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Article

Some New Inverse Hilbert Inequalities on Time Scales

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Abstract: Several inverse integral inequalities were proved in 2004 by Yong. It is our aim in this paper to extend these inequalities to time scales. Furthermore, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Our results are proved using some algebraic inequalities, inverse Hölder's inequality and inverse Jensen's inequality on time scales. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Hilbert's inequality; dynamic inequality; time scale

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1. Introduction

The form of the established classical discrete Hardy–Hilbert double series inequality [1] is given as follows: If $\{a_m\} \geqslant 0$, $\{b_n\} \geqslant 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n} \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} b_m^q\right)^{\frac{1}{q}}, \tag{1}$$

where $p > 1$, $q = p/p-1$.

The continuous versions of inequality (1) is given by:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^{p'}(x)dx\right)^{\frac{1}{q}}, \tag{2}$$

unless $f \equiv 0$ or $g \equiv 0$, where f and g are measurable non-negative functions such that $\int_0^{\infty} f^p(x)dx < \infty$ and $\int_0^{\infty} g^p(x)dx < \infty$. The constant $\frac{\pi}{\sin\frac{\pi}{p}}$, in (1) and (2), is the best possible.

In [2], Pachpatte proved that if $f \in C^1[[0,x],\mathbb{R}^+]$, $g \in C^1[[0,y],\mathbb{R}^+]$ with $f(0) = g(0) = 0$ and p, q are two positive functions defined for $t \in [0,x)$ and $\tau \in [0,y)$, with $P(t) = \int_0^t p(\tau)d\tau$ and $Q(t) = \int_0^t q(\tau)d\tau$ for $s \in [0,x)$ and $t \in [0,y)$, where x, y are positive real numbers. Let Φ and Ψ be two real-valued non-negative, convex and sub-multiplicative functions defined on $[0,\infty)$. Then,

$$\begin{aligned}\int_0^x\int_0^y\frac{\Phi(f(s))\Psi(g(t))}{s+t}dsdt \leqslant\ & L(x,y)\left(\int_0^x (x-s)\left(p(s)\Phi\left(\frac{f'(s)}{p(s)}\right)\right)^2 ds\right)^{\frac{1}{2}} \\ & \times\left(\int_0^y (y-t)\left(q(t)\Psi\left(\frac{g'(t)}{q(t)}\right)\right)^2 dt\right)^{\frac{1}{2}},\end{aligned} \tag{3}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\left(\frac{\Phi(P(s))}{P(s)}\right)^2ds\right)^{\frac{1}{2}}\left(\int_0^y\left(\frac{\Psi(Q(t))}{Q(t)}\right)^2dt\right)^{\frac{1}{2}}.$$

In 2004, Yong [3] studied the following integral inequality:

Theorem 1. *Let $l,m\geqslant 1$ $r\leqslant 0$ and $f(\sigma)\geqslant 0$, $g(\tau)\geqslant 0$ for $\sigma\in(0,\xi)$, $\tau\in(0,\zeta)$, where ξ,ζ are positive real numbers and define*

$$\Theta(\Im):=\int_0^{\Im}f(\sigma)d\sigma,\ \ and\ \ \Xi(\varsigma):=\int_0^{\varsigma}g(\tau)d\tau,$$

for $\Im\in(0,\xi)$ and $\varsigma\in(0,\zeta)$. Then, for $p^{-1}+q^{-1}=1$, $p<0$ or $0<p<1$

$$\begin{aligned}\int_0^{\xi}\int_0^{\zeta}\frac{\Theta^l(\Im)\Xi^m(\varsigma)}{\left(\frac{(\Im^r+\varsigma^r)}{2}\right)^{\frac{2}{rp}}}d\Im d\varsigma &\geqslant lm(\xi\zeta)^{\frac{1}{p}}\left(\int_0^{\xi}(\xi-\Im)(\Theta_f(\Im))^q d\Im\right)^{\frac{1}{q}}\\ &\times\left(\int_0^{\zeta}(\zeta-\varsigma)(\Xi_g(\varsigma))^q d\varsigma\right)^{\frac{1}{q}}\end{aligned}\tag{4}$$

unless $f\equiv 0$ or $g\equiv 0$, where $\Theta_f(\Im)=\Theta^{l-1}(\Im)f(\Im)$, $\Xi_g(\varsigma)=\Xi^{m-1}(\varsigma)g(\varsigma)$.

In 2009, Yang [4] studied the following integral inequality:

Theorem 2. *Let $p,q\geqslant 0$, $\alpha>1$, $\gamma>1$ and $f(\sigma)\geqslant 0$, $g(\tau)\geqslant 0$ for $\sigma\in(0,\xi)$, $\tau\in(0,\zeta)$, where ξ,ζ are positive real numbers and define*

$$\Theta(\Im):=\int_0^{\Im}f(\sigma)d\sigma,\ \ and\ \ \Xi(\varsigma):=\int_0^{\varsigma}g(\tau)d\tau,$$

for $\Im\in(0,\xi)$ and $\varsigma\in(0,\zeta)$. Then,

$$\begin{aligned}\int_0^{\xi}\int_0^{\zeta}\frac{\Theta^p(\Im)\Xi^q(\varsigma)}{\gamma\Im^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha\varsigma^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}d\Im d\varsigma &\leqslant D(p,q,\xi,\zeta,\alpha,\gamma)\left(\int_0^{\xi}(\xi-\Im)\left(\Theta^{p-1}(\Im)f(\Im)\right)^{\alpha}d\Im\right)^{\frac{1}{\alpha}}\\ &\times\left(\int_0^{\zeta}(\zeta-\varsigma)\left(\Xi^{q-1}(\varsigma)g(\varsigma)\right)^{\gamma}d\varsigma\right)^{\frac{1}{\gamma}}\end{aligned}\tag{5}$$

unless $f\equiv 0$ or $g\equiv 0$, where $D(p,q,\xi,\zeta,\alpha,\gamma)=\frac{pq}{\alpha+\gamma}\xi^{\frac{\alpha-1}{\alpha}}\zeta^{\frac{\gamma-1}{\gamma}}$.

In this paper, we prove some new dynamic inequalities of Hilbert type and their converses on time scales. From these inequalities, as special cases, we formulate some special integral and discrete inequalities. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Now, we present some fundamental concepts and effects on time scales which are beneficial for deducing our main results. In 1988, S. Hilger [5] presented time scale theory to unify continuous and discrete analysis. For some Hilbert-type integral, dynamic inequalities and other types of inequalities on time scales, see the papers [2,3,6–16]. For more details on time scale calculus see [17].

