symmetry

MDPI

Article

Online and Connected Online Ramsey Numbers of a Matching versus a Path

Ruyu Song [1,2] and Yanbo Zhang [1,2,*]

1. School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China
2. Hebei International Joint Research Center for Mathematics and Interdisciplinary Science, Shijiazhuang 050024, China

* Correspondence: ybzhang@hebtu.edu.cn

Abstract: The (G_1, G_2)-online Ramsey game is a two-player turn-based game between a builder and a painter. Starting from an empty graph with infinite vertices, the builder adds a new edge in each round, and the painter colors it red or blue. The builder aims to force either a red copy of G_1 or a blue copy of G_2 in as few rounds as possible, while the painter's aim is the opposite. The online Ramsey number $\tilde{r}(G_1, G_2)$ is the minimum number of edges that the builder needs to win the (G_1, G_2)-online Ramsey game, regardless of the painter's strategy. Furthermore, we initiate the study of connected online Ramsey game, which is identical to the usual one, except that at any time the graph induced by all edges should be connected. In this paper, we show a general bound of the online Ramsey number of a matching versus a path and determine its exact value when the path has an order of three or four. For the connected version, we obtain all connected online Ramsey numbers of a matching versus a path.

Keywords: Ramsey number; online Ramsey number; connected online Ramsey number; paths; stars

check for updates

Citation: Song, R.; Zhang, Y. Online and Connected Online Ramsey Numbers of a Matching versus a Path. *Symmetry* **2022**, *14*, 2277. https://doi.org/10.3390/sym14112277

Academic Editor: Michel Planat

Received: 11 October 2022
Accepted: 27 October 2022
Published: 31 October 2022

1. Introduction

Let G_1 and G_2 be two finite simple graphs. The (G_1, G_2)-*online Ramsey game* is a two-player turn-based game between a builder and a painter. It begins from an empty graph with infinite vertices. In each round, the builder draws an edge joining two nonadjacent vertices, and the painter immediately colors it red or blue. The builder wins the game if there is either a red copy of G_1 or a blue copy of G_2, while the painter's goal is to delay the builder's victory in as many rounds as possible. We are interested in the minimum number of rounds that the builder can win the game. Formally speaking, the *online Ramsey number* $\tilde{r}(G_1, G_2)$ is the minimum number of edges that the builder needs to guarantee a win in the (G_1, G_2)-online Ramsey game, regardless of the painter's strategy.

The online Ramsey game was first introduced by Beck [1], whereas the online Ramsey number owes its name to Kurek and Ruciński [2]. The number can be viewed as an online version of the size Ramsey number, which is defined as follows. We write $G \to (G_1, G_2)$ if for any partition (E_1, E_2) of $E(G)$, either $G_1 \subseteq E_1$ or $G_2 \subseteq E_2$. The *Ramsey number* $r(G_1, G_2)$ and the *size Ramsey number* $\hat{r}(G_1, G_2)$ are the smallest number of vertices and edges, respectively, in a graph G satisfying $G \to (G_1, G_2)$. It follows that $\tilde{r}(G_1, G_2) \leq \hat{r}(G_1, G_2) \leq \binom{r(G_1,G_2)}{2}$, and hence $\tilde{r}(G_1, G_2)$ is well-defined.

Now we introduce some graphs that will be used in the sequel. A complete graph K_m is a graph on m vertices such that any two vertices are adjacent. A path P_m is a graph on m vertices, say $v_1, v_2, \ldots, v_m$, such that v_i is adjacent to v_{i+1} for all i with $1 \leq i \leq m-1$. If v_m is also adjacent to v_1, this is a cycle of length m, denoted by C_m. A matching nK_2 is a graph with n edges such that any two edges share no endpoints. We use [3] for terminology and notation not defined here.

