact fiber* under the topological projections, which enables us to analyze the formation of the Zeno sequence within topological spaces depending upon the varieties of spaces. The multidimensional ($n > 1$) topological space helps to attain generalizations of: (1) fibered topological decomposition in the presence of quasi-compact subspaces, and (2) topological analyses of the analytic behavior of continuous $f : I \rightarrow (X, \tau_X)$, where X is a Hausdorff n-space. We illustrate that the topological concepts of *local homogeneity* and *local compactness* are different as compared to the *quasi-compactness* in a fibered space, which helps in analyzing the natures of compactness, the formation of a Zeno sequence, and the convergence of a sequence in any topologically decomposable multidimensional space. Finally, this paper presents the case studies specifically considering Minkowski 4-space and Euclidean 4-space by applying the proposed topological analysis in reduced dimensions. We show that the quasi-compact fibering of a Minkowski 4-space can retain the strict partial ordering in standard form under projections, and the fibered, as well as decomposed, topological space determines the admissibility of a convergent sequence depending upon the specific variety of a Hausdorff topological space (i.e., either (M^4, τ_M) or (E^4, τ_E)). Moreover, it is shown that the topological compactness of an n-space can be determined locally or globally through the employment of the concept of quasi-compact fibers in the topologically fibered n-space.

The rest of the paper is organized as follows: Section 2 presents the preliminary concepts and relevant classical results. Sections 3 and 4 present the definitions and main results, respectively. Section 5 details the analytical case-studies (applications) of the proposed concepts to Minkowski and Euclidean 4-spaces. Section 6 presents a comprehensive discussion outlining the general applicabilities and distinctions of the proposed concepts. Finally, Section 7 concludes the paper.

2. Preliminary Concepts

Let us consider a topological space of n-manifold denoted by (X, τ_X) such that $\dim(X) = n > 0$. It was shown by Urysohn that a second-countable topological space is metrizable if it is a T_3 topological space. As a consequence, the n-manifold (X, τ_X) is a metrizable space, in general, and the local Euclidean subspaces play an important role in determining the local (sequential) compactness, which is defined as follows [11].

Definition 1. *An n-manifold (X, τ_X) is locally Euclidean if $\forall A \subset X, \exists B \subset R^n$ the continuous function $h : A \to B$ is a homeomorphism.*

The aforesaid definition can be further generalized as: if (X, τ_X) is a connected, compact, and metrizable topological space, then (X, τ_X) is an n-manifold which is locally Euclidean.

Remark 1. *If we consider a decomposable n-manifold (X, τ_X) such that $X = X_1\#X_2$, then the composition is commutative, i.e., $X_2\#X_1 = X_1\#X_2$. Moreover, if it is decomposable as $X = (X_1\#X_2)\#X_3$, then it maintains the associativity as $(X_1\#X_2)\#X_3 = X_1\#(X_2\#X_3)$.*

Recall that the Minkowski 4-manifold is decomposable into two topological subspaces [12]. Note that the decomposition of an n-manifold results in the formation of submanifolds. This invites the notion of topological immersion [13].

Definition 2. *If (X, τ_X) is a decomposable n-manifold such that $X = X_1\#X_2$, then the submanifolds X_1, X_2 are embeddings of the respective manifolds M, N such that $i(M) \cong X_1$ and $i(N) \cong X_2$ under the topological embedding $i : \{M, N\} \to X$.*

The classification of n-manifolds though homeomorphisms is due to Poincaré (i.e., Poincaré conjecture and its generalization for higher dimensions), which is presented in the following theorem [14].

Theorem 1. *If (X, τ_X) is a simply connected and compact topological 3-manifold without boundary, then it is homeomorphic to S^3 (i.e., 3-sphere in R^4), and if it is an n-manifold ($n \geq 5$), then it is $(n-1)/2$-connected for $n -$ odd and $n/2$-connected for $n -$ even.*

Suppose (X, τ_X) is a closed aspherical n-manifold with universal covering generating $K(\pi, 1) - space$ such that $\pi = \pi_1(X)$. The following theorem establishes the interrelationship between the homeomorphism between Euclidean space and the covering of (X, τ_X) in view of a finitely generated fundamental subgroup [3].