We need the following important relations between calculus on time scales $\mathbb{T}$ and either continuous calculus on $\mathbb{R}$ or discrete calculus on $\mathbb{Z}$. Note that:

(*i*) If $\mathbb{T}=\mathbb{R}$, then

$$\sigma(\varsigma)=\varsigma,\quad \mu(\varsigma)=0,\quad f^{\Delta}(\varsigma)=f'(\varsigma),\quad \int_a^b f(\varsigma)\Delta\varsigma=\int_a^b f(\varsigma)d\varsigma.\tag{6}$$

(*ii*) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\varsigma) = \varsigma + 1, \quad \mu(\varsigma) = 1, \quad f^{\Delta}(\varsigma) = f(\varsigma + 1) - f(\varsigma), \quad \int_a^b f(\varsigma)\Delta\varsigma = \sum_{\varsigma=a}^{b-1} f(\varsigma). \tag{7}$$

Next, we write Hölder's inequality and Jensen's inequality on time scales.

Lemma 1 (Dynamic Hölder's Inequality [18])**.** *Let $a, b \in \mathbb{T}$ and $f, g \in C_{rd}([a,b]_{\mathbb{T}}, [0,\infty))$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b f(\varsigma)g(\varsigma)\Delta\varsigma \leq \left[\int_a^b f^p(\varsigma)\Delta\varsigma\right]^{\frac{1}{p}} \left[\int_a^b g^q(\varsigma)\Delta\varsigma\right]^{\frac{1}{q}}. \tag{8}$$

This inequality is reversed if $0 < p < 1$ and if $p < 0$ or $q < 0$.

Lemma 2 (Dynamic Jensen's Inequality [18])**.** *Let a, $b \in \mathbb{T}$ and c, $d \in \mathbb{R}$. Assume that $g \in C_{rd}\big([a,b]_{\mathbb{T}}, [c,d]\big)$ and $r \in C_{rd}\big([a,b]_{\mathbb{T}}, \mathbb{R}\big)$ are non-negative with $\int_a^b r(\varsigma)\Delta\varsigma > 0$. If $\phi \in C_{rd}\big((c,d), \mathbb{R}\big)$ is a convex function, then*

$$\phi\left(\frac{\int_a^b g(\varsigma)r(\varsigma)\Delta\varsigma}{\int_a^b r(\varsigma)\Delta\varsigma}\right) \leqslant \frac{\int_a^b r(\varsigma)\phi(g(\varsigma))\Delta\varsigma}{\int_a^b r(\varsigma)\Delta\varsigma}. \tag{9}$$

This inequality is reversed if $\phi \in C_{rd}\big((c,d), \mathbb{R}\big)$ is concave.

Moreover, we use the following definition and lemma as we see in the proof of our results:

Definition 1. *Λ is called a supermultiplicative function on $[0,\infty)$ if*

$$\Lambda(xy) \geqslant \Lambda(\xi)\Lambda(\zeta), \text{ for all } \xi, \zeta \geqslant 0. \tag{10}$$

Lemma 3 ([19])**.** *Let $\mathbb{T}$ be a time scale with $\xi, a \in \mathbb{T}$ such that $\xi \geqslant a$. If $f \geqslant 0$ and $\alpha \geqslant 1$, then*

$$\left(\int_a^{\xi} f(\tau)\Delta\tau\right)^{\alpha} \geqslant \alpha \int_a^{\xi} f(\eta)\left(\int_a^{\eta} f(\tau)\Delta\tau\right)^{\alpha-1} \Delta\eta. \tag{11}$$

Now, we present the formula that reduces double integrals to single integrals, which is desired in [9].

Lemma 4. *Let $\chi : \mathbb{T} \longrightarrow \mathbb{R}$ and $u, \Im, \theta \in \mathbb{T}$. Then,*

$$\int_u^{\Im} \int_u^{\theta} \chi(\tau)\Delta\tau\Delta\theta = \int_u^{\Im} (\Im - \sigma(\theta))\chi(\theta)\Delta\theta \text{ for } \Im \in \mathbb{T}, \tag{12}$$

assuming the integrals considered exist.

The following section contains our main results:

2. Main Results

In the next theorems, we assume that $p < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 3. *Let $\mathbb{T}$ be time scale with $L, K \geqslant 1$ and $\Im, \varsigma, \varsigma_0, \xi, \zeta \in \mathbb{T}$. Assume $a(\tau)$ and $b(\tau)$ are two non-negative and right-dense continuous functions on $[\varsigma_0, \xi]$ and $[\varsigma_0, \zeta]$, respectively, and define*

$$\psi(\Im) := \int_{\varsigma_0}^{\Im} a(\tau)\Delta\tau \;\; and \;\; \varphi(\varsigma) := \int_{\varsigma_0}^{\varsigma} b(\tau)\Delta\tau,$$

then, for $\Im \in [\varsigma_0, \xi]$ and $\varsigma \in [\varsigma_0, \zeta]$, we have that

$$\begin{aligned}
&\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta} \frac{\psi^K(\Im)\varphi^L(\varsigma)}{\left(p_*(\Im-\varsigma_0)^{p-1} + p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \\
&\geqslant C_1(K,L,p,p_*)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\left(a(\Im)\psi^{K-1}(\Im)\right)^p\Delta\Im\right)^{\frac{1}{p}} \\
&\times\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\left(b(\varsigma)\varphi^{L-1}(\varsigma)\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}, \qquad (13)
\end{aligned}$$

where

$$C_1(K,L,p,p_*) = \frac{KL}{(p+p_*)^{\frac{p+p_*}{pp_*}}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}.$$

Proof. By using the inequality (11), we obtain

$$\psi^K(\Im) \geqslant K\int_{\varsigma_0}^{\Im} a(\eta)\psi^{K-1}(\eta)\Delta\eta, \qquad (14)$$

$$\varphi^L(\varsigma) \geqslant L\int_{\varsigma_0}^{\varsigma} b(\eta)\varphi^{L-1}(\eta)\Delta\eta. \qquad (15)$$

Applying inverse Hölder's inequality on the right hand side of (14) with indices p and $\frac{p}{p-1}$, we have

$$\psi^K(\Im) \geqslant K(\Im-\varsigma_0)^{\frac{p-1}{p}}\left(\int_{\varsigma_0}^{\Im}\left(a(\eta)\psi^{K-1}(\eta)\right)^p\Delta\eta\right)^{\frac{1}{p}}. \qquad (16)$$

Applying inverse Hölder's inequality on the right hand side of (15) with indices p_* and $\frac{p_*}{p_*-1}$, we also have that

$$\varphi^L(\varsigma) \geqslant L(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\varsigma}\left(b(\eta)\varphi^{L-1}(\eta)\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}. \qquad (17)$$