The online Ramsey theory for graphs has been well studied. It is true that $\tilde{r}(G_1, G_2) = \tilde{r}(G_2, G_1)$ by symmetry. If both G_1 and G_2 are complete graphs, we refer the reader to [2,4–6]. Determining the exact values of online Ramsey numbers $\tilde{r}(K_m, K_n)$ has proved to be even more difficult than determining the classical Ramsey numbers exactly. Only two nontrivial ones were obtained up to symmetries: $\tilde{r}(K_3, K_3) = 8$ [2] and $\tilde{r}(K_3, K_4) = 17$ [6]. For sparse graphs, the online Ramsey numbers involving paths, stars, trees, and cycles have been studied [7–14]. If G_1 is a small fixed graph and G_2 is a class of sparse graphs, most results of $\tilde{r}(G_1, G_2)$ are upper or lower bounds, between which there is a large gap. The only known exact values of this type are $\tilde{r}(P_3, P_\ell)$ and $\tilde{r}(P_3, C_\ell)$.

Theorem 1 (Cyman et al. [8])**.** *We have* $\tilde{r}(P_3, P_\ell) = \lceil 5(\ell-1)/4 \rceil$ *for* $\ell \geq 3$; $\tilde{r}(P_3, C_\ell) = \lceil 5\ell/4 \rceil$ *for* $\ell \geq 5$; *and* $\tilde{r}(P_3, C_\ell) = \ell + 2$ *for* $\ell = 3, 4$.

We derive more online Ramsey numbers by considering a matching versus a path. Before that, let us first review the corresponding results of its Ramsey number and size Ramsey number. Faudree and Schelp [15] calculated the Ramsey numbers of all linear forests in 1976. In particular, $r(nK_2, P_m) = \max\{2n + \lceil m/2 \rceil - 1, n + m - 1\}$. However, its size Ramsey number is far from being completely confirmed. Erdős and Faudree [16] obtained the exact value of $\hat{r}(nK_2, P_m)$ when $m \leq 5$. They showed a general bound for other cases.

Theorem 2 (Erdős and Faudree [16])**.** *For* $n \geq 1$, *we have* $\hat{r}(nK_2, P_3) = 2n$, $\hat{r}(nK_2, P_4) = \lceil 5n/2 \rceil$, $\hat{r}(nK_2, P_5) = 3n + \epsilon$, *where* $\epsilon = 0$ *if* n *is even and* $\epsilon = 1$ *otherwise.*

Theorem 3 (Erdős and Faudree [16])**.** *For a given integer* n *with* $n \geq 3$, *there are positive constants* c_1 *and* c_2 *such that for all* $m \geq 3$,

$$m + c_1\sqrt{m} \leq \hat{r}(nK_2, P_m) \leq m + c_2\sqrt{m}.$$

Now we turn to exploring the online Ramsey number $\tilde{r}(nK_2, P_m)$. If a P_m has at most four vertices, its exact value can be obtained. Otherwise, we obtain general lower and upper bounds where the gap is not very large. Particularly, if $m \geq 5$ and $m = o(n)$, then $\tilde{r}(nK_2, P_m) = 2n + o(n)$.

Theorem 4. *For* $n \geq 2$, *we have* $\tilde{r}(nK_2, P_3) = \lceil 3n/2 \rceil$ *and* $\tilde{r}(nK_2, P_4) = \lceil 9n/5 \rceil$.

Theorem 5. *For* $n \geq 2$ *and* $m \geq 5$, *we have*

$$2n + \min\{n/2, (m-5)/2\} \leq \tilde{r}(nK_2, P_m) \leq 2n + m - 4.$$

In addition, the upper bound can be attained for $n = 2, 3$.