Theorem 2. *If S is a universal covering of (X, τ_X), then S is homeomorphic to Euclidean space if $\pi = \pi_1(X)$ contains a finitely generated non-trivial Abelian subgroup.*

Note that the aforesaid property may not be generalized for all dimensions, as it considers that $\dim(X) \neq 3$ and $\dim(X) \neq 4$ without restricting to a base point in π [3]. The topological n-space (X, τ_X) is a T-space if it can be partitioned into $\{C_m : m \in \Lambda\}$, where $m \to +\infty$, preserving certain properties [13]. It is important to note that a topological T-space (X, τ_X) does *not* admit complete topological separation as a necessary condition. A topological space can be discretely fibered through the lifting and covering spaces, which is defined as follows [11,15].

Definition 3. *If the continuous function* $p : A \to X$ *is a covering map of* (X, τ_X)*, then* $\forall x \in X$*,* $p^{-1}(x)$ *is a discrete fiber at* $\{x\}$*. The topological space* (X, τ_X) *is weakly locally contractible if* $\forall x \in X$*,* $\exists N_x \subset X$ *such that* $\gamma(\overline{N_x}) = \{x\}$*, where* $\gamma : (X, \tau_X) \to (X, \tau_X)$ *is a topological contraction.*

The space denoted by (X, A, p) is called a Hurewicz fiber space. If we restrict to the topological space of Euclidean n-manifold E^n, then the following theorem provides needed insight [10].

Theorem 3. *A Euclidean n-manifold* $E^n, n > 1$ *cannot be fibered by any 1-dimensional fiber if the fiber is compact.*

Interestingly, the lifting in fiber space (X, A, p) preserves the topological projections.

3. Generalizations in Topological *n*-Spaces

In this section, we present the generalization of the interplay of the topological decomposition of a fibered space and the admissibility of a Zeno sequence in a fibered n-space. The resulting concept of quasi-compactness of a topological subspace is defined along with the formulation of topological projections in such an n-space. Let the topological n-space (X, τ_X) be a decomposable space such that $\dim(X) = n$ and $X = X_m \times X_k$, where $n = m + k, m < k$. Suppose the functions $\pi_m : (X, \tau_X) \to X_m$ and $\pi_k : (X, \tau_X) \to X_k$ are two topological projections maintaining continuity within the respective subspaces. This results in the possibility of the incorporation of refined partial ordering (i.e., relaxing the strict ordering in (M^4, τ_M)), which is defined as follows.

Definition 4. *Let* (X, τ_X) *be a topological n-space and* $f : I \to (X, \tau_X)$ *be a continuous function such that* $A \subset f(I)$*. The topological projections are defined to maintain refined partial ordering in the subspace* $A_m \subset X_m$ *if the following conditions are preserved.*

$$\begin{aligned} &\pi_m(A) = A_m, \\ &\forall a, b \in I, [a < b] \Rightarrow [(\pi_m \circ f)(a) \leq (\pi_m \circ f)(b)]. \end{aligned} \tag{1}$$

It is important to note that the aforesaid property allows the possibility of non-analytic pathological behavior of $f(I)$ within $A_k \subset X_k$ if (X, τ_X) is a non-compact n-space and $\tau_X \equiv \tau_E$, maintaining the generality. Moreover, we can preserve the total ordering of $f(I)$ in $A_m \subset X_m$ if we consider that $\tau_X \equiv \tau_M$ in a 4-space. If we consider that (X, τ_X) is a topologically fibered n-space, then the analysis of the admissibility of a Zeno sequence and compactness become facilitated. The definition of a topological fiber in an n-space is defined as follows.

