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Novel Identities of Bernoulli Polynomials Involving Closed Forms for Some Definite Integrals

Waleed Mohamed Abd-Elhameed [1,*] and Amr Kamel Amin [2,3]

1 Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
2 Department of Basic Sciences, Adham University College, Umm AL-Qura University, Mecca 24382, Saudi Arabia; akgadelrab@uqu.edu.sa
3 Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef 65211, Egypt
* Correspondence: waleed@cu.edu.eg

Abstract: This paper presents new results of Bernoulli polynomials. New derivative expressions of some celebrated orthogonal polynomials and other polynomials are given in terms of Bernoulli polynomials. Hence, some new connection formulas between these polynomials and Bernoulli polynomials are also deduced. The linking coefficients involve hypergeometric functions of different arguments that can be summed in some cases. Formulas that express some celebrated numbers in terms of Bernoulli numbers are displayed. Based on the new connection formulas between different polynomials and Bernoulli polynomials, along with some well-known integrals involving these polynomials, new closed forms for some definite integrals are given.

Keywords: Bernoulli polynomials; Bernoulli numbers; orthogonal polynomials; connection formulas; hypergeometric functions

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1. Introduction

Different areas of practical mathematics, including mathematical physics and numerical analysis, place great emphasis on special functions in general and orthogonal polynomials in particular (see, for example, [1–3]). Therefore, many articles have been devoted to investigating these functions and polynomials. The classical Jacobi polynomials are among the most significant special functions because of their significance in numerical analysis and approximation theory. It is well-known that Jacobi polynomials involve two parameters, so they have several special polynomials. There are six famous special classes of polynomials of the class of Jacobi polynomials. All these six classes of polynomials have their parts in approximating solutions of different types of differential and integral equations, see for example [4–8].

The connection problems between different polynomials and their related numbers are important problems. These problems have been investigated by many authors. For example, the connection problems between orthogonal polynomials were investigated in [9]. Another study was presented in [10]. A comprehensive study for some formulas concerned with the classical orthogonal polynomials involving connection formulas was given in [11]. The authors of [12] established connection formulas between Fibonacci polynomials and Chebyshev polynomials. In [13], the authors derived other connection formulas between Chebyshev polynomials of the fifth kind with some other polynomials. The authors of [14] studied a connection problem for sums of finite products of Chebyshev polynomials of the third and fourth kinds.

The derivative formulas of different polynomials in terms of their original ones can also be classified as connection problems. The solutions to these problems are very useful in obtaining spectral solutions to different types of differential equations. In this direction, the authors of [15] found explicit expressions for the derivatives for the third- and fourth-kinds of Chebyshev polynomials. Additionally, they utilized them to solve certain differential equations. Additionally, the authors of [16] obtained new explicit formulas for certain Jacobi polynomials that generalize the third-kind Chebyshev polynomials and employed them to obtain spectral solutions for even-order BVPs. In [17], Abd-Elhameed found explicit formulas for the derivatives of Chebyshev polynomials of the sixth-kind and employed them to develop a numerical solution to a certain type of Burgers' equation.

We are aware that Bernoulli polynomials and numbers are frequently used in mathematics, As a result, various old and recent investigations have focused on these polynomials and numbers. In this direction, in [18], the authors developed some integrals of products of Bernoulli polynomials. Other identities regarding Bernoulli polynomials were developed in [19]. Relations between Fibonacci and Bernoulli numbers were displayed in [20]. In [21], the author obtained some recurrences for Bernoulli polynomials and their associated numbers. Investigations concerning the generalizations of Bernoulli polynomials and their related numbers have also been studied by many authors. For example, the authors of [22] found an explicit formula for the generalized Bernoulli polynomials. In [23], the author found formulas for the sums of products of the generalized Bernoulli polynomials. For some other contributions regarding Bernoulli polynomials and some of their related polynomials, one can be referred to [24–28].

The principal purpose of this article is to solve connection problems between different polynomials and the Bernoulli polynomials. In fact, we will establish new formulas for the high-order derivatives of various polynomials including the classical orthogonal polynomials, and this will yield new connection formulas between the different polynomials with Bernoulli polynomials. We will show that several orthogonal polynomials can be written as simple combinations of Bernoulli polynomials. In addition, and as an important application to the connection formulas, four new general formulas for definite integrals will be displayed. These general formulas produce a variety of new formulas for definite integrals of different polynomials and also formulas for definite integrals of the products of different polynomials with Bernoulli polynomials. We do believe that most of these formulas are new and they may be useful. This motivates our interest in such a study.

The paper is organized as follows. Section 2 reviews some fundamental properties of Bernoulli polynomials and some other celebrated polynomials including some orthogonal polynomials. In Section 3, new expressions as combinations of Bernoulli polynomials are given to express the derivatives of various polynomials including the classical orthogonal polynomials. In addition, connection problems between different polynomials and Bernoulli polynomials are solved in this section. Section 4 is devoted to obtaining derivative expressions of Bernoulli polynomials in terms of different polynomials. Section 5 focuses on utilizing the derived connection formulas to find four general formulas of certain definite integrals of some polynomials in closed forms. Some concluding remarks are given in Section 6.

2. An Overview of Some Celebrated Polynomials

We present in this section some fundamental properties of Bernoulli polynomials as well as some characteristics of some celebrated polynomials.

2.1. Some Properties of Bernoulli Polynomials

Bernoulli polynomials $B_\ell(x)$ may be generated using the following generating function [29]:

$$\frac{t\,e^{x\,t}}{e^t-1}=\sum_{\ell=0}^{\infty} B_\ell(x)\,\frac{t^\ell}{\ell!},\quad |t|<2\,\pi,$$

and their associated numbers are given by:

$$B_\ell=B_\ell(0)=(-1)^\ell B_\ell(1). \tag{1}$$

Bernoulli polynomials can be explicitly given by:

$$B_\ell(x) = \sum_{r=0}^{\ell} \binom{\ell}{r} B_r x^{\ell-r}, \tag{2}$$

where B_r are the well-known Bernoulli numbers.

The inversion formula of Bernoulli polynomials is given explicitly by:

$$x^i = \sum_{r=0}^{j} \frac{\binom{j+1}{j-r}}{j+1} B_{j-r}(x). \tag{3}$$

In many books and tables on this subject, there are numerous fascinating and useful formulas regarding these polynomials and numbers, see, for example, [30,31].