Furthermore, we initiate the study of a variant called the *connected online Ramsey game* by adding the restriction that at any round, the graph induced by all edges should be connected. In other words, except for the first edge, the builder is not allowed to draw an edge joining two isolated vertices. This notion can be viewed as a special case of the definition in [17], where the constructed graph is always asked to be in a prescribed class of graphs at any time. The *connected online Ramsey number* $\tilde{r}_c(G_1, G_2)$ is the smallest possible number of edges that the builder needs to guarantee a win in the connected (G_1, G_2)-online Ramsey game. It can be seen that $\tilde{r}(G_1, G_2) \leq \tilde{r}_c(G_1, G_2)$.

Here we are concerned with the connected online Ramsey number of a matching versus a path. Even though the corresponding online Ramsey problem has not been solved yet, its connected online Ramsey number can be determined completely. We have the following surprising result.

Theorem 6. *For* $n \geq 1$ *and* $m \geq 2$, *we have* $\tilde{r}_c(nK_2, P_m) = 2n + m - 3$.

The remainder of this paper is organized as follows. Section 2 includes two lemmas that will be used for the online Ramsey result. Proofs of Theorems 4 and 5 are presented in Section 3 and Section 4, respectively. Section 5 shows the connected online Ramsey result, which is Theorem 6.

2. Lemmas

We first state a lemma given by Cyman, Dzido, Lapinskas, and Lo [8] (Lemma 16), which is crucial for the proofs of lower bounds.

Lemma 1. *Let $m, \ell \in \mathbb{N}$ with $m \geq 3$. Let B be a forest that has ℓ edges, no isolated vertices, and does not contain P_m as a subgraph. Let X be the set of all endpoints of P_{m-1} in B. Then*

$$|V(B)| + |X| \leq \begin{cases} 4\ell & \text{if } m = 3, \\ 5\ell/2 & \text{if } m = 4, \\ 2\ell & \text{if } m \geq 5. \end{cases}$$

Moreover, if $m \geq 5$, and there exists an edge e such that $B + e$ contains a P_m, then $|V(B)| + |X| \leq 2\ell - m + 5$.

The following symmetric edge-colored path appears a couple of times in the sequel. A path with at least four vertices is called an *$rb\cdots br$ path* if its two pendant edges are red and all internal edges are blue. That is, $P_t := v_1 v_2 \dots v_t$ is an $rb\cdots br$ path if $t \geq 4$, $v_1 v_2$ and $v_{t-1} v_t$ are red and all other edges are blue.

Lemma 2. *During the (nK_2, P_m)-game, to avoid a blue P_m, every blue path P_ℓ with $2 \leq \ell \leq m-1$ can be lengthened to an $rb\cdots br$ path, which has at most $m+1$ vertices.*

Proof. Let v_1 and v_2 denote the two ends of P_ℓ. The builder extends P_ℓ to a longer path, whose length is increased by one in each round. He first joins v_1 to a new vertex u_1. If $v_1 u_1$ is blue, he joins u_1 to a new vertex u_2. The builder continues lengthening the path as above until the first red edge appears. We denote the path by $P_s := u_k u_{k-1} \cdots u_1 v_1 P_\ell v_2$, where the edge $u_k u_{k-1}$ is red, and all other edges are blue. Then, beginning from v_2, the builder continues extending P_s to a longer path until the second red edge appears. Assume now the path is $u_k P_s v_2 v_3 \cdots v_t$, where only two edges $u_k u_{k-1}$ and $v_{t-1} v_t$ are red. This path is an $rb\cdots br$ path. Since we need to avoid a blue P_m, this $rb\cdots br$ path can always be obtained and has at most $m+1$ vertices. □

3. Exact Values of $\tilde{r}(nK_2, P_3)$ and $\tilde{r}(nK_2, P_4)$

We first show $\tilde{r}(nK_2, P_3) = \lceil 3n/2 \rceil$. In each round, the builder chooses a new edge such that the uncovered edges form a matching. The procedure stops when $p + 2q \geq n$. Here, p denotes the number of red edges, and q denotes the number of blue edges. Since $p + 2(q-1) \leq n-1$, we have $q \leq \lceil n/2 \rceil$. Let $v_1, v_2, \dots, v_{2q}$ denote the ends of all blue edges. In the next ith round for $1 \leq i \leq n-p$, the builder draws an edge joining v_i to a new vertex. Since $n - p \leq 2q$, this process can be realized. If at least one edge in these $n-p$ rounds is blue, then we have a blue P_3. If all edges in these $n-p$ rounds are red, then we obtain a red matching with n edges. Thus, $\tilde{r}(nK_2, P_3) \leq p + q + (n-p) \leq \lceil 3n/2 \rceil$.