Definition 5. *The continuous function* $f : I \to (X, \tau_X)$ *is a topological fiber in an n-space if the following properties are preserved.*

$$\begin{aligned} &\pi_B : X_m \to X_k, \\ &\exists a \in I, \{(\pi_m \circ f)(a)\} \cap X_m \neq \phi, \\ &(\pi_B^{-1} \circ (\pi_k \circ f))(I) \cong \{(\pi_m \circ f)(a)\}. \end{aligned} \tag{2}$$

Note that the fibering point $a \in I$ is considered to be unique while examining the admission of Zeno sequence in (X, τ_X). This results in the concept of the topological fibering of an n-space, which is defined as follows.

Definition 6. *A space $A \subseteq X$ in a topological n-space (X, τ_X) is called as a topologically fibered space if $F_A = \{f_i : I^o \to A, i \in \Lambda\}$ is a set of topological fibers such that $\underset{i \in \Lambda}{\cup} f_i(I) = \overline{A}$.*

Remark 2. *Note that, if a topological subspace $A \subset X$ can be compactly fibered, then $\overline{A} \subset X$ and $A^o \in \tau_X$, preserving the local compactness.*

If we relax the condition for the preservation of local compactness of a topological subspace in a fibered n-space, then it results in the concept of the quasi-compactness of a topological subspace. The definition of a topologically quasi-compact subspace is presented as follows.

Definition 7. *A topological space $A \subseteq X$ in a topological n-space (X, τ_X) is defined to be quasi-compact if there exists a topological fiber $f : I \to A$ maintaining the following conditions.*

$$\begin{aligned} & f(I^o) \subset A, \\ & (\pi_k \circ f)(I) \cong (\pi_B \circ f)(I), \\ & \{(\pi_B \circ (\pi_m \circ f))(\sup(I))\} \cup \{(\pi_B \circ (\pi_m \circ f))(\inf(I))\} \subseteq \{-\infty, +\infty\}. \end{aligned} \tag{3}$$

Remark 3. *Suppose we consider a subspace $A \subset X$ in a topologically fibered n-space. It is important to note that if $A \subset X$ is quasi-compact, then it is possible that $A \neq \overline{A}$ and $\pi_m(A)$ is closed in X_m. In other words, the projective subspace $\pi_k(A)$ is not closed in X_k, admitting the non-analytic behavior of the corresponding fiber in the topological n-space (X, τ_X).*

4. Main Results

This section presents a set of topological properties in generalized forms in a fibered n-space for analyzing Zeno sequences (and convergence of sequences), where specific references to topological 4-spaces are made if it is necessary for maintaining the clarity. In the following theorem, we show that the existence of a quasi-compact fiber affects the existence of the Zeno sequence irrespective of the nature of underlying topological 4-spaces.

Theorem 4. *If $f : Z^+ \to A$ forms a sequence within the subspace $A \subset X$ of a fibered 4-space, then it is not a Zeno sequence in (X, τ_M) and (X, τ_E) if $A \subset X$ does not contain any quasi-compact fiber.*

Proof. We prove the aforesaid theorem by considering two cases representing fibered (X, τ_M) and (X, τ_E), respectively. □

Case-I: Let us first consider the topological 4-space (X, τ_M) endowed with Minkowski topology under fibering. The topological decomposition results in $X = X_m \times X_k$, where $\dim(X_m) = 1$ and $\dim(X_k) = 3$. Suppose $f : Z^+ \to A$ forms a sequence within the fibered subspace $A \subset X$ such that $F_X = \{h_i : I \to X, i \in \Lambda\}$ is a set of topological fibers preserving the relaxed ordering property, given as $\forall a, b \in I, [a < b] \Rightarrow [(\pi_m \circ h_i)(a) \leq (\pi_m \circ h_i)(b)]$. If $A \subset X, A = A^o$ does not contain any quasi-compact fiber, then $\overline{A} = \underset{i \in \Lambda}{\cup} g_i(I)$ such that $\forall i \in \Lambda, g_i(I) \subset h_i(I)$, where $g_i : I \to \overline{A}$. As a result, we can conclude that $\forall i \in \Lambda$, $(\pi_m \circ g_i)(I) \subset (R \equiv X_m)$ is compact. Moreover, the fibered subspace under topological projections maintains the following two conditions, which are given as:

(1) $\underset{i \in \Lambda}{\cup} \{(\pi_B \circ (\pi_m \circ g_i))(\sup(I))\} \subset R \backslash \{-\infty, +\infty\}$ and,

(2) $\underset{i \in \Lambda}{\cup} \{(\pi_B \circ (\pi_m \circ g_i))(\inf(I))\} \subset R \backslash \{-\infty, +\infty\}$.