2.2. Essential Properties of Some Celebrated Polynomials

This section is confined to presenting some properties of symmetric and non-symmetric classes of polynomials. Let $\phi_k(x)$, $k \geq 0$ be a symmetric polynomial of degree k and $\psi_k(x)$, $k \geq 0$ be a non-symmetric polynomial of degree k. These polynomials have, respectively, the following expressions:

$$\phi_k(x) = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} A_{m,k}\, x^{k-2m}, \tag{4}$$

$$\psi_k(x) = \sum_{m=0}^{k} B_{m,k}\, x^{k-m}. \tag{5}$$

Among the most important orthogonal polynomials are the normalized Jacobi polynomials [32,33]. These polynomials can be expressed in the form:

$$R_\ell^{(\gamma,\mu)}(x) = {}_2F_1\left(\begin{matrix} -\ell, \ell+\gamma+\mu+1 \\ \gamma+1 \end{matrix} \,\middle|\, \frac{1-x}{2} \right).$$

Six celebrated kinds of polynomials can be extracted from $R_n^{(\gamma,\mu)}(x)$ as special cases. In fact, the polynomials $T_\ell(x), U_\ell(x), V_\ell(x)$ and $W_\ell(x)$ that represent, respectively, the first-, second-, third-, and fourth-kinds of Chebyshev polynomials, and the polynomials $P_\ell(x), C_\ell^{(\lambda)}(x)$ that represent, respectively, the Legendre and ultraspherical polynomials are special ones of Jacobi polynomials. Furthermore, the following relations hold:

$$\begin{aligned} T_\ell(x) &= R_\ell^{(-\frac{1}{2},-\frac{1}{2})}(x), & U_\ell(x) &= (\ell+1)\, R_\ell^{(\frac{1}{2},\frac{1}{2})}(x), \\ V_\ell(x) &= R_\ell^{(-\frac{1}{2},\frac{1}{2})}(x), & W_\ell(x) &= (2\ell+1)\, R_\ell^{(\frac{1}{2},-\frac{1}{2})}(x), \\ P_\ell(x) &= R_\ell^{(0,0)}(x), & C_\ell^{(\lambda)}(x) &= R_\ell^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \end{aligned}$$

Note that $T_\ell(x), U_\ell(x), P_\ell(x)$ and $C_\ell^{(\lambda)}(x)$ are symmetric polynomials, so they can be expressed as in (4); however, the polynomials $V_\ell(x)$ and $W_\ell(x)$ are non-symmetric polynomials, so they can be represented as in (5).

For an extensive survey on Jacobi polynomials and their different classes, one can consult the useful books of Andrews et al. [34] and Mason and Handscomb [35].

We denote by $\tilde{R}_j^{(\alpha,\beta)}(x)$ the shifted Jacobi polynomials on $[0,1]$. These polynomials can be given explicitly as:

$$\tilde{R}_j^{(\alpha,\beta)}(x) = R_j^{(\alpha,\beta)}(2x-1), \tag{6}$$

and therefore, six special shifted polynomials of Jacobi polynomials can be extracted as special ones.

The shifted Jacobi polynomials defined in (6) can be explicitly represented as powers of x in the form [36]:

$$\tilde{R}_{\ell}^{(\alpha,\beta)}(x) = \frac{\ell!\,\Gamma(\alpha+1)}{\Gamma(\ell+\alpha+1)} \sum_{j=0}^{\ell} \frac{(-1)^j\,(\beta+1)_{\ell}\,(\alpha+\beta+1)_{2\ell-j}}{(\ell-j)!\,j!\,(\beta+1)_{\ell-j}\,(1+\alpha+\beta)_{\ell}}\,x^{\ell-j}. \tag{7}$$

There are other classes of non-symmetric polynomials such as the generalized Laguerre polynomials $L_{\ell}^{(\alpha)}(x)$ and Schröder polynomials $S_n(x)$. The polynomials $L_j^{(\alpha)}(x)$ have the following expression [37]:

$$L_j^{(\alpha)}(x) = \frac{\Gamma(j+\alpha+1)}{j!} \sum_{m=0}^{j} \frac{(-1)^{j-m}\binom{j}{m}}{\Gamma(j+\alpha-m+1)}\,x^{j-m}, \tag{8}$$

while the Schröder polynomials $S_n(x)$ have the following expression ([38]):

$$S_j(x) = \sum_{m=0}^{j} \frac{\binom{2m}{m}\binom{j+m}{j-m}}{m+1}\,x^m. \tag{9}$$

Here, we give examples of symmetric polynomials. Among the celebrated symmetric polynomials are the ultraspherical and Hermite polynomials (see [37]). Additionally, the Fibonacci and Lucas polynomials are symmetric ones (see, [39]). There are many generalizations of Fibonacci and Lucas polynomials. Recently, Abd-Elhameed et al. in [40] investigated two generalized Fibonacci and Lucas polynomials and introduced new formulas concerned with them. The generalized Fibonacci polynomials $F_j^{a,b}(x)$, $j \geq 0$ can be generated using the following recurrence relation:

$$F_j^{a,b}(x) = a\,x\,F_{j-1}^{a,b}(x) + b\,F_{j-2}^{a,b}(x), \quad F_0^{a,b}(x) = 1,\ F_1^{a,b}(x) = a\,x, \quad j \geq 2, \tag{10}$$

while the generalized Lucas polynomials $L_j^{c,d}(x)$ can be generated using the following recurrence relation:

$$L_j^{c,d}(x) = c\,x\,L_{j-1}^{c,d}(x) + d\,L_{j-2}^{c,d}(x), \quad L_0^{c,d}(x) = 2,\ L_1^{c,d}(x) = c\,x, \quad j \geq 2. \tag{11}$$

It is important to note that the two generalized classes of polynomials $F_j^{a,b}(x)$ and $L_j^{c,d}(x)$ involve various well-known polynomial classes (see, [40]). For instance, the Fibonacci polynomials $F_{j+1}(x)$ and Lucas polynomials $L_j(x)$ can be thought of as special cases $F_j^{a,b}(x)$ and $L_j^{c,d}(x)$. In fact, we have:

$$F_{j+1}(x) = F_j^{1,1}(x), \quad L_j(x) = L_j^{1,1}(x).$$

Additionally, the generalized polynomials $F_j^{a,b}(x)$ and $L_j^{c,d}(x)$ each have a power form representation that is given as [40]:

$$F_j^{a,b}(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \binom{j-r}{r}\,b^r\,a^{j-2r}\,x^{j-2r}, \quad j \geq 0,$$

$$L_j^{c,d}(x) = j \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{d^m\,c^{i-2m}\binom{j-r}{r}}{j-r}\,x^{j-2r}, \quad j \geq 1,$$

where the well-known floor function is represented by the symbol $\lfloor z \rfloor$.

We comment here on the generalized Fibonacci and generalized Lucas numbers defined as:

$$F_j^{a,b} = F_j^{a,b}(1), \qquad L_j^{c,d} = L_j^{c,d}(1). \tag{12}$$

3. New Derivative Formulas of Symmetric and Non-Symmetric Polynomials

This section focuses on establishing new Bernoulli polynomial expressions for the high-order derivatives of a few symmetric and non-symmetric polynomials. Formulas connecting various polynomials with Bernoulli polynomials are deduced as special cases of the derivative expressions.

3.1. Derivative Expressions for Some Non-Symmetric Polynomials

In this section, we provide expressions for a few well-known non-symmetric polynomials. To be more specific, Bernoulli polynomials will be used to express the derivatives of the shifted Jacobi, Laguerre, and Schröder polynomials.