From (16) and (17), we obtain

$$\begin{aligned}
\psi^K(\Im)\varphi^L(\varsigma) &\geqslant KL(\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}} \\
&\times\left(\int_{\varsigma_0}^{\Im}\left(a(\eta)\psi^{K-1}(\eta)\right)^p\Delta\eta\right)^{\frac{1}{p}} \\
&\times\left(\int_{\varsigma_0}^{\varsigma}\left(b(\eta)\varphi^{L-1}(\eta)\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}. \qquad (18)
\end{aligned}$$

Using the following inequality

$$\lambda_1^{\alpha_1'}\lambda_2^{\alpha_2'} \geqslant \left(\frac{1}{\alpha'}\left(\alpha_1'\lambda_1 + \alpha_2'\lambda_2\right)\right)^{\alpha'}, \qquad (19)$$

where $\alpha_1', \alpha_2' < 0$ and $\lambda_1, \lambda_2 > 0$. Now, by setting $\lambda_1 = (\Im - \varsigma_0)^{(p-1)}$, $\lambda_2 = (\varsigma - \varsigma_0)^{(p_*-1)}$, $\alpha_1' = \frac{1}{p}$, $\alpha_2' = \frac{1}{p_*}$ and $\alpha' = \alpha_1' + \alpha_2'$. we obtain that

$$(\Im - \varsigma_0)^{\frac{p-1}{p}}(\varsigma - \varsigma_0)^{\frac{p_*-1}{p_*}} \geqslant \left(\frac{pp_*}{p+p_*}\left(\frac{(\Im - \varsigma_0)^{p-1}}{p} + \frac{(\varsigma - \varsigma_0)^{p_*-1}}{p_*}\right)\right)^{\frac{p+p_*}{pp_*}}. \tag{20}$$

Substituting (20) in (18), yields

$$\begin{aligned}\psi^K(\Im)\varphi^L(\varsigma) \geqslant \frac{KL}{(p+p_*)^{\frac{p+p_*}{pp_*}}}&\left(p_*(\Im - \varsigma_0)^{p-1} + p(\varsigma - \varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}} \\ &\times\left(\int_{\varsigma_0}^{\Im}\left(a(\eta)\psi^{K-1}(\eta)\right)^p\Delta\eta\right)^{\frac{1}{p}} \\ &\times\left(\int_{\varsigma_0}^{\varsigma}\left(b(\eta)\varphi^{L-1}(\eta)\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}.\end{aligned} \tag{21}$$

Dividing both side of (21) by $\left(p_*(\Im - \varsigma_0)^{p-1} + p(\varsigma - \varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}$, we obtain that

$$\begin{aligned}\frac{\psi^K(\Im)\varphi^L(\varsigma)}{\left(p_*(\Im - \varsigma_0)^{p-1} + p(\varsigma - \varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant \frac{KL}{(p+p_*)^{\frac{p+p_*}{pp_*}}}&\left(\int_{\varsigma_0}^{\Im}\left(a(\eta)\psi^{K-1}(\eta)\right)^p\Delta\eta\right)^{\frac{1}{p}} \\ &\times\left(\int_{\varsigma_0}^{\varsigma}\left(b(\eta)\varphi^{L-1}(\eta)\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}.\end{aligned} \tag{22}$$

Integrating both sides of (22) from ς_0 to ξ and from ς_0 to ζ, and applying inverse Hölder's inequality with indices p, $\dfrac{p}{p-1}$ and p_*, $\dfrac{p_*}{p_*-1}$, we obtain

$$\begin{aligned}&\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\psi^K(\Im)\varphi^L(\varsigma)}{\left(p_*(\Im - \varsigma_0)^{p-1} + p(\varsigma - \varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \\ &\geqslant \frac{KL}{(p+p_*)^{\frac{p+p_*}{pp_*}}}(\xi - \varsigma_0)^{\frac{p-1}{p}}(\zeta - \varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\xi}\left(\int_{\varsigma_0}^{\Im}\left(a(\eta)\psi^{K-1}(\eta)\right)^p\Delta\eta\right)\Delta\Im\right)^{\frac{1}{p}} \\ &\times\left(\int_{\varsigma_0}^{\zeta}\left(\int_{\varsigma_0}^{\varsigma}\left(b(\eta)\varphi^{L-1}(\eta)\right)^{p_*}\Delta\eta\right)\Delta\varsigma\right)^{\frac{1}{p_*}}.\end{aligned} \tag{23}$$

Applying Lemma 4 on the right hand side of (23), we have

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\psi^K(\Im)\varphi^L(\varsigma)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant \frac{KL}{(p+p_*)^{\frac{p+p_*}{pp_*}}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\left(a(\Im)\psi^{K-1}(\Im)\right)^p\Delta\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\left(b(\varsigma)\varphi^{L-1}(\varsigma)\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}$$
$$= C_1(K,L,p,p_*)\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\left(a(\Im)\psi^{K-1}(\Im)\right)^p\Delta\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\left(b(\varsigma)\varphi^{L-1}(\varsigma)\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

By using the facts $\xi \geqslant \rho(\xi)$ and $\zeta \geqslant \rho(\zeta)$, we obtain

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\psi^K(\Im)\varphi^L(\varsigma)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant C_1(K,L,p,p_*)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\left(a(\Im)\psi^{K-1}(\Im)\right)^p\Delta\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\left(b(\varsigma)\varphi^{L-1}(\varsigma)\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

This completes the proof. □

Theorem 4. *Let $a(\tau)$, $b(\eta)$, $\psi(\Im)$ and $\varphi(\varsigma)$ be defined as Theorem 3. Then, we have*

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\psi(\Im)\varphi(\varsigma)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant \frac{(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}}{(p+p_*)^{\frac{p+p_*}{pp_*}}}\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))(a(\Im))^p\Delta\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))(b(\varsigma))^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

Proof. Put $K = L = 1$ in (13). This completes the proof. □

As a special case of Theorem 3, when $\mathbb{T} = \mathbb{R}$, we have $\rho(\xi) = \xi$, $\rho(\zeta) = \zeta$, $\sigma(\Im) = \Im$, $\sigma(\varsigma) = \varsigma$, and we obtain the following result:

Corollary 1. *Assume that* $a(\Im)$ *and* $b(\varsigma)$ *are non-negative functions and define* $\psi(\Im) := \int_0^{\Im} a(\eta)d\eta$ *and* $\varphi(\varsigma) := \int_0^{\varsigma} b(\eta)d\eta$. *Then,*

$$\int_0^{\xi}\int_0^{\zeta} \frac{\psi^K(\Im)\varphi^L(\varsigma)}{\left(p_*\ \Im^{p-1} + p\ \varsigma^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} d\Im d\varsigma$$

$$\geqslant C_3(K,L,p,p_*)\left(\int_0^{\xi}(\xi-\Im)\left(a(\Im)\psi^{K-1}(\Im)\right)^p d\Im\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\zeta}(\zeta-\varsigma)\left(b(\varsigma)\varphi^{L-1}(\varsigma)\right)^{p_*} d\varsigma\right)^{\frac{1}{p_*}},$$

where

$$C_3(L,K,p,p_*) = \frac{KL\xi^{\frac{p-1}{p}}\zeta^{\frac{p_*-1}{p_*}}}{(p+p_*)^{\frac{p+p_*}{pp_*}}}.$$

As special case of Theorem 3, when $\mathbb{T} = \mathbb{Z}$, we have $\rho(\xi) = \xi - 1$, $\rho(\zeta) = \zeta - 1$, $\sigma(\Im) = \Im + 1$, $\sigma(\varsigma) = \varsigma + 1$, and we obtain the following result:

Corollary 2. *Assume that* $a(n)$ *and* $b(m)$ *are non-negative sequences and define*

$$\psi(n) = \sum_{\Im=0}^{n} a(\Im) \ \ and \ \ \varphi(m) = \sum_{k=0}^{m} b(k).$$

Then,

$$\sum_{n=1}^{N}\sum_{m=1}^{M} \frac{\psi^L(n)\varphi^K(m)}{\left(p_*\ n^{p-1} + p\ m^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant C_4(K,L,p,p_*)\left(\sum_{n=1}^{N}((N-1)-(n+1))(a(n)\psi^{L-1}(n))^p\right)^{\frac{1}{p}}$$

$$\times\left(\sum_{m=1}^{M}((M-1)-(m+1))(b(m)\varphi^{L-1}(m))^{p_*}\right)^{\frac{1}{p_*}},$$

where

$$C_4(K,L,p) = \frac{KLN^{\frac{p-1}{p}}M^{\frac{p_*-1}{p_*}}}{(p+p_*)^{\frac{p+p_*}{pp_*}}}.$$

Theorem 5. *Let* $\mathbb{T}$ *be a time scale with* $\Im$, ς, ς_0, ξ, $\zeta \in \mathbb{T}$, $\psi(\Im)$ *and* $\varphi(\varsigma)$ *be as defined in Theorem 3. Let* $f(\tau)$ *and* $g(\eta)$ *be two non-negative and right-dense continuous functions on* $[\varsigma_0, \xi]$ *and* $[\varsigma_0, \zeta]$, *respectively. Suppose that* Λ *and* Υ *are non-negative, concave and supermultiplicative functions defined on* $[0, \infty)$. *Furthermore, assume that*

$$\Theta(\Im) := \int_{\varsigma_0}^{\Im} f(\tau)\Delta\tau \ \ and \ \ \Xi(\varsigma) := \int_{\varsigma_0}^{\varsigma} g(\eta)\Delta\eta, \tag{24}$$

then, for $\Im \in [\varsigma_0, \xi]$ *and* $\varsigma \in [\varsigma_0, \zeta]$*, we have that*

$$\begin{aligned}
&\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta} \frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \\
&\geqslant M_1(p)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\left(f(\Im)\Lambda\left[\frac{a(\Im)}{f(\Im)}\right]\right)^{p}\Delta\Im\right)^{\frac{1}{p}} \\
&\times\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\left(g(\varsigma)\Upsilon\left[\frac{b(\varsigma)}{g(\varsigma)}\right]\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}},
\end{aligned} \tag{25}$$

where

$$M_1(p)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left\{\int_{\varsigma_0}^{\xi}\left(\frac{\Lambda(\Theta(\Im))}{\Theta(\Im)}\right)^{\frac{p}{p-1}}\Delta\Im\right\}^{\frac{p-1}{p}}\left\{\int_{\varsigma_0}^{\zeta}\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\right)^{\frac{p_*}{p_*-1}}\Delta\varsigma\right\}^{\frac{p_*-1}{p_*}}.$$

Proof. Since Λ is a concave and supermultiplicative function, we obtain by applying inverse Jensen's inequality that

$$\begin{aligned}
\Lambda(\psi(\Im)) &= \Lambda\left(\frac{\Theta(\Im))\int_{\varsigma_0}^{\Im} f(\tau)\frac{a(\tau)}{f(\tau)}\Delta\tau}{\int_{\varsigma_0}^{\Im} f(\tau)\Delta\tau}\right) \\
&\geqslant \Lambda(\Theta(\Im))\Lambda\left(\frac{\int_{\varsigma_0}^{\Im} f(\tau)\frac{a(\tau)}{f(\tau)}\Delta\tau}{\int_{\varsigma_0}^{\Im} f(\tau)\Delta\tau}\right) \\
&\geqslant \frac{\Lambda(\Theta(\Im))}{\Theta(\Im)}\int_{\varsigma_0}^{\Im} f(\tau)\Lambda\left(\frac{a(\tau)}{f(\tau)}\right)\Delta\tau.
\end{aligned} \tag{26}$$

Applying inverse Hölder's inequality with indices p and $\frac{p}{p-1}$ on the right hand side of (26), we see that

$$\Lambda(\psi(\Im)) \geqslant \frac{\Lambda(\Theta(\Im)}{\Theta(\Im)}(\Im-\varsigma_0)^{\frac{p-1}{p}}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda\left[\frac{a(\tau)}{f(\tau)}\right]\right)^{p}\Delta\tau\right)^{\frac{1}{p}}. \tag{27}$$

Moreover, since Υ is a concave and supermultiplicative function, we obtain by applying inverse Jensen's inequality and inverse Hölder's inequality with indices p_* and $\frac{p_*}{p_*-1}$ that we have

$$\Upsilon(\varphi(\varsigma)) \geqslant \frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon\left[\frac{b(\eta)}{g(\eta)}\right]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}. \tag{28}$$

From (27) and (28), we have

$$\begin{aligned}
\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma)) \geqslant\; &(\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\frac{\Lambda(\Theta(\Im)}{\Theta(\Im)}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda\left[\frac{a(\tau)}{f(\tau)}\right]\right)^{p}\Delta\tau\right)^{\frac{1}{p}}\right) \\
&\times\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon\left[\frac{b(\eta)}{g(\eta)}\right]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}\right).
\end{aligned} \tag{29}$$

By using inequality (19), we obtain that

$$(\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}} \geqslant \left(\frac{pp_*}{p+p_*}\left(\frac{(\Im-\varsigma_0)^{p-1}}{p}+\frac{(\varsigma-\varsigma_0)^{p_*-1}}{p_*}\right)\right)^{\frac{p+p_*}{pp_*}}. \tag{30}$$