Thus, we can conclude that $A \subset X$ is locally compact in fibered (X, τ_M) if it does not contain any quasi-compact topological fiber, and, as a result, it cannot admit a non-convergent sequence in $A \subset X$ because $f : Z^+ \to A$ converges in $\overline{A}$.

Case-II: Let us consider the topological 4-space (X, τ_E) endowed with Euclidean topology under fibering $F_X = \{h_i : I \to X, i \in \Lambda\}$. Suppose we consider a topologically fibered subspace $A \subset X, A = A^o$, where $F_A = \{g_i : I^o \to A, i \in \Lambda\}$ is a set of topological fibers such that $\overline{A} = \underset{i\in\Lambda}{\cup} g_i(I)$ and $\forall i \in \Lambda, g_i(I) \subset h_i(I)$. In other words, $A \subset X$ does not contain any quasi-compact fiber. Note that, in the topological space (X, τ_E), we do not need to impose the ordering relation under topological projection. Suppose that we are not decomposing the space X. As a result, if we consider that $A \in \tau_E$, then $A \cup \partial A = \overline{A}$, which is locally compact in (X, τ_E). Hence, the subspace $A \subset X$ within the fibered topological 4-space cannot admit the Zeno sequence because $f : Z^+ \to A$ converges in $\overline{A}$. On the other hand, if we consider topological decomposition as $X = X_m \times X_k$ by following the principles of product topology, then it results in the conclusion that $\pi_m(\overline{A}) \subset X_m$ and $\pi_k(\overline{A}) \subset X_k$ are both locally compact subspaces. Hence, the function $f : Z^+ \to A$ converges in $\overline{A}$ within (X, τ_E), and, as a result, $f : Z^+ \to A$ is not a Zeno sequence in (X, τ_M) and (X, τ_E). □

We can further generalize this observation in a fibered n-space to determine compactness, which is presented in the following theorem.

Theorem 5. *If (X, τ_X) is a topologically fibered n-space with at least one quasi-compact fiber, then (X, τ_X) is not a compact topological space.*

Proof. The proof is relatively straightforward. If we consider a quasi-compact fiber $h_i(I) \in F_X$, then $f : Z^+ \to X$ is not a convergent sequence if $f(Z^+) \subset h_i(I)$ such that either $(\pi_B \circ (\pi_m \circ h_i))(\inf(I)) \in f(Z^+)$, $(\pi_B \circ (\pi_m \circ h_i))(\sup(I)) \in f(Z^+)$, or both properties are preserved within the topologically fibered n-space (X, τ_X). Hence, the topologically fibered n-space (X, τ_X), containing a quasi-compact fiber, is not compact. □

Interestingly, the admission of a non-convergent sequence irrespective of the nature of topological 4-spaces can be examined in a generalized form in the respective fibered 4-spaces, (X, τ_M) and (X, τ_E), which also examines the existence of a Zeno sequence. This observation is presented in the following lemma.

Lemma 1. *If the subspaces $A \subset X$ within the fibered topological 4-spaces (X, τ_M) and (X, τ_E) have at least one quasi-compact fiber, then $A \subset X$ cannot admit Zeno sequence.*