Theorem 1. *Consider any two non-negative integers j and q such that $j \geq q$. The following is the expression of the qth derivative of the shifted Jacobi polynomials $\tilde{R}_j^{(\alpha,\beta)}(x)$ in terms of Bernoulli polynomials.*

$$\begin{aligned} D^q \tilde{R}_j^{(\alpha,\beta)}(x) &= \frac{j!\,\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+\beta+1)} \times \\ &\sum_{p=0}^{j-q} \frac{\Gamma(2j-p+\alpha+\beta)\big((-1)^p\,\Gamma(j-p+\alpha)\Gamma(j+\beta+1)+\Gamma(j+\alpha+1)\Gamma(j-p+\beta)\big)}{(p+1)!\,(j-p-q)!\,\Gamma(j-p+\alpha)\,\Gamma(j-p+\beta)}\, B_{j-q-p}(x). \end{aligned} \tag{13}$$

Proof. The well-known analytic form of the shifted Jacobi polynomials in (7) enables one to express their derivatives in the form:

$$D^q \tilde{R}_j^{(\alpha,\beta)}(x) = \frac{j!\,\Gamma(\alpha+1)\Gamma(j+\beta+1)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+\beta+1)} \sum_{r=0}^{j-q} \frac{(-1)^r\,\Gamma(2j-r+\alpha+\beta+1)}{(j-q-r)!\,r!\,\Gamma(j-r+\beta+1)}\, x^{j-r-q}.$$

Thanks to the inversion formula of Bernoulli polynomials (3), the last formula turns into the following form:

$$\begin{aligned} D^q \tilde{R}_j^{(\alpha,\beta)}(x) =& \frac{j!\,\Gamma(\alpha+1)\Gamma(j+\beta+1)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+\beta+1)} \sum_{r=0}^{j-q} \frac{(-1)^r\,\Gamma(2j-r+\alpha+\beta+1)}{r!\,\Gamma(1+j-r+\beta)} \times \\ & \sum_{t=0}^{j-q-r} \frac{1}{(j-q-r-t)!\,(t+1)!}\, B_{j-r-q-t}(x). \end{aligned} \tag{14}$$

Performing some lengthy manipulation on (14) leads to the following formula:

$$\begin{aligned} D^q \tilde{R}_j^{(\alpha,\beta)}(x) &= \frac{j!\,\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)\,\Gamma(j+\alpha+\beta+1)} \times \\ &\sum_{p=0}^{j-q} \frac{(-1)^p\,\big(\Gamma(j+\beta+1)\Gamma(2j-p+\alpha+\beta)+(-1)^p\,\Gamma(j-p+\beta)\,\Gamma(2j+\alpha+\beta+1)M_{p,j}\big)}{(p+1)!\,(j-p-q)!\,\Gamma(j-p+\beta)}\, B_{j-p-q}(x), \end{aligned}$$

with

$$M_{p,j} = {}_2F_1\left(\begin{matrix} -p-1, -j-\beta \\ -2j-\alpha-\beta \end{matrix} \,\middle|\, 1 \right).$$

Based on Chu–Vandermonde identity, one has

$$ {}_2F_1\left(\begin{matrix} -p-1, -j-\beta \\ -2j-\alpha-\beta \end{matrix} \middle| 1 \right) = \frac{(j-p+\alpha)_{p+1}}{(2j-p+\alpha+\beta)_{p+1}}, $$

and, consequently, the following formula can be obtained:

$$ D^q \tilde{R}_j^{(\alpha,\beta)}(x) = \frac{j!\,\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+\beta+1)} \times $$
$$ \sum_{p=0}^{j-q} \frac{\Gamma(2j-p+\alpha+\beta)((-1)^p\,\Gamma(j-p+\alpha)\Gamma(j+\beta+1)+\Gamma(j+\alpha+1)\Gamma(j-p+\beta))}{(p+1)!\,(j-p-q)!\,\Gamma(j-p+\alpha)\,\Gamma(j-p+\beta)}\, B_{j-q-p}(x). $$

□

Six special formulas of Formula (13) can be produced by noting that the class of the shifted Jacobi polynomials includes six well-known sub-classes. The next two corollaries demonstrate these formulas.

Corollary 1. *Consider any two non-negative integers j and q such that $j \geq q$. The following derivative formulas are valid:*

$$ D^q \tilde{C}_j^{(\alpha)}(x) = \frac{2\,j!\,\Gamma\left(\frac{1}{2}+\alpha\right)}{\Gamma(j+2\alpha)} \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{\Gamma(2j-2p+2\alpha-1)}{(j-2p-q)!\,(2p+1)!\,\Gamma\left(j-2p+\alpha-\frac{1}{2}\right)} B_{j-q-2p}(x), $$
$$ D^q \tilde{P}_j(x) = 2 \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(2j-2p-1)!}{(j-2p-1)!\,(2p+1)!(j-2p-q)!} B_{j-q-2p}(x), $$
$$ D^q \tilde{T}_j(x) = 2j\sqrt{\pi} \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(2j-2p-2)!}{\Gamma\left(-\frac{1}{2}+j-2p\right)(2p+1)!\,(j-2p-q)!} B_{j-q-2p}(x), $$
$$ D^q \tilde{U}_j(x) = \sqrt{\pi} \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(2j-2p)!}{\Gamma\left(\frac{1}{2}+j-2p\right)(2p+1)!\,(j-2p-q)!} B_{j-q-2p}(x). $$

Corollary 2. *Consider any two non-negative integers j and q such that $j \geq q$. The following derivative formulas are valid:*

$$ D^q \tilde{V}_j(x) = \frac{1}{2}\sqrt{\pi} \sum_{p=0}^{j-q} \frac{(-1)^{p+1}\,(-1-2j+(-1)^p\,(1-2j+2p))(2j-p-1)!}{(p+1)!\,\Gamma\left(\frac{1}{2}+j-p\right)(j-p-q)!} B_{j-q-p}(x), $$
$$ D^q \tilde{W}_j(x) = \frac{1}{2}\sqrt{\pi} \sum_{p=0}^{j-q} \frac{(1+2j+(-1)^p\,(2j-2p-1))(2j-p-1)!}{(p+1)!\,\Gamma\left(\frac{1}{2}+j-p\right)(j-p-q)!} B_{j-q-p}(x). $$

Performing similar procedures to those followed to derive Formula (13), several derivative formulas of non-symmetric polynomials can be derived. The following theorem gives the corresponding derivative expressions of the generalized Laguerre and Schröder polynomials.