The painter uses the strategy that she will always color an edge blue unless doing so would create a blue P_3. Therefore, every red edge is forced to avoid a blue P_3. Hence, each red edge shares a common end with a blue edge. If there are $\lceil n/2 \rceil - 1$ blue edges, they can force a red matching with at most $2\lceil n/2 \rceil - 2$ edges, which is less than n. That is to say, to force a red nK_2, at least $\lceil n/2 \rceil$ blue edges are needed. Thus, $\tilde{r}(nK_2, P_3) \geq n + \lceil n/2 \rceil = \lceil 3n/2 \rceil$.

Now we turn to prove that $\tilde{r}(nK_2, P_4) \leq \lceil 9n/5 \rceil$ for $n \geq 2$. To show the upper bound, the builder first constructs some special components, each of which is either a red P_2 or an

$rb\cdots br$ path as defined before Lemma 2. In the beginning, and each time after a red P_2 or an $rb\cdots br$ path has been constructed, the builder forces another red P_2 or another $rb\cdots br$ path as follows. The builder joins two isolated vertices, say, v_1, v_2. If v_1v_2 is colored red, it is a red P_2 that we need. If v_1v_2 is colored blue, by Lemma 2, it can be lengthened to an $rb\cdots br$ path, which has at most five vertices. For simplicity, the $rb\cdots br$ paths of orders four and five are denoted by *rbr* path and *rbbr* path, respectively. We see that each component is a red P_2, an *rbr* path, or an *rbbr* path. Additionally, all red edges form a matching.

The procedure stops when there are either $\lceil 9n/5 \rceil$ edges or $\lceil 4n/5 \rceil$ blue edges. In the former case, there are at most $\lceil 4n/5 \rceil$ blue edges and thus at least n red edges. By the builder's strategy, all red edges form a matching. Thus, we have a red nK_2. Therefore, we need only consider the latter case when $\lceil 4n/5 \rceil$ blue edges have appeared, but the total number of edges is less than $\lceil 9n/5 \rceil$.

Because the last appeared edge is blue, by Lemma 2 and the builder's strategy, the last component is either a blue P_2, a blue P_3, or a P_4 with one pendant edge red and two other edges blue. For this P_4 and P_3, the builder extends them to an *rbbr* path in the next one or two steps, respectively. All edges in the one or two steps have to be red to avoid a blue P_4. If the number of red edges is at least n after this extension, our proof is done. Now let p denote the number of components that is a red P_2. Additionally, let q, s denote the number of *rbr*-paths and *rbbr*-paths, respectively. We see that the number of red edges is $p + 2q + 2s$ up to now, which is assumed to be less than n. Since every *rbr* path contains one blue edge, and every *rbbr* path contains two blue edges, we have $q + 2s + 1 = \lceil 4n/5 \rceil$ if the last component is a blue P_2, and $q + 2s = \lceil 4n/5 \rceil$ otherwise. We see at once that $s \geq 1$, since otherwise $2q \geq n$, and thus we already have a red nK_2.