Proof. The proof is a direct consequence of theorem 5, considering (X, τ_M) and (X, τ_E), where $\dim(X) = 4$. If the fibered topological subspaces $A \subset X$ in either (X, τ_M) or (X, τ_E) are selected such that $\exists h_i(I) \in F_X, h_i : I \to A$ is a quasi-compact fiber, then $A \subset X$ is not locally compact in (X, τ_M) and (X, τ_E). As a result, we can formulate a sequence by $f : Z^+ \to A$, such that $f(Z^+) \subset h_i(I)$, where either $(\pi_B \circ (\pi_m \circ h_i))(\inf(I)) \in f(Z^+)$, $(\pi_B \circ (\pi_m \circ h_i))(\sup(I)) \in f(Z^+)$, or both the conditions are preserved. Thus, the function $f : Z^+ \to A$ is not convergent in $\overline{A}$ considering either $A \in \tau_M$ or $A \in \tau_E$; hence, it cannot be a Zeno sequence (note that we have considered $A = A^o$). □

Interestingly, the quasi-compactness of a topological subspace determines the compactness of the topological space globally. This observation is presented in the following corollary.

Corollary 1. *In a fibered topological n-space (X, τ_X), if $A \subset X$ is quasi-compact, then X is not a compact topological n-space.*

Let us take a different view. If we consider the relationship between the topological compactness and quasi-compact fibers in a fibered n-space, then it can be shown that the local as well as the global compactness of a fibered n-space can be determined through the existence of quasi-compact fibers. This observation is presented in the following theorem as a generalization.

Theorem 6. *If every subspace $A \subset X$ of a fibered topological n-space (X, τ_X) do not have any quasi-compact fiber, then (X, τ_X) is compact everywhere.*

Proof. Let us consider a fibered topological n-space (X, τ_X) and an arbitrary subspace $A_u \subset X$ such that $\tau_A = A_u \cap \tau_X$. If $\forall B \in \tau_A, i \in \Lambda, f_i : I^o \to B$ and it is true that $(\pi_m \circ f_i)(I)$ and $(\pi_k \circ f_i)(I)$ are compact, then $\overline{B} = \underset{i\in\Lambda}{\cup} f_i(I)$ is a locally compact topological n-subspace. As a consequence, $\overline{A_u} \subset X$ is also a locally compact n-subspace in (X, τ_X). Furthermore, if $X = \underset{u\in\Lambda}{\cup} A_u$, then $\overline{X}$ is a globally compact topological n-space. Hence, the topologically fibered n-space (X, τ_X) does not contain any quasi-compact fiber preserving local as well as global compactness (i.e., (X, τ_X) is compact everywhere). □

5. Applications to Minkowski and Euclidean 4-Spaces

In this section, we present the applications of the generalizations and the corresponding topological analyses considering Minkowski and Euclidean 4-spaces. The topological analyses examine the cases of admissibility of Zeno sequences and conditions affecting compactness.

5.1. Fibered Minkowski 4-Space

It was mentioned earlier that, in the topologically fibered Minkowski 4-space $(X = M^4, \tau_M)$, a quasi-compact fiber $f : I \to (X, \tau_M)$ admits the Zeno sequence, and the proposed generalizations do not induce any influence preserving strict ordering in $X_t \equiv R$. In other words, the sequence generated by $h : Z^+ \to (X, \tau_M)$ admits the Zeno sequence such that $h(Z^+) \subset f(I)$ and, additionally, $\forall a, b \in I, [a < b] \Rightarrow [(\pi_t \circ h)(a) < (\pi_t \circ h)(b)]$. However, the proposed generalizations of topological decomposition ensure that the subspace $A \subset X$ in $(X = M^4, \tau_M)$ is compact, eliminating the formation of a Zeno sequence (i.e., non-converging sequence) if $A \subset X$ is locally compact without any existence of a quasi-compact fiber $f|_A : I \to (X, \tau_M)$.