Theorem 2. *Consider any two non-negative integers j and q such that $j \geq q$. The following derivative formulas for the generalized Laguerre and Schröder polynomials are, respectively, given as follows:*

$$D^q L_j^{(\alpha)}(x) = \Gamma(j+\alpha+1) \sum_{p=0}^{j-q} \frac{(-1)^{j+p+1}(-1+{}_1F_1(-1-p;j-p+\alpha;1))}{(p+1)!\,(j-p-q)!\,\Gamma(j-p+\alpha)}\, B_{j-q-p}(x), \tag{15}$$

$$D^q S_j(x) = \frac{1}{(j+1)!} \sum_{p=0}^{j-q} \frac{-(j+1)!(2j-p-1)! + (2j)!(j-p)!\,{}_2F_1\left(\begin{matrix} -1-j,-1-p \\ -2j \end{matrix}\,\middle|\,-1\right)}{(j-p)!\,(p+1)!\,(j-p-q)!} \times B_{j-q-p}(x). \tag{16}$$

Proof. The derivation of (15) is based on the power form representation of the Laguerre polynomials given in (8), along with the inversion Formula (3), and the derivation of (16) is based on the power form representation of Schröder polynomials given by (9), along with Formula (3). □

3.2. Derivative Expressions for Some Symmetric Polynomials

This section focuses on finding new explicit formulas for several symmetric polynomials' derivatives in terms of Bernoulli polynomials. The derivative formulas for the ultraspherical, Hermite, as well as the generalized Fibonacci and generalized Lucas polynomials that are respectively generated by the two recurrence relations (10) and (11), will be provided.

Theorem 3. *Consider any two non-negative integers j and q such that $j \geq q$, and let $C_j^{(\lambda)}(x)$ be the ultraspherical polynomials. The derivatives of $C_j^{(\lambda)}(x)$ can be expressed in terms of Bernoulli polynomials as:*

$$\begin{aligned} D^q C_j^{(\lambda)}(x) &= \frac{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(j+2\lambda)} \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2^{-j+2p+1}\,\Gamma(2j-2p+2\lambda-1)}{(2p+1)!\,(j-2p-q)!\,\Gamma\left(j-2p+\lambda-\frac{1}{2}\right)}\, B_{j-q-2p}(x) \\ &+ \frac{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{2^{j-2p+2\lambda-3}}{(p+1)!\,(2p+2)!\,(j-2p-q-1)!} \times \\ &\left((-1)^p\,(2p+2)!\,\Gamma(j-p+\lambda-1) + \frac{4^{p+1}\,(p+1)!\,\Gamma(j+\lambda)\left(j-2p+\lambda-\frac{3}{2}\right)_{p+1}}{(j-p+\lambda-1)_{p+1}} \right) B_{j-q-2p-1}(x). \end{aligned} \tag{17}$$

Proof. Based on the power form representation of the ultraspherical polynomials [37], we have

$$D^q C_j^{(\lambda)}(x) = \frac{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^r\,2^{j-2r+2\lambda-1}\,\Gamma(j-r+\lambda)}{r!\,(j-q-2r)!}\, x^{j-2r-q}.$$

The inversion formula of the Bernoulli polynomials in (3) converts the last equation to the following one:

$$\begin{aligned} D^q C_j^{(\lambda)}(x) = &\frac{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \sum_{r=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(-1)^r\,2^{j-2r+2\lambda-1}\,\Gamma(j-r+\lambda)}{r!} \times \\ &\sum_{t=0}^{j-2r-q} \frac{1}{(t+1)!\,(j-q-2r-t)!} B_{j-2r-q-t}(x). \end{aligned} \tag{18}$$

Formula (18) can be rearranged to give the following alternative form:

$$D^q C_j^{(\lambda)}(x) = \frac{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(j+2\lambda)} \left(\sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{1}{(j-2p-q)!} \sum_{r=0}^{p} \frac{(-1)^r\, 2^{j-2r+2\lambda-1}\,\Gamma(j-r+\lambda)}{(2p-2r+1)!\,r!} B_{j-q-2p}(x) + \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{1}{(j-2p-q-1)!} \sum_{r=0}^{p} \frac{(-1)^r\, 2^{j-2r+2\lambda-1}\,\Gamma(j-r+\lambda)}{(2p-2r+2)!\,r!} B_{j-q-2p-1}(x) \right).$$

Noting the following two identities:

$$\sum_{r=0}^{p} \frac{(-1)^r\, 2^{j-2r+2\lambda-1}\,\Gamma(j-r+\lambda)}{(2p-2r+1)!\,r!} = \frac{2^{j+2\lambda-1}\,\Gamma(j+\lambda)}{(2p+1)!}\ {}_2F_1\left(\begin{matrix} -p, -p-\frac{1}{2} \\ 1-j-\lambda \end{matrix} \middle| 1 \right),$$

$$\sum_{r=0}^{p} \frac{(-1)^r\, 2^{j-2r+2\lambda-1}\,\Gamma(j-r+\lambda)}{(2p-2r+2)!\,r!} = 2^{-3+j+2\lambda} \times \left(\frac{\left(-\frac{1}{4}\right)^p \Gamma(j-p+\lambda-1)}{(p+1)!} + \frac{4\,\Gamma(j+\lambda)}{(2p+2)!}\ {}_2F_1\left(\begin{matrix} -p-1, -p-\frac{1}{2} \\ 1-j-\lambda \end{matrix} \middle| 1 \right) \right),$$

together with the following two reductions by Chu–Vandermonde identities:

$${}_2F_1\left(\begin{matrix} -p, -p-\frac{1}{2} \\ 1-j-\lambda \end{matrix} \middle| 1 \right) = \frac{\left(-\frac{1}{2}+j-2p+\lambda\right)_p}{(j-p+\lambda)_p},$$

$${}_2F_1\left(\begin{matrix} -p-1, -p-\frac{1}{2} \\ 1-j-\lambda \end{matrix} \middle| 1 \right) = \frac{\left(-\frac{3}{2}+j-2p+\lambda\right)_{p+1}}{(j-p+\lambda-1)_{p+1}},$$

Formula (17) can be obtained. □

Corollary 3. *Consider any two non-negative integers j and q such that $j \geq q$. The derivatives of the Legendre, Chebyshev polynomials of the first and second kinds, are expressed in terms of Bernoulli polynomials as follows:*

$$\begin{aligned} D^q P_j(x) = & \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2^{1-j+2p}\,(2j-2p-1)!}{(j-2p-1)!\,(2p+1)!(j-2p-q)!} B_{j-q-2p}(x) \\ & + \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{2^{-2-j-2p}\left(\frac{(-1)^p\, 4^j\, \Gamma\left(-\frac{1}{2}+j-p\right)}{\sqrt{\pi}\,(p+1)!} + \frac{16^{p+1}\,(2j-2p-2)!}{(j-2p-2)!\,(2p+2)!} \right)}{(j-2p-q-1)!} B_{j-q-2p-1}(x), \\ D^q T_j(x) = & j\sqrt{\pi} \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2^{1-j+2p}\,(2j-2p-2)!}{\Gamma\left(-\frac{1}{2}+j-2p\right)(2p+1)!\,(j-2p-q)!} B_{j-q-2p}(x) \\ & + j \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{2^{-3-j-2p}\left(\frac{(-1)^p\, 4^j\,(j-p-2)!}{(p+1)!} + \frac{2^{5+4p}\sqrt{\pi}\,(2j-2p-3)!}{\Gamma\left(-\frac{3}{2}+j-2p\right)(2p+2)!} \right)}{(j-2p-q-1)!} B_{j-q-2p-1}(x), \end{aligned} \tag{19}$$