From (29) and (30), we have that

$$\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma)) \geqslant \left(\frac{pp_*}{p+p_*}\left(\frac{(\Im-\varsigma_0)^{p-1}}{p}+\frac{(\varsigma-\varsigma_0)^{p_*-1}}{p_*}\right)\right)^{\frac{p+p_*}{pp_*}} \tag{31}$$
$$\times\left(\frac{\Lambda(\Theta(\Im)}{\Theta(\Im)}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right)^{\frac{1}{p}}\right)\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon\left[\frac{b(\eta)}{g(\eta)}\right]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}\right).$$

Then,

$$\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left(\frac{\Lambda(\Theta(\Im)}{\Theta(\Im)}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right)^{\frac{1}{p}}\right)$$
$$\times\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon\left[\frac{b(\eta)}{g(\eta)}\right]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}\right). \tag{32}$$

Integrating both sides of (32) from ς_0 to ξ and from ς_0 and ζ, we obtain

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\int_{\varsigma_0}^{\xi}\left(\frac{\Lambda(\Theta(\Im)}{\Theta(\Im)}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right)^{\frac{1}{p}}\Delta\Im\right) \tag{33}$$
$$\times\int_{\varsigma_0}^{\zeta}\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon\left[\frac{b(\eta)}{g(\eta)}\right]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}\Delta\varsigma\right).$$

Applying inverse Hölder's inequality with indices p, $\frac{p}{p-1}$ and p_*, $\frac{p_*-1}{p_*}$ on the right hand of side (33), we have

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left\{\int_{\varsigma_0}^{\xi}\left(\frac{\Lambda(\Theta(\Im)}{\Theta(\Im)}\right)^{\frac{p}{p-1}}\Delta\Im\right\}^{\frac{p-1}{p}}\left(\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\Delta\Im\right)^{\frac{1}{p}} \tag{34}$$
$$\times\left\{\int_{\varsigma_0}^{\zeta}\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\right)^{\frac{p_*}{p_*-1}}\Delta\varsigma\right\}^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\zeta}\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon\left[\frac{b(\eta)}{g(\eta)}\right]\right)^{p_*}\Delta\eta\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

Applying Lemma 4 on the right hand side of (34), we obtain

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant M_1(p)\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\left(f(\Im)\Lambda\left[\frac{a(\Im)}{f(\Im)}\right]\right)^p\Delta\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\left(g(\varsigma)\Upsilon\left[\frac{b(\varsigma)}{g(\varsigma)}\right]\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

By using the facts $\xi \geqslant \rho(\xi)$ and $\zeta \geqslant \rho(\zeta)$, we obtain

$$\begin{aligned}
&\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta} \frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \\
&\geqslant M_1(p)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\left(f(\Im)\Lambda\left[\frac{a(\Im)}{f(\Im)}\right]\right)^p\Delta\Im\right)^{\frac{1}{p}} \\
&\times\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\left(g(\varsigma)\Upsilon\left[\frac{b(\varsigma)}{g(\varsigma)}\right]\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}},
\end{aligned}$$

where

$$M_1(p) = \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left\{\int_{\varsigma_0}^{\xi}\left(\frac{\Lambda(\Theta(\Im))}{\Theta(\Im)}\right)^{\frac{p}{p-1}}\Delta\Im\right\}^{\frac{p-1}{p}}\left\{\int_{\varsigma_0}^{\zeta}\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\right)^{\frac{p_*}{p_*-1}}\Delta\varsigma\right\}^{\frac{p_*-1}{p_*}}.$$

This completes the proof. □

As a special case of Theorem 5, when $\mathbb{T}=\mathbb{R}$, we have $\rho(\xi)=\xi$ $\rho(\zeta)=\zeta$, $\sigma(\Im)=\Im$, $\sigma(\varsigma)=\varsigma$, and we obtain the following result:

Corollary 3. *Assume that $a(\Im)$, $b(\varsigma)$, $f(\tau)$ and $g(\eta)$ are non-negative functions and define*

$$\psi(\Im) := \int_0^{\Im} a(\eta)d\eta, \quad \varphi(\varsigma) := \int_0^{\varsigma} b(\eta)d\eta, \quad \Theta(\Im) := \int_0^{\Im} f(\tau)d\tau, \quad \text{and} \quad \Xi(\varsigma) := \int_0^{\varsigma} g(\eta)d\eta.$$

Then,

$$\begin{aligned}
&\int_0^{\xi}\int_0^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))}{\left(p_*\Im^{p-1}+p\varsigma^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}d\Im d\varsigma \\
&\geqslant M_2(p)\left(\int_0^{\xi}(\xi-\Im)\left(f(\Im)\Lambda\left[\frac{a(\Im)}{f(\Im)}\right]\right)^p d\Im\right)^{\frac{1}{p}} \\
&\times\left(\int_0^{\zeta}(\zeta-\varsigma)\left(g(\varsigma)\Upsilon\left[\frac{b(\varsigma)}{g(\varsigma)}\right]\right)^{p_*}d\varsigma\right)^{\frac{1}{p_*}},
\end{aligned}$$

where

$$M_2(p) = \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left\{\int_0^{\xi}\left(\frac{\Lambda(\Theta(\Im))}{\Theta(\Im)}\right)^{\frac{p}{p-1}}d\Im\right\}^{\frac{p-1}{p}}\left\{\int_0^{\zeta}\left(\frac{\Upsilon(\Xi(\varsigma))}{\Xi(\varsigma)}\right)^{\frac{p_*}{p_*-1}}d\varsigma\right\}^{\frac{p_*-1}{p_*}}.$$

As a special case of Theorem 5, when $\mathbb{T}=\mathbb{Z}$, we have $\rho(\xi)=\xi-1$, $\rho(\zeta)=\zeta-1$, $\sigma(\Im)=\Im+1$, $\sigma(\varsigma)=\varsigma+1$, and we obtain the following result.