5.2. Fibered Euclidean 4-Space

In the case of topologically fibered Euclidean 4-space $(X = E^4, \tau_E)$, the existence of a quasi-compact fiber $f : I \to (X, \tau_E)$ indicates the possibility of admission of a non-converging sequence in $(X = E^4, \tau_E)$ through $h(Z^+) \subset f(I)$. However, the distinction in this case is that the quasi-compact fiber $f : I \to (X, \tau_E)$ does not preserve the ordering property in $X_m \equiv R$ under the topological projection such that $\exists a \in I, [a < b] \Rightarrow [(\pi_m \circ h)(a) = (\pi_m \circ h)(b)]$. Note that it is *not* necessary to impose the restriction that $a = 0$ always. Suppose we consider a quasi-compact fiber $f_E : I \to (X, \tau_E)$ such that $\forall a \in (0, 1), \forall b_i > a$; the topological projection maintains the condition given by: $(\pi_m \circ f_E)(b_i) = (\pi_m \circ f_E)(b_{i+1})$. Evidently, the sequence $h(Z^+) \subset f_E(I)$ is non-convergent in $(X = E^4, \tau_E)$. It is important to note that the function $f_E : I \to (X, \tau_E)$ is admissible in $(X = E^4, \tau_E)$, but $f_{E\equiv M} : I \to (X, \tau_M)$ is *not* admissible in $(X = M^4, \tau_M)$.

Finally, the behavior of a rectifying curve $f : I^o \to (X = M^3, \tau_M)$ in the open interval is an interesting subject with physical interpretations in a Minkowski 3-space [16,17]. The proposed quasi-compact fibering in a Minkowski 4-space may facilitate the extension of related results in higher dimensions through the generalizations.

6. Discussions

The presence of Zeno effects (i.e., effects due to existence of Zeno sequence) can be found in gravitational physics (space-time geometry), quantum mechanical physics (quantum Zeno phase), and in the modeling of hybrid systems [1,18–22]. Interestingly, in the physical world, the existence of spin and the spin-state of an elementary particle (i.e., electron) are due to the Zeeman topological effects under certain conditions [21]. In order to mitigate the topologically inconsistent behaviors of Minkowski space-time 4-manifold due to Zeno effects, the ambient cosmological metric is proposed based on a $(M^4 \times R)$

5-manifold structure [18]. The $(M^4 \times R)$ 5-manifold embeds Minkowski 4-space, admitting the Einstein equation, and it restricts topological degeneracy of (M^4, τ_M) by eliminating singularities. It aims to resolve the formation of Zeno sequence-like non-convergent causal curves by considering a 5-manifold with a boundary. However, the concept proposed in this paper does not require the bounded n-manifold, and it considers the existence of oriented singularities in an n-space to maintain generality. Moreover, if we apply the proposed concept to $(M^4 \times R)$, then we can essentially form a compact fiber out of a quasi-compact fiber due to the existence of bounded Cauchy sequences in $(M^4 \times R)$. Similarly, the concept of quasi-compact fibers can be admitted in a Minkowski 4-manifold for analyzing the local compactness, as explained earlier in this paper. Note that the concept of the quasi-compact fiber does not depend on the nature of homogeneity of topological spaces or its decomposability. The proposed generalizations assist the topological analysis of local compactness without paying any specific attention to the local homogeneity of various topological spaces, including Minkowski and Euclidean spaces.

7. Conclusions

The determination of compactness and the formation of a Zeno sequence in various topological spaces become difficult due to the differences in local homogeneities of the respective topological spaces. The fiberings of different topological n-manifolds have specific requirements that affect the topological analysis to establish common properties and the preservation of distinctions. The generalizations of topological decomposition, algebraic ordering, and projections under fibering facilitate the topological analysis of the local as well as the global compactness of topological n-spaces. The concept of a quasi-compact fiber assists in analyzing the formation of Zeno sequences in various topological spaces. The analytic behavior of a continuous function can be clearly understood for various topological spaces in the presence of fibered topological decompositions and projections. The generalizations can be applied to Minkowski and Euclidean 4-spaces to analyze various distinguishing topological properties, preserving the fundamental results such as the retention of the algebraic total ordering of a continuous function in Minkowski 4-space under decomposition. The dimension reduction does not greatly alter the topological properties of the proposed generalizations. Moreover, the topological 5-manifold with a boundary embedding Minkowski 4-space can admit quasi-compact fiber by transforming it into a compact fiber preserving generality. However, the behavior of the sequence of the continuous causal curves in a quasi-compactly fibered n-space without boundary and the rectifiabilities of such curves need further investigation.

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