$$
\begin{aligned}
D^q U_j(x) =& \sqrt{\pi} \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{2^{-j+2p}\,(2j-2p)!}{\Gamma\left(\frac{1}{2}+j-2p\right)(2p+1)!\,(j-2p-q)!} B_{j-q-2p}(x) \\
&+ \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{2^{-2-j-2p}\left(\frac{(-1)^p\,4^j\,\Gamma(j-p)}{(p+1)!} + \frac{2^{3+4p}\,\sqrt{\pi}\,(2j-2p-1)!}{\Gamma\left(-\frac{1}{2}+j-2p\right)(2p+2)!}\right)}{(j-2p-q-1)!} B_{j-q-2p-1}(x).
\end{aligned} \tag{20}
$$

Proof. Substitution by $\frac{1}{2}, 0, 1,$ into (17) yields, respectively, (19) and (20). □

Remark 1. *Performing similar procedures to those followed to derive Formula (17), several derivative formulas of symmetric polynomials can be obtained. The following theorem displays the corresponding derivative expressions of Hermite, and the generalized Fibonacci and generalized Lucas polynomials that are generated respectively by the two recurrence relations (10), and (11).*

Theorem 4. *Consider any two non-negative integers j and q such that $j \geq q$. The derivatives of the generalized Fibonacci, generalized Lucas and Hermite polynomials are expressed in terms of Bernoulli polynomials as follows:*

$$
\begin{aligned}
D^q F_j^{a,b}(x) =& a^j\, j! \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{{}_2F_1\left(\begin{matrix} -p, -p-\frac{1}{2} \\ -j \end{matrix} \,\middle|\, -\frac{4b}{a^2}\right)}{(2p+1)!(j-2p-q)!} B_{j-q-2p}(x) \\
&+ a^j \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{-\frac{a^{-2(p+1)}\,B^{p+1}(j-p-1)!}{(p+1)!} + \frac{j!\,{}_2F_1\left(\begin{matrix} -p-1, -p-\frac{1}{2} \\ -j \end{matrix} \,\middle|\, -\frac{4b}{a^2}\right)}{(2p+2)!}}{(j-2p-q-1)!} B_{j-q-2p-1}(x),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
D^q L_j^{c,d}(x) =& c^j\, j! \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{{}_2F_1\left(\begin{matrix} -p, -p-\frac{1}{2} \\ 1-j \end{matrix} \,\middle|\, -\frac{4d}{c^2}\right)}{(2p+1)!(j-2p-q)!} B_{j-q-2p}(x) \\
&+ c^j\, j \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{-\frac{c^{-2(p+1)}\,d^{p+1}(j-p-2)!}{(p+1)!} + \frac{(j-1)!\,{}_2F_1\left(\begin{matrix} -p-1, -p-\frac{1}{2}, -p \\ 1-j \end{matrix} \,\middle|\, -\frac{4d}{c^2}\right)}{(2p+2)!}}{(j-2p-q-1)!} B_{j-q-2p-1}(x),
\end{aligned}
$$

$$
\begin{aligned}
D^q H_j(x) =& 2^j\, j! \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{U\left(-p, \frac{3}{2}, 1\right)}{(2p+1)!\,(j-2p-q)!} B_{j-q-2p}(x) \\
&+ j! \sum_{p=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{(-1)^{p+1}\,2^{j-2p-2}\left(-1+{}_1F_1\left(-p-1; \frac{1}{2}; 1\right)\right)}{(p+1)!(j-2p-q-1)!} B_{j-q-2p-1}(x),
\end{aligned} \tag{22}
$$

where $U(a, b; z)$ represents the confluent hypergeometric function ([37]).

Proof. Similar to the proof of Theorem 3. □

3.3. Some Connection Formulas between Different Polynomials and Bernoulli Polynomials

This section is limited to providing some connection formulas between various symmetric and non-symmetric polynomials. For $q = 0$, the formulas found in Sections 3.1 and 3.2 that express the q-th derivative for various polynomials are valid. This means that every derivative formula for a particular polynomial expressed in terms of Bernoulli polynomials produces a connection formula between this polynomial and Bernoulli polynomials.

Corollary 4. *For every non-negative integer j, the ultraspherical-Bernoulli connection formula is*

$$C_j^{(\lambda)}(x) = \frac{j!\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(j+2\lambda)}\left(\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{2^{-j+2p+1}\,\Gamma(2j-2p+2\lambda-1)}{(2p+1)!\,(j-2p)!\,\Gamma\left(-\frac{1}{2}+j-2p+\lambda\right)}B_{j-2p}(x)+\right.$$
$$\frac{1}{\sqrt{\pi}}\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{2^{j-2p+2\lambda-3}\left((-1)^p\,(2p+2)!\,\Gamma(j-p+\lambda-1)+\frac{4^{p+1}\,(p+1)!\,\Gamma(j+\lambda)\,\left(-\frac{3}{2}+j-2p+\lambda\right)_{p+1}}{(j-p+\lambda-1)_{p+1}}\right)}{(p+1)!\,(2p+2)!\,(j-2p-1)!}\times \tag{23}$$
$$\left.B_{j-2p-1}(x)\right).$$

Proof. simple by setting $q=0$ in Formula (17). □

As special cases of the connection Formula (23), three particular connection formulas can be deduced.

Corollary 5. *For every non-negative integer j, the following are the Legendre–Bernoulli, first kind Chebyshev–Bernoulli, and second kind Chebyshev–Bernoulli connection formulas:*

$$\begin{aligned}P_j(x) = &\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{2^{-j+2p+1}\,(2j-2p-1)!}{(j-2p-1)!\,(j-2p)!\,(2p+1)!}B_{j-2p}(x)\\ &+\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{2^{-j-2p-2}\left(\frac{16^{p+1}\,(2j-2p-2)!}{(j-2p-2)!\,(2p+2)!}+\frac{(-1)^p\,4^j\,\Gamma\left(-\frac{1}{2}+j-p\right)}{\sqrt{\pi}\,(p+1)!}\right)}{(j-2p-1)!}B_{j-2p-1}(x),\quad j\geq 1,\end{aligned} \tag{24}$$

$$\begin{aligned}T_j(x) =&j\sqrt{\pi}\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{2^{-j+2p+1}\,(2j-2p-2)!}{(j-2p)!\,(2p+1)!\,\Gamma\left(-\frac{1}{2}+j-2p\right)}B_{j-2p}(x)\\ &+j\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{2^{-j-2p-3}\left(\frac{(-1)^p\,4^j\,(j-p-2)!}{(p+1)!}+\frac{2^{4p+5}\,\sqrt{\pi}\,(-3+2j-2p)!}{(2p+2)!\,\Gamma\left(-\frac{3}{2}+j-2p\right)}\right)}{(j-2p-1)!}B_{j-2p-1}(x),\quad j\geq 2,\end{aligned} \tag{25}$$

$$\begin{aligned}U_j(x) =&\sqrt{\pi}\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{2^{-j+2p}(2j-2p)!}{(j-2p)!\,(2p+1)!\Gamma\left(\frac{1}{2}+j-2p\right)}B_{j-2p}(x)\\ &+\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{2^{-j-2p-2}\left(\frac{2^{4p+3}\,\sqrt{\pi}\,(2j-2p-1)!}{(2+2p)!\,\Gamma\left(-\frac{1}{2}+j-2p\right)}+\frac{(-1)^p\,4^j\,(j-p-1)!}{(p+1)!}\right)}{(j-2p-1)!}B_{j-2p-1}(x).\end{aligned} \tag{26}$$