Corollary 4. *Assume that $a(n)$, $b(m)$, $f(n)$ and $g(m)$ are non-negative sequences and define*

$$\psi(n) = \sum_{\Im=0}^{n} a(\Im), \quad \varphi(m) = \sum_{k=0}^{m} b(k), \quad \Theta(n) = \sum_{\Im=0}^{n} f(\Im) \quad \text{and} \quad \Xi(m) = \sum_{k=0}^{m} g(k).$$

Then,

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\Lambda(\psi(n))\Upsilon(\varphi(m))}{\left(p_* n^{p-1}+p m^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant M_3(p)\left\{\sum_{n=1}^{N}((N-1)-(n+1))\left(f(n)\Lambda\left[\frac{a(n)}{f(n)}\right]\right)^{p}\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{m=1}^{M}((M-1)-(m+1))\left(g(m)\Upsilon\left[\frac{b(m)}{g(m)}\right]\right)^{p_*}\right\}^{\frac{1}{p_*}},$$

where

$$M_3(p)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left\{\sum_{n=1}^{N}\left(\frac{\Lambda(\Theta(n))}{\Theta(n)}\right)^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\left\{\sum_{m=1}^{M}\left(\frac{\Upsilon(\Xi(m))}{\Xi(m)}\right)^{\frac{p_*}{p_*-1}}\right\}^{\frac{p_*-1}{p_*}}.$$

Theorem 6. *Let $\mathbb{T}$ be a time scale with $\Im$, ς, ς_0, ξ, $\zeta \in \mathbb{T}$. Let f and g be two non-negative and right-dense continuous functions on $[\varsigma_0,\xi]$ and $[\varsigma_0,\zeta]$, respectively. Suppose that Λ and Υ are non-negative, concave and supermultiplicative functions defined on $[0,\infty)$ and define*

$$\Theta(\Im):=\frac{1}{\Im-\varsigma_0}\int_{\varsigma_0}^{\Im}f(\tau)\Delta\tau \ \text{ and } \ \Xi(\varsigma):=\frac{1}{\varsigma-\varsigma_0}\int_{\varsigma_0}^{\varsigma}g(\tau)\Delta\tau, \tag{35}$$

then, for $\Im\in[\varsigma_0,\xi]$ and $\varsigma\in[\varsigma_0,\zeta]$, we have that

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant E(p,p_*)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\left[\Lambda(f(\Im))\right]^{p}\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\left[\Upsilon(g(\varsigma))\right]^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}, \tag{36}$$

where

$$E(p,p_*)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}.$$

Proof. By assumption and by using the inverse Jensen inequality, we see that

$$\begin{aligned}\Lambda(\Theta(\Im)) &= \Lambda\left(\frac{1}{\Im-\varsigma_0}\int_{\varsigma_0}^{\Im}f(\tau)\Delta\tau\right)\\ &\geqslant \frac{1}{\Im-\varsigma_0}\int_{\varsigma_0}^{\Im}\Lambda(f(\tau))\Delta\tau.\end{aligned} \tag{37}$$

By applying inverse Hölder's inequality on (37) with indices, p, $\frac{p}{p-1}$, we have

$$\Lambda(\Theta(\Im))\geqslant\frac{1}{\Im-\varsigma_0}(\Im-\varsigma_0)^{\frac{p-1}{p}}\left(\int_{\varsigma_0}^{\Im}\left[\Lambda(f(\tau))\right]^{p}\Delta\tau\right)^{\frac{1}{p}}. \tag{38}$$

This implies that

$$\Lambda(\Theta(\Im))(\Im-\varsigma_0)\geqslant(\Im-\varsigma_0)^{\frac{p-1}{p}}\left(\int_{\varsigma_0}^{\Im}\left[\Lambda(f(\tau))\right]^{p}\Delta\tau\right)^{\frac{1}{p}}. \tag{39}$$

Analogously,

$$\Upsilon(\Xi(\varsigma))(\varsigma-\varsigma_0)\geqslant(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\varsigma}\left[\Upsilon(g(\tau))\right]^{p_*}\Delta\tau\right)^{\frac{1}{p_*}}. \tag{40}$$

From (39) and (40), we obtain

$$\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0) \geqslant (\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\Im}[\Lambda(f(\tau))]^p\Delta\tau\right)^{\frac{1}{p}} \times\left(\int_{\varsigma_0}^{\varsigma}[\Upsilon(g(\tau))]^{p_*}\Delta\tau\right)^{\frac{1}{p_*}}. \quad (41)$$

By using inequality (19), we obtain that

$$(\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}} \geqslant \left(\frac{pp_*}{p+p_*}\left(\frac{(\Im-\varsigma_0)^{p-1}}{p}+\frac{(\varsigma-\varsigma_0)^{p_*-1}}{p_*}\right)\right)^{\frac{p+p_*}{pp_*}}. \quad (42)$$

Then,

$$\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0) \geqslant \left(\frac{pp_*}{p+p_*}\left(\frac{(\Im-\varsigma_0)^{p-1}}{p}+\frac{(\varsigma-\varsigma_0)^{p_*-1}}{p_*}\right)\right)^{\frac{p+p_*}{pp_*}} \left(\int_{\varsigma_0}^{\Im}[\Lambda(f(\tau))]^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\varsigma}[\Upsilon(g(\tau))]^{p_*}\Delta\tau\right)^{\frac{1}{p_*}}. \quad (43)$$

From (43), we have

$$\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left(\int_{\varsigma_0}^{\Im}[\Lambda(f(\tau))]^p\Delta\tau\right)^{\frac{1}{p}} \times\left(\int_{\varsigma_0}^{\varsigma}[\Upsilon(g(\tau))]^{p_*}\Delta\tau\right)^{\frac{1}{p_*}}. \quad (44)$$

Taking delta integrating on both sides of (44), first over $\Im$ from ς_0 to ξ and then over ς from ς_0 to ζ, we find that

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \geqslant \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left(\int_{\varsigma_0}^{\xi}\left(\int_{\varsigma_0}^{\Im}[\Lambda(f(\tau))]^p\Delta\tau\right)^{\frac{1}{p}}\Delta\Im\right) \times\left(\int_{\varsigma_0}^{\zeta}\left(\int_{\varsigma_0}^{\varsigma}[\Upsilon(g(\tau))]^{p_*}\Delta\tau\right)^{\frac{1}{p_*}}\Delta\varsigma\right). \quad (45)$$

By applying inverse Hölder'sinequality on (45) with indices p, $\dfrac{p}{p-1}$ and p_*, $\dfrac{p_*}{p_*-1}$, we get

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \geqslant \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}} \times\left(\int_{\varsigma_0}^{\xi}\left(\int_{\varsigma_0}^{\Im}[\Lambda(f(\tau))]^p\Delta\tau\right)\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}\left(\int_{\varsigma_0}^{\varsigma}[\Upsilon(g(\tau))]^{p_*}\Delta\tau\right)\Delta\varsigma\right)^{\frac{1}{p_*}}. \quad (46)$$

Applying Lemma 4 on (46), we fined that

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \geqslant \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}$$
$$\times\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\big[\Lambda\big(f(\Im)\big)\big]^{p}\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\big[\Upsilon\big(g(\varsigma)\big)\big]^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}$$
$$= E(p,p_*)\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\big[\phi\big(f(\Im)\big)\big]^{p}\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\big[\Upsilon\big(g(\varsigma)\big)\big]^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

By using the facts $\xi \geqslant \rho(\xi)$ and $\zeta \geqslant \rho(\zeta)$, we obtain

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))(\Im-\varsigma_0)(\varsigma-\varsigma_0)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant E(p,p_*)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\big[\Lambda\big(f(\Im)\big)\big]^{p}\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\big[\Upsilon\big(g(\varsigma)\big)\big]^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}},$$

where

$$E(p,p_*) = \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}.$$

This completes the proof. □

As a special case of Theorem 6, when $\mathbb{T}=\mathbb{R}$, we obtain the following conclusion.