Each *rbbr* path has a center, which is the vertex incident to two blue edges. Let C be the set of centers from all *rbbr* paths. Then $|C| = s$. Set $t = n - (p + 2q + 2s)$, which is the number of red edges we still need to form an nK_2. If the last component is not a blue P_2, then $2p + 4q + 5s \geq 5(q + 2s)/2 = 5\lceil 4n/5 \rceil/2 \geq 2n$, which implies $2t \leq s$. In the next t rounds, the builder draws a matching with t edges, each of which has both ends in C. If one of the t edges is blue, we have a blue P_4, and our proof is done. If all t edges are red, there are totally n red edges which form a red nK_2. If the last component is a blue P_2, denoted by w_1w_2, then $2p + 4q + 5s + 5/2 \geq 5(q + 2s + 1)/2 = 5\lceil 4n/5 \rceil/2 \geq 2n$. It follows that $2p + 4q + 5s + 2 \geq 2n$, which implies $2t \leq s + 2$. In the next t rounds, the builder draws a matching with t edges such that each edge has both ends in $C \cup \{w_1, w_2\}$, and this matching does not contain w_1w_2. Since $s \geq 2t - 2$ and $s \geq 1$, this matching can always be obtained. If one of the t edges is blue, we have a blue P_4. If all t edges are red, there are totally n red edges which form a red nK_2. In both cases, $\tilde{r}(nK_2, P_4) \leq \lceil 9n/5 \rceil$.

To prove $\tilde{r}(nK_2, P_4) \geq \lceil 9n/5 \rceil$, the painter uses the strategy that she will always color an edge blue unless doing so would create either a blue P_4 or a blue triangle. Therefore, all blue edges form a forest, denoted by B, and every red edge is forced to avoid a blue P_4 or a blue triangle. When the builder wins the game, a red nK_2 must appear. Let X be the set of all endpoints of P_3 in B, and denote by ℓ the number of blue edges. From the painter's strategy, we see that any red edge has either at least one end in X or both ends in $V(B) \setminus X$. Thus, $|X| + |V(B) \setminus X|/2 \geq n$, which implies $2n \leq |V(B)| + |X|$. By Lemma 1, $|V(B)| + |X| \leq 5\ell/2$. Thus, $\ell \geq 4n/5$, and so $\tilde{r}(nK_2, P_4) \geq n + \ell \geq \lceil 9n/5 \rceil$.

4. A General Bound of $\tilde{r}(nK_2, P_m)$

We now show a general bound of $\tilde{r}(nK_2, P_m)$. Here, $n \geq 2$, since the result $\tilde{r}(K_2, P_m) = m - 1$ is trivial. We also have $m \geq 5$, since the cases $m = 3, 4$ have been given in Section 3.

We first bound $\tilde{r}(nK_2, P_m)$ from below. The painter uses a blocking strategy which is defined as follows. Denote by B_i the graph induced by all uncovered blue edges immediately before the ith move of the game. The builder then chooses the ith edge e_i. If $B_i + e_i$ contains no path P_m and no cycle, then the painter colors e_i blue. Otherwise, the painter colors e_i red. In this way, a blue path P_m will never appear, and every red edge is forced to appear to avoid either a blue P_m or a blue cycle.

When the builder wins the game, a red nK_2 must appear. Let B be the graph induced by all blue edges, let X be the set of all endpoints of P_{m-1} in B, and denote by ℓ the number of blue edges. From the painter's strategy, we see that any red edge has either at least one end in X or both ends in $V(B) \setminus X$. Thus, $|X| + |V(B) \setminus X|/2 \geq n$, which implies $2n \leq |V(B)| + |X|$. If there exists an edge e such that $B + e$ contains a P_m, by Lemma 1, $|V(B)| + |X| \leq 2\ell - m + 5$. It follows that $2n \leq 2\ell - m + 5$, and hence, $\ell \geq n + (m-5)/2$. Therefore, $\tilde{r}(nK_2, P_m) \geq 2n + (m-5)/2$. If such an edge e does not exist, then every red edge is forced to avoid a blue cycle. Let $C_1, C_2, \ldots, C_k$ be the components of B. Assume that C_i has ℓ_i edges for $1 \leq i \leq k$. If $\ell_i = 1$, no red edge with two ends in C_i can be forced. If $\ell_i \geq 2$, the number of red edges with two ends in C_i is at most $2\ell_i/3$. Thus, the total number of red edges is at most $2\ell/3$. To force a red nK_2, we need at least $3n/2$ blue edges. Therefore, $\tilde{r}(nK_2, P_m) \geq 5n/2$, and so $\tilde{r}(nK_2, P_m) \geq 2n + \min\{n/2, (m-5)/2\}$.