Proof. Formulas (24)–(26) can be immediately obtained by setting, respectively, $\lambda=\frac{1}{2},0,1,$ in Formula (23). □

Remark 2. *The two connection Formulas (25) and (26) lead to two trigonometric identities. The following corollary exhibits these identities.*

Corollary 6. *The following two trigonometric identities hold:*

$$
\begin{aligned}
&j\sqrt{\pi}\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{2^{-j+2p+1}\,(2j-2p-2)!}{(j-2p)!\,(2p+1)!\,\Gamma\left(-\frac{1}{2}+j-2p\right)}\,B_{j-2p}(\cos(\theta))\\
&+j\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{2^{-j-2p-3}\left(\frac{(-1)^p\,4^j\,(j-p-2)!}{(p+1)!}+\frac{2^{4p+5}\sqrt{\pi}(-3+2j-2p)!}{(2p+2)!\,\Gamma\left(-\frac{3}{2}+j-2p\right)}\right)}{(j-2p-1)!}\,B_{j-2p-1}(\cos(\theta))=\cos(j\,\theta),\quad j\geq 2,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\sqrt{\pi}\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{2^{-j+2p}\,(2j-2p)!}{(j-2p)!\,(2p+1)!\Gamma\left(\frac{1}{2}+j-2p\right)}B_{j-2p}(\cos(\theta))\\
&+\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{2^{-j-2p-2}\left(\frac{2^{4p+3}\sqrt{\pi}\,(2j-2p-1)!}{(2p+2)!\,\Gamma\left(-\frac{1}{2}+j-2p\right)}+\frac{(-1)^p\,4^j\,(j-p-1)!}{(p+1)!}\right)}{(j-2p-1)!}\,B_{j-2p-1}(\cos(\theta))=\frac{\sin((j+1)\theta)}{\sin(\theta)},\quad j\geq 0.
\end{aligned}
\tag{28}
$$

Proof. Formulas (27) and (28) are direct consequences of Formulas (25) and (26) together with the two well-known trigonometric definitions for $T_j(x)$ and $U_j(x)$:

$$
T_j(x)=\cos(j\,\theta),\qquad U_j(x)=\frac{\sin((j+1)\,\theta)}{\sin\theta},
$$

with $\theta=\cos^{-1}(x)$. □

The following corollary exhibits three connection formulas between three symmetric polynomials and Bernoulli polynomials.

Corollary 7. *For every non-negative integer j, the following are the generalized Fibonacci–Bernoulli, generalized Lucas– Bernoulli, and Hermite–Bernoulli connection formulas:*

$$
\begin{aligned}
F_j^{a,b}(x)=&a^j\,j!\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{{}_2F_1\left(\begin{array}{c}-p,-p-\frac{1}{2}\\-j\end{array}\middle|-\frac{4\,b}{a^2}\right)}{(2p+1)!(j-2p)!}B_{j-2p}(x)\\
&+a^j\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{-\frac{a^{-2(p+1)}b^{p+1}(j-p-1)!}{(p+1)!}+\frac{j!\,{}_2F_1\left(\begin{array}{c}-p-1,-p-\frac{1}{2}\\-j\end{array}\middle|-\frac{4\,b}{a^2}\right)}{(2p+2)!}}{(j-2p-1)!}B_{j-2p-1}(x),
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
L_j^{c,d}(x)=&c^j\,j!\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{{}_2F_1\left(\begin{array}{c}-p,-p-\frac{1}{2}\\1-j\end{array}\middle|-\frac{4\,d}{c^2}\right)}{(2p+1)!(j-2p)!}B_{j-2p}(x)\\
&+c^j\,j\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{-\frac{c^{-2(p+1)}d^{p+1}(j-p-2)!}{(p+1)!}+\frac{(j-1)!\,{}_2F_1\left(\begin{array}{c}-p-1,-p-\frac{1}{2}\\1-j\end{array}\middle|-\frac{4\,d}{c^2}\right)}{(2p+2)!}}{(j-2p-1)!}B_{j-2p-1}(x),
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
H_j(x)=&2^j\,j!\sum_{p=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{U\left(-p,\frac{3}{2},1\right)}{(2p+1)!\,(j-2p)!}B_{j-2p}(x)\\
&+j!\sum_{p=0}^{\left\lfloor\frac{j-1}{2}\right\rfloor}\frac{(-1)^{p+1}2^{j-2p-2}\left(-1+{}_1F_1\left(-p-1;\frac{1}{2};1\right)\right)}{(p+1)!\,(j-2p-1)!}B_{j-2p-1}(x).
\end{aligned}
\tag{31}
$$

Proof. Formulas (29)–(31) are direct special cases of, respectively, Formulas (21) and (22) for $q = 0$. $\square$

Now, we are going to write connection formulas between some non-symmetric polynomials and Bernoulli polynomials.

Corollary 8. *For every non-negative integer j, the shifted Jacobi–Bernoulli connection formula is:*

$$\begin{aligned}\tilde{R}_j^{(\alpha,\beta)}(x) = {} & \frac{j!\,\Gamma(\alpha+1)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+\beta+1)} \times \\ & \sum_{p=0}^{j} \frac{\Gamma(2j-p+\alpha+\beta)\left((-1)^p\,\Gamma(j-p+\alpha)\,\Gamma(j+\beta+1)+\Gamma(j+\alpha+1)\Gamma(j-p+\beta)\right)}{(p+1)!\,(j-p)!\,\Gamma(j-p+\alpha)\,\Gamma(j-p+\beta)}\,B_{j-p}(x).\end{aligned} \tag{32}$$

Proof. Formula (32) is a direct special case of Formula (13) for $q = 0$. $\square$

The following are six special connection formulas of Formula (32).