Corollary 5. *Assume that f and g are non-negative functions and define*

$$\Theta(\Im) := \frac{1}{\Im}\int_0^{\Im} f(\tau)d\tau \;\; and \;\; \Xi(\varsigma) := \frac{1}{\varsigma}\int_0^{\varsigma} g(\tau)d\tau,$$

then, for $\Im\in[0,\xi]$ and $\varsigma\in[0,\zeta]$, we have that

$$\int_0^{\xi}\int_0^{\zeta}\frac{\Lambda(\Theta(\Im))\Upsilon(\Xi(\varsigma))\Im\varsigma}{\left(p_*\Im^{p-1}+p\varsigma^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}d\Im d\varsigma \;\; \geqslant \;\; E_1(p,p_*)\left(\int_0^{\xi}(\xi-\Im)\big[\Lambda\big(f(\Im)\big)\big]^{p}d\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^{\zeta}(\zeta-\varsigma)\big[\Upsilon\big(g(\varsigma)\big)\big]^{p_*}d\varsigma\right)^{\frac{1}{p_*}},$$

where

$$E_1(p,p_*) = \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\xi^{\frac{p-1}{p}}\zeta^{\frac{p_*-1}{p_*}}.$$

As a special case of Theorem 6, when $\mathbb{T}=\mathbb{Z}$, we obtain the following conclusion.

Corollary 6. *Assume that $f(n)$ and $g(m)$ are two non-negative sequences of real numbers and define*

$$\Theta(n) := \frac{1}{n}\sum_{\Im=1}^{n} f(\Im) \;\; and \;\; \Xi(m) := \frac{1}{m}\sum_{k=1}^{m} g(k),$$

then,

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\Lambda(\Theta(n))\Upsilon(\Xi(m))nm}{\left(p_*n^{p-1}+pm^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant E_2(p,p_*)\left(\sum_{n=1}^{N}((N-1)-(n+1))\left[\Lambda\left(f(n)\right)\right]^{p}\right)^{\frac{1}{p}} \times\left(\sum_{m=1}^{M}((M-1)-(m+1))\left[\Upsilon\left(g(m)\right)\right]^{p_*}\right)^{\frac{1}{p_*}},$$

where

$$E_2(p,p_*)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}N^{\frac{p-1}{p}}M^{\frac{p_*-1}{p_*}}.$$

Theorem 7. *Let $\mathbb{T}$ be a time scale with $\Im, \varsigma, \varsigma_0, \xi, \zeta \in \mathbb{T}$. Let $a(\tau)$ and $b(\tau)$ be two non-negative and right-dense continuous functions on $[\varsigma_0,\xi]$ and $[\varsigma_0,\zeta]$, respectively. Let Θ, Ξ, f, g, Λ and Υ be as assumed in Theorem 5. Furthermore, assume that*

$$\psi(\Im):=\frac{1}{\Theta(\Im)}\int_{\varsigma_0}^{\Im}a(\tau)f(\tau)\Delta\tau \ \ and \ \ \varphi(\varsigma):=\frac{1}{\Xi(\varsigma)}\int_{\varsigma_0}^{\varsigma}b(\eta)g(\eta)\Delta\eta, \tag{47}$$

then, for $\Im\in[\varsigma_0,\xi]$ and $\varsigma\in[\varsigma_0,\zeta]$, we have that

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant D_1(p)\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\left(f(\Im)\Lambda[a(\Im)]\right)^{p}\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\left(g(\varsigma)\Upsilon[b(\varsigma)]\right)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}},$$

where

$$D_1(p)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}.$$

Proof. From (47), we see that

$$\Lambda(\psi(\Im))=\Lambda\left(\frac{1}{\Theta(\Im)}\int_{\varsigma_0}^{\Im}f(\tau)a(\tau)\Delta\tau\right). \tag{48}$$

Applying inverse Hölder's inequality with indices p and $\frac{p}{p-1}$ on the right hand side of (48), we obtain

$$\Lambda(\psi(\Im))\geqslant\frac{(\Im-\varsigma_0)^{\frac{p-1}{p}}}{\Theta(\Im)}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^{p}\Delta\tau\right)^{\frac{1}{p}}. \tag{49}$$

From (49), we obtain that

$$\Lambda(\psi(\Im))\Theta(\Im)\geqslant(\Im-\varsigma_0)^{\frac{p-1}{p}}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^{p}\Delta\tau\right)^{\frac{1}{p}}. \tag{50}$$

Similarly, we obtain

$$\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\geqslant(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}. \tag{51}$$

From (50) and (51), we observe that

$$\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im) \geqslant (\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}} \times\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}. \tag{52}$$

Applying the inequality (19) on the term $(\Im-\varsigma_0)^{\frac{p-1}{p}}(\varsigma-\varsigma_0)^{\frac{p_*-1}{p_*}}$, we obtain the following inequality

$$\begin{aligned}\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im) \geqslant & \left(\frac{pp_*}{p+p_*}\left(\frac{(\Im-\varsigma_0)^{p-1}}{p}+\frac{(\varsigma-\varsigma_0)^{p_*-1}}{p_*}\right)\right)^{\frac{p+p_*}{pp_*}} \\ & \times\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}.\end{aligned} \tag{53}$$

Dividing both sides of (53) by $\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}$, we obtain that

$$\begin{aligned}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}} \geqslant & \left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^p\Delta\tau\right)^{\frac{1}{p}} \\ & \times\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}.\end{aligned} \tag{54}$$

Integrating both sides of (54) from ς_0 to ξ and ς_0 to ζ, we obtain

$$\begin{aligned}&\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \\ &\geqslant\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}\left(\int_{\varsigma_0}^{\xi}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^p\Delta\tau\right)^{\frac{1}{p}}\Delta\Im\right)\left(\int_{\varsigma_0}^{\zeta}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)^{\frac{1}{p_*}}\Delta\varsigma\right).\end{aligned} \tag{55}$$