To prove the upper bound $\tilde{r}(nK_2, P_m) \leq 2n + m - 4$, we show the following stronger claim.

Claim 1. *Let m and n be integers with $n \geq 2, m \geq 5$. For every positive integer ℓ with $\ell \leq m-1$, if there is already a blue path P_ℓ, then in the next $2n + m - \ell - 3$ rounds, the builder can force either a red nK_2 or a blue P_m.*

Proof. First, consider the case $n = 2$. If $\ell \geq 2$, from Lemma 2, we see that the builder can lengthen the P_ℓ to an $rb \cdots br$ path, which has at most $m + 1$ vertices. That is to say, in the next $m - \ell + 1$ rounds, a red $2K_2$ and a blue P_m cannot be both avoided. If $\ell = 1$, the builder picks one edge, v_1v_2. If v_1v_2 is colored blue, then the proof is the same as above. If v_1v_2 is colored red, since $\tilde{r}(K_2, P_m) = m - 1$, in the next $m - 1$ rounds, the builder can force either a red $2K_2$ containing v_1v_2 or a blue P_m. The case $n = 2$ is done.

Now we apply induction on n. Assume that Claim 1 holds for $n = k - 1$, where $k \geq 3$. We show that it also holds for $n = k$. We run a similar argument as in the case $n = 2$. Beginning from an end of the blue P_ℓ, the builder extends it to a longer path, whose length is increased by one in each round. The procedure stops when the first red edge appears. Assume that the path is $P_t : v_1v_2 \ldots v_t$, with the edge $v_{t-1}v_t$ red and all others blue. Then, $t \leq m$, since otherwise there is already a blue P_m. If $t \geq 3$, we leave the red edge $v_{t-1}v_t$ alone and consider the blue segment $v_1v_2 \ldots v_{t-2}$. By the inductive hypothesis, in the next $2(k-1) + m - (t-2) - 3$ rounds, the builder can force either a red $(k-1)K_2$ which together with $v_{t-1}v_t$ forms a red kK_2 or a blue P_m. The total number of rounds is $2k + m - \ell - 3$, proving our claim. If $t = 2$, then $\ell = 1$. We leave the red edge $v_{t-1}v_t$ alone and consider an isolated vertex as a blue P_1. Using the inductive hypothesis again, in the required rounds, the builder can force either a red $(k-1)K_2$, which together with $v_{t-1}v_t$ forms a red kK_2 or a blue P_m. This proves our claim. □

Since any vertex can be viewed as a blue path P_1, the upper bound $\tilde{r}(nK_2, P_m) \leq 2n + m - 4$ is a particular case of Claim 1 when $\ell = 1$.