Corollary 9. *Let j be a non-negative integer. The following six connection formulas hold:*

$$\begin{aligned}
\tilde{C}_j^{(\alpha)}(x) &= \frac{2j!\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(j+2\alpha)} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{\Gamma(2j-2p+2\alpha-1)}{(2p+1)!\,(j-2p)!\,\Gamma\left(j-2p+\alpha-\frac{1}{2}\right)} B_{j-2p}(x),\\
\tilde{P}_n(x) &= 2 \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(2j-2p-1)!}{(2p+1)!\,(j-2p-1)!\,(j-2p)!} B_{j-2p}(x),\\
\tilde{T}_n(x) &= 2j\sqrt{\pi} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(2j-2p-2)!}{(2p+1)!\,(j-2p)!\,\Gamma\left(j-2p-\frac{1}{2}\right)} B_{j-2p}(x),\\
\tilde{U}_n(x) &= \sqrt{\pi} \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(2j-2p)!}{(2p+1)!\,(j-2p)!\Gamma\left(j-2p+\frac{1}{2}\right)} B_{j-2p}(x),\\
\tilde{V}_n(x) &= \frac{1}{2}\sqrt{\pi} \sum_{p=0}^{j} \frac{(-1)^{p+1}\,(2j-p-1)!\,(-1-2j+(-1)^p\,(-2j+2p+1))}{(p+1)!\,(j-p)!\,\Gamma\left(j-p+\frac{1}{2}\right)} B_{j-p}(x),\\
\tilde{W}_n(x) &= \frac{1}{2}\sqrt{\pi} \sum_{p=0}^{j} \frac{(2j-p-1)!\,(1+2j+(-1)^p\,(2j-2p-1))}{(p+1)!\,(j-p)!\,\Gamma\left(j-p+\frac{1}{2}\right)} B_{j-p}(x).
\end{aligned}$$

Proof. The results of Corollary 9 are special ones of Formula (32) by choosing the two parameters α and β suitably. $\square$

Corollary 10. *The following are, respectively, the generalized Laguerre–Bernoulli and Schröder–Bernoulli connection formulas:*

$$L_j^{(\alpha)}(x) = \sum_{p=0}^{j} \frac{(-1)^{j+p+1}\Gamma(j+\alpha+1)(-1+{}_1F_1(-p-1;j-p+\alpha;1))}{(p+1)!\,(j-p)!\,\Gamma(j-p+\alpha)} B_{j-p}(x), \tag{33}$$

$$\begin{aligned}S_j(x) = {} & \frac{1}{(j+1)!} \sum_{p=0}^{j} \frac{-(j+1)!\,(2j-p-1)!+(2j)!\,(j-p)!\,{}_2F_1\left(\left.\begin{matrix}-p-1,-1-j\\ -2j\end{matrix}\,\right|\,-1\right)}{(j-p)!\,(p+1)!\,(j-p)!} \times \\ & B_{j-p}(x).\end{aligned} \tag{34}$$

Proof. Formulas (33) and (34) are special ones of (15) and (16) for $q = 0$. □

3.4. Connections between Some Celebrated Numbers

In this section, we account for some formulas that express some celebrated numbers in terms of Bernoulli numbers. In fact, if we set $x = 1$ in the connection formulas that are derived in Section 3.3, some new expressions between these numbers can be deduced. In the following, we give some of these expressions. Other expressions can be also deduced.

Corollary 11. *For every non-negative integer j, the generalized Fibonacci numbers that are given in (12) are expressed in terms of the Bernoulli numbers as:*

$$\begin{aligned} F_j^{a,b} =& (-a)^j\, j! \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{{}_2F_1\left(\begin{matrix} -p, -p-\frac{1}{2} \\ -j \end{matrix} \middle| -\frac{4\,b}{a^2}\right)}{(2p+1)!(j-2p)!} B_{j-2p} \\ &+ (-a)^j \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{\frac{a^{-2(p+1)} b^{p+1} (j-p-1)!}{(p+1)!} - \frac{j!\,{}_2F_1\left(\begin{matrix} -p-1, -p-\frac{1}{2} \\ -j \end{matrix} \middle| -\frac{4\,b}{a^2}\right)}{(2p+2)!}}{(j-2p-1)!} B_{j-2p-1}, \end{aligned} \tag{35}$$

and in particular

$$\begin{aligned} F_j =& (-1)^j\, j! \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} \frac{{}_2F_1\left(\begin{matrix} -p, -p-\frac{1}{2} \\ -j \end{matrix} \middle| -4\right)}{(2p+1)!(j-2p)!} B_{j-2p} \\ &+ (-1)^j \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \frac{\frac{(j-p-1)!}{(p+1)!} - \frac{j!\,{}_2F_1\left(\begin{matrix} -p-1, -p-\frac{1}{2} \\ -j \end{matrix} \middle| -4\right)}{(2p+2)!}}{(j-2p-1)!} B_{j-2p-1}. \end{aligned} \tag{36}$$

Proof. Formula (35) is a direct consequence of Formula (29) by setting $x = 1$ and noting the two identities (12) and (1). The particular case (36) is a special case of (35) for the case corresponds to $a = b = 1$. □

4. Expressions for the Derivatives of Bernoulli Polynomials in Terms of Different Polynomials

This section is devoted to explaining how to derive the derivative expressions for Bernoulli polynomials in terms of other polynomials; that is, we describe how to obtain the inversion formulas to the connection formulas derived in Section 3. We give here one of these expressions. We will give the formula that expresses the derivatives of Bernoulli polynomials in terms of the shifted Jacobi polynomials. Other expressions can be similarly obtained.

Theorem 5. *For all non-negative integers j and q with $j \geq q$. The following expression for the derivatives of Bernoulli polynomials holds:*

$$\begin{aligned} D^q B_j(x) = & \frac{j!\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{m=0}^{\lfloor \frac{j-q}{2} \rfloor} \frac{(j-2m-q+\alpha)\,\Gamma(j-2m-q+2\alpha)}{(j-2m-q)!} \times \\ & \sum_{r=0}^{m} \frac{2^{-j+q+2r+1}}{(m-r)!\,(2r)!\,\Gamma(1+j-m-q-r+\alpha)} B_{2r}\, C_{j-q-2m}^{(\alpha)}(x) \\ & - \frac{2^{-j+q+1}\, j!\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{m=0}^{\lfloor \frac{1}{2}(j-q-1) \rfloor} \frac{(j-2m-q+\alpha-1)\,\Gamma(j-2m-q+2\alpha-1)}{m!\,(j-2m-q-1)!\,\Gamma(j-m-q+\alpha)} C_{j-q-2m-1}^{(\alpha)}(x), \end{aligned} \tag{37}$$

where B_r are the well-known Bernoulli numbers.