Applying inverse Hölder's inequality again with indices p, $\frac{p}{p-1}$ and p_*, $\frac{p_*}{p_*-1}$ on the right hand side of (55), we have

$$\begin{aligned}&\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma \\ &\geqslant\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi-\varsigma_0)^{\frac{p-1}{p}}(\zeta-\varsigma_0)^{\frac{p_*-1}{p_*}}\left(\int_{\varsigma_0}^{\xi}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^p\Delta\tau\right)\Delta\Im\right)^{\frac{1}{p}} \\ &\times\left(\int_{\varsigma_0}^{\zeta}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)\Delta\varsigma\right)^{\frac{1}{p_*}} \\ &= D_1(p)\left(\int_{\varsigma_0}^{\xi}\left(\int_{\varsigma_0}^{\Im}\left(f(\tau)\Lambda[a(\tau)]\right)^p\Delta\tau\right)\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}\left(\int_{\varsigma_0}^{\varsigma}\left(g(\eta)\Upsilon[b(\eta)]\right)^{p_*}\Delta\eta\right)\Delta\varsigma\right)^{\frac{1}{p_*}}.\end{aligned} \tag{56}$$

Applying Lemma4 on the right hand side of (56), we obtain that

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant D_1(p)\left(\int_{\varsigma_0}^{\xi}(\xi-\sigma(\Im))\Big(f(\Im)\Lambda[a(\Im)]\Big)^p\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\zeta-\sigma(\varsigma))\Big(g(\varsigma)\Upsilon[b(\varsigma)]\Big)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

By using the facts $\xi \geqslant \rho(\xi)$ and $\zeta \geqslant \rho(\zeta)$, we obtain

$$\int_{\varsigma_0}^{\xi}\int_{\varsigma_0}^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Xi(\varsigma)\Theta(\Im)}{\left(p_*(\Im-\varsigma_0)^{p-1}+p(\varsigma-\varsigma_0)^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}\Delta\Im\Delta\varsigma$$
$$\geqslant D_1(p)\left(\int_{\varsigma_0}^{\xi}(\rho(\xi)-\sigma(\Im))\Big(f(\Im)\Lambda[a(\Im)]\Big)^p\Delta\Im\right)^{\frac{1}{p}}\left(\int_{\varsigma_0}^{\zeta}(\rho(\zeta)-\sigma(\varsigma))\Big(g(\varsigma)\Upsilon[b(\varsigma)]\Big)^{p_*}\Delta\varsigma\right)^{\frac{1}{p_*}}.$$

This completes the proof. □

As a special case of Theorem 7, when $\mathbb{T}=\mathbb{R}$, we have $\rho(\xi)=\xi$, $\rho(\zeta)=\zeta$, $\sigma(\Im)=\Im$, $\sigma(\varsigma)=\varsigma$, and we obtain the following result:

Corollary 7. *Assume that $a(\Im)$, $b(\varsigma)$, $f(\Im)$ and $g(\varsigma)$ are non-negative functions and define*

$$\psi(\Im):=\frac{1}{\Theta(\Im)}\int_0^{\Im}f(\tau)a(\tau)d\tau \ \ and \ \ \varphi(\varsigma):=\frac{1}{\Xi(\varsigma)}\int_0^{\varsigma}g(\tau)b(\tau)d\tau,$$

$$\Theta(\Im):=\int_0^{\Im}f(\tau)d\tau \ \ and \ \ \Xi(\varsigma):=\int_0^{\varsigma}g(\tau)d\tau.$$

Then,

$$\int_0^{\xi}\int_0^{\zeta}\frac{\Lambda(\psi(\Im))\Upsilon(\varphi(\varsigma))\Theta(\Im)\Xi(\varsigma)}{\left(p_*\Im^{p-1}+p\varsigma^{p_*-1}\right)^{\frac{p+p_*}{pp_*}}}d\Im d\varsigma \geqslant D_2(p)\left(\int_0^{\xi}(\xi-\Im)\Big(f(\Im)\Lambda\Big(a(\Im)\Big)\Big)^p d\Im\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^{\zeta}(\zeta-\varsigma)\Big(g(\varsigma)\Upsilon\Big(b(\varsigma)\Big)\Big)^{p_*}d\varsigma\right)^{\frac{1}{p_*}},$$

where

$$D_2(p)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(\xi)^{\frac{p-1}{p}}(\zeta)^{\frac{p_*-1}{p_*}}.$$

As a special case of Theorem 7, when $\mathbb{T}=\mathbb{Z}$, we have $\rho(\xi)=\xi-1$, $\rho(\zeta)=\zeta-1$, $\sigma(\Im)=\Im+1$, $\sigma(\varsigma)=\varsigma+1$, and we obtain the following result:

Corollary 8. *Assume that $a(n)$, $b(m)$, $f(n)$ and $g(m)$ are non-negative sequences and define*

$$\psi(n):=\frac{1}{\Theta(n)}\sum_{\Im=0}^{n}f(\Im)a(\Im) \ \ and \ \ \varphi(m):=\frac{1}{\Xi(m)}\sum_{k=0}^{m}g(k)b(k),$$

$$\Theta(n):=\sum_{\Im=0}^{n}f(\Im) \ \ and \ \ \Xi(m):=\sum_{k=0}^{m}g(k).$$

Then

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\Lambda(\psi(n))\Upsilon(\varphi(m))\Theta(n)\Xi(m)}{\Big(p_*n^{p-1}+pm^{p_*-1}\Big)^{\frac{p+p_*}{pp_*}}} \geqslant D_3(p)\left(\sum_{n=1}^{N}((N-1)-(n+1))\Big(f(n)\Lambda\Big(a(n)\Big)\Big)^{p}\right)^{\frac{1}{p}}$$

$$\times\left(\sum_{m=1}^{M}((M-1)-(m+1))\Big(g(m)\Upsilon\Big(b(m)\Big)\Big)^{p_*}\right)^{\frac{1}{p_*}},$$

where

$$D_3(p)=\left(\frac{1}{p+p_*}\right)^{\frac{p+p_*}{pp_*}}(N)^{\frac{p-1}{p}}(M)^{\frac{p_*-1}{p_*}}.$$

3. Conclusions and Discussion

In this article, with the help of the inverse Hölder's inequality and inverse Jensen's inequality on time scales, we discussed and proved several new generalizations of the integral retarded inequalities given in [3]. Moreover, we generalized a number of other inequalities to a general time scale. Finally, as a special case, we studied the discrete and continuous inequalities. As a future work, we intend to generalize these inequalities by using alpha-conformable fractional derivatives on time scales. Furthermore, we will extend these results to diamond alpha calculus.

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