We are left to prove that the upper bound can be attained for $n = 2, 3$, that is, to show $\tilde{r}(nK_2, P_m) \geq 2n + m - 4$ for $n = 2, 3$. For the lower bound of $\tilde{r}(2K_2, P_m)$, the painter would color $m - 2$ edges blue and one edge red during the first $m - 1$ rounds. There is neither a blue P_m nor a red $2K_2$. Thus, $\tilde{r}(2K_2, P_m) \geq m$. For the lower bound of $\tilde{r}(3K_2, P_m)$, the painter would color the first $m - 2$ edges blue. We now consider three cases. If these blue edges form a path P_{m-1}, we use v_1, v_2 to denote its two ends. In the next three rounds, if a new edge has at least one end in $\{v_1, v_2\}$, the painter colors it red; if a new edge has no end in $\{v_1, v_2\}$, the painter colors the first such edge blue and the others red. It is easy to check that both red $3K_2$ and blue P_m do not exist. If the first $m - 2$ edges form two blue paths P_x and P_y such that $x + y = m$, $x \geq 2$, $y \geq 2$, then we use v_3, v_4 to denote the ends of P_x, and v_5, v_6 to denote that of P_y. In the next three rounds, if a new edge has both ends in $\{v_3, v_4, v_5, v_6\}$, the painter colors it red; if a new edge has at least one end not in $\{v_3, v_4, v_5, v_6\}$, the painter colors the first such edge blue and the others red. Again, both red $3K_2$ and blue P_m do not exist. If neither of the above two cases happens, the painter colors the $(m-1)$th edge blue. It follows that there is no blue P_m. The painter then colors the next two edges red. Thus, $\tilde{r}(3K_2, P_m) \geq m+2$.

5. Exact Value of $\tilde{r}_c(nK_2, P_m)$

In this section, we prove that $\tilde{r}_c(nK_2, P_m) = 2n+m-3$ for $m \geq 2$. For inductive reasons, it will be easier to prove the following stronger claim to indicate the upper bound.

Claim 2. *For every positive integer ℓ with $\ell \leq m-1$ and $m \geq 2$, if there is a blue path P_ℓ, then in the next $2n+m-\ell-2$ rounds, the builder can construct a connected graph which forces either a red nK_2 or a blue P_m.*

Since any vertex can be viewed as a blue path P_1, the upper bound is a particular case of Claim 2 when $\ell = 1$.

Proof. We apply induction on n. If $n = 1$, the builder extends the blue path P_ℓ to a path P_m in $m-\ell$ moves. The painter is forced to color either a red K_2 or a blue P_m, proving our claim. Assume that Claim 2 holds for $n = k-1$. We show that it also holds for $n = k$.

We consider two cases: $\ell \geq 2$ or $\ell = 1$. If $\ell \geq 2$, then $m \geq 3$, since otherwise our proof is done. Beginning from an end of the blue P_ℓ, the builder extends it to a longer path, whose length is increased by one in each round. The procedure stops when the first red edge appears. Suppose now the path is $P_t : v_1v_2 \ldots v_t$ with the edge $v_{t-1}v_t$ red and all other edges blue, where $3 \leq \ell+1 \leq t \leq m$. We leave the red edge $v_{t-1}v_t$ alone and consider the blue segment $v_1v_2 \ldots v_{t-2}$, where $1 \leq t-2 \leq m-2$. By the inductive hypothesis, in the next $2(k-1)+m-(t-2)-2$ rounds, the builder can force either a red $(k-1)K_2$, which together with $v_{t-1}v_t$ forms a red kK_2, or a blue P_m. The total number of rounds is $2k+m-\ell-2$, proving our claim.

If $\ell = 1$, P_ℓ is a single vertex, denoted by v_1. The builder chooses one edge v_1v_2. If v_1v_2 is colored blue, then the proof is the same as the above case. If v_1v_2 is colored red, the builder joins v_2 to a new vertex v_3. We leave the red edge v_1v_2 alone and consider the blue path $P_1 = v_3$. By the inductive hypothesis, in the next $2(k-1)+m-1-2$ rounds, the builder can force either a red $(k-1)K_2$, which together with v_1v_2 forms a red kK_2, or a blue P_m. The total number of rounds is $2k+m-\ell-2$, proving our claim. □

To prove the lower bound, consider the following strategy for the painter. In the first $2n-2$ rounds, the painter colors all edges red. Since these edges form a connected graph, there are at most $2n-1$ vertices that are incident to red edges. Because an nK_2 occupies $2n$ vertices, the graph does not contain a red nK_2. In the next $m-2$ rounds, the painter colors all edges blue. There is no blue P_m, since a P_m has $m-1$ edges. Thus, $\tilde{r}_c(nK_2, P_m) \geq 2n+m-3$ for $m \geq 2$.

Author Contributions: Conceptualization, Y.Z.; methodology, Y.Z.; validation, R.S.; formal analysis, R.S. and Y.Z.; investigation, R.S.; resources, R.S.; writing—original draft preparation, R.S.; writing—review and editing, Y.Z.; supervision, Y.Z.; funding acquisition, Y.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The research was supported by the National Natural Science Foundation of China (Grant No. 11601527 and Grant No. 11971011).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors appreciate the anonymous reviewers sincerely for their valuable suggestions, which improved the original manuscript.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Beck, J. Achievement games and the probabilistic method. In *Combinatorics, Paul Erdős Is Eighty*; Bolyai Society of Mathematical Studies; János Bolyai Mathematical Society: Budapest, Hungary, 1993; Volume 1, pp. 51–78.
2. Kurek, A.; Ruciński, A. Two variants of the size Ramsey number. *Discuss. Math. Graph Theory* **2005**, *25*, 141–149. [CrossRef]
3. Bondy J.A.; Murty U.S.R. *Graph Theory*; Springer: Berlin/Heidelberg, Germany, 2008.
4. Conlon, D. On-line Ramsey numbers. *SIAM J. Discrete Math.* **2009**, *23*, 1954–1963. [CrossRef]
5. Conlon, D.; Fox, J.; Grinshpun, A.; He, X. Online Ramsey numbers and the subgraph query problem. In *Building Bridges II*; Bárány, I., Katona, G. O. H., Sali, A., Eds.; Springer, Berlin/Heidelberg, Germany, 2019; pp. 159–194.
6. Prałat, P. $\overline{R}(3;4) = 17$. *Electron. J. Combin.* **2008**, *15*, R67. [CrossRef]
7. Cyman J.; Dzido T. A note on on-line Ramsey numbers for quadrilaterals. *Opuscula Math.* **2014**, *34*, 463–468. [CrossRef]
8. Cyman J.; Dzido T.; Lapinskas J.; Lo, A. On-line Ramsey numbers of paths and cycles. *Electron. J. Combin.* **2015**, *22*, 15. [CrossRef]
9. Dybizbański, J.; Dzido, T.; Zakrzewska, R. On-line Ramsey numbers for paths and short cycles. *Discrete Appl. Math.* **2020**, *282*, 265–270. [CrossRef]
10. Dzido, T.; Zakrzewska, R. A note on on-line Ramsey numbers for some paths. *Mathematics* **2021**, *9*, 735. [CrossRef]
11. Grytczuk, J.; Kierstead, H.; Prałat, P. On-line Ramsey numbers for paths and stars. *Discrete Math. Theor. Comput. Sci.* **2008**, *10*, 63–74. [CrossRef]
12. Prałat, P. A note on small on-line Ramsey numbers for paths and their generalization. *Australas. J. Combin.* **2008**, *40*, 27–36.
13. Prałat, P. A note on off-diagonal small on-line Ramsey numbers for paths. *Ars Combin.* **2012**, *107*, 295–306.
14. Mohd Latip, F.N.N.B.; Tan, T.S. A note on on-line Ramsey numbers of stars and paths. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 3511–3521. [CrossRef]
15. Faudree, R.J.; Schelp, R.H. Ramsey numbers for all linear forests. *Discrete Math.* **1976**, *16*, 149–155. [CrossRef]
16. Erdős, P.; Faudree, R.J. Size Ramsey numbers involving matchings. In *Finite and Infinite Sets*; Elsevier B.V.: Amsterdam, The Netherlands, 1984; pp. 247–264.
17. Grytczuk, J.; Hałuszczak, M.; Kierstead, H.A. On-line Ramsey theory. *Electron. J. Combin.* **2004**, *11*, R57. [CrossRef]