Proof. We start with the power form representation of Bernoulli polynomials in (2) to write:

$$D^q B_j(x) = \sum_{r=0}^{j-q} \binom{j}{r} (j-q-r+1)_q \, B_r \, x^{j-r-q}.$$

In virtue of the inversion formula of the ultraspherical polynomials [37], one gets:

$$\begin{aligned} D^q B_j(x) =& \frac{j!\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{r=0}^{j-q} \frac{2^{-j+q+r+1}\,B_r}{r!} \times \\ & \sum_{t=0}^{\left\lfloor \frac{1}{2}(j-r-q) \right\rfloor} \frac{(j-q-r-2t+\alpha)\,\Gamma(j-q-r-2t+2\alpha)}{(j-q-r-2t)!\,t!\,\Gamma(j-q-r-t+\alpha+1)}\, C^{(\alpha)}_{j-r-q-2t}(x), \end{aligned}$$

which can be written again in the form:

$$\begin{aligned} D^q B_j(x) = &= \frac{j!\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{m=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{(j-2m-q+\alpha)\,\Gamma(j-2m-q+2\alpha)}{(j-2m-q)!} \times \\ & \sum_{r=0}^{m} \frac{2^{-j+q+2r+1}}{(m-r)!\,(2r)!\,\Gamma(j-m-q-r+\alpha+1)}\, B_{2r}\, C^{(\alpha)}_{j-q-2m}(x) \\ & + \frac{j!\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{m=0}^{\left\lfloor \frac{1}{2}(j-q-1) \right\rfloor} \frac{(j-2m-q+\alpha-1)\,\Gamma(j-2m-q+2\alpha-1)}{(j-2m-q-1)!} \times \\ & \sum_{r=0}^{m} \frac{2^{-j+q+2r+2}}{(m-r)!\,(2r+1)!\,\Gamma(j-m-q-r+\alpha)}\, B_{2r+1}\, C^{(\alpha)}_{j-q-2m-1}(x). \end{aligned} \tag{38}$$

Now, regarding the following summation:

$$U_{m,j,q,\alpha} = \sum_{r=0}^{m} \frac{2^{-j+q+2r+2}\,B_{2r+1}}{(m-r)!\,(2r+1)!\,\Gamma(j-m-q-r+\alpha)},$$

can be summed. In fact, the employment of Zeilberger's algorithm [41] leads to the following recurrence relation:

$$U_{m+1,j,q,\alpha} - \frac{-m+j-q+\alpha-1}{m+1}\, U_{m,j,q,\alpha} = 0, \quad U_{0,j,q,\alpha} = 1.$$

Since the last recurrence relation is of order one, it can be immediately solved to give:

$$U_{m,j,q,\alpha} = \frac{-2^{-j+q+1}}{m!\,\Gamma(j-m-q+\alpha)}.$$

Based on the above reduction, Formula (38) turns into Formula (37). This proves Theorem 5. □

5. New Closed Forms for Some Definite Integrals

This section is interested in the evaluation of some new definite integrals of some polynomials based on the connection formulas between different polynomials and Bernoulli polynomials. To be more precise, we will present two applications to the connection formulas derived in this paper. The following definite integrals will be computed:

- The definite integral for different polynomials on the interval $[0,1]$;
- The definite integral for the product of different polynomials with Bernoulli polynomials on $[0,1]$.

In this regard, four theorems concerned with the above two mentioned applications will be stated and proved. Furthermore, these theorems will be utilized to compute definite integrals for different polynomials.

5.1. First Application to the Connection Formulas

This section concerns the first application to the connection formulas that are stated in Section 3.3. We will present two theorems in this respect to compute the definite integrals $\int_0^1 \phi_j(x)\,dx$ and $\int_0^1 \psi_j(x)\,dx$, where $\phi_j(x)$ is any symmetric polynomial that takes the form in (4), and $\psi_j(x)$ is any non-symmetric polynomial that takes the form in (5).

Theorem 6. *Let $\psi_i(x)$ by any non-symmetric polynomial whose power form can be written as in (5), and let it have the following connection formula with Bernoulli polynomials:*

$$\psi_j(x) = \sum_{p=0}^{j} A_{p,j}.\, B_{j-p}(x). \tag{39}$$

The following integral formula applies:

$$\int_0^1 \psi_j(x)\,dx = A_{j,j}.$$

Proof. Starting from the connection formula (39), we integrate both sides from 0 to 1, to get

$$\int_0^1 \psi_j(x)\,dx = \sum_{p=0}^{j} A_{p,j} \int_0^1 B_{j-p}(x)\,dx. \tag{40}$$

If we make use of the well-known identity [29]:

$$\int_0^1 B_n(x)\,dx = \delta_{0,n}, \quad n \geq 0, \tag{41}$$

where $\delta_{i,j}$ is the well-known Kronecker delta function, then Formula (40) reduces to the following formula:

$$\int_0^1 \psi_j(x)\,dx = A_{j,j}.$$

This ends the proof of Theorem 6. □

The following theorem exhibits closed forms for the definite integrals of three non-symmetric polynomials.

Corollary 12. *The following integral formulas hold for every non-negative integer j:*

$$\begin{aligned}
\int_0^1 \tilde{R}_j^{(\alpha,\beta)}(x)\,dx &= \frac{\alpha\Gamma(j+\alpha+1)\,\Gamma(\beta) + (-1)^j\,\Gamma(\alpha+1)\,\Gamma(j+\beta+1)}{(j+1)\,(j+\alpha+\beta)\,\Gamma(j+\alpha+1)\,\Gamma(\beta)}, \\
\int_0^1 L_j^{(\alpha)}(x)\,dx &= \frac{\Gamma(j+\alpha+1)(1 - {}_1F_1(-1-j;\alpha;1))}{(j+1)!\,\Gamma(\alpha)},
\end{aligned} \tag{42}$$

$$\int_0^1 S_j(x)\,dx = \frac{-(j-1)!\,(j+1)! + (2j)!\;{}_2F_1\left(\begin{matrix} -j-1, -j-1 \\ -2j \end{matrix}\,\middle|\, -1\right)}{((j+1)!)^2}. \tag{43}$$

Proof. Formulas (42) and (43) can be obtained by the application to Theorem 6 together with the three connection Formulas (32)–(34). □

Theorem 7. *Let $\phi_j(x)$ be any symmetric polynomial whose power form can be written as in (4), and let it have the following connection formula with Bernoulli polynomials:*

$$\phi_j(x) = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} G_{p,j}\, B_{p-2j}(x) + \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \bar{G}_{p,j}\, B_{p-2j-1}(x). \tag{44}$$

The following integral formula applies:

$$\int_0^1 \phi_j(x)\, dx = \begin{cases} G_{\frac{j}{2},j}, & j \text{ even}, \\ \bar{G}_{\frac{j-1}{2},j}, & j \text{ odd}. \end{cases}$$

Proof. If we start from the connection Formula (44) and integrate both sides from 0 to 1, then the following formula is obtained:

$$\int_0^1 \phi_j(x)\, dx = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} G_{p,j} \int_0^1 B_{p-2j}(x)\, dx + \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \bar{G}_{p,j} \int_0^1 B_{p-2j-1}(x)\, dx. \tag{45}$$

The utilization of Identity (41) transforms (45) into:

$$\int_0^1 \phi_j(x)\, dx = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} G_{p,j}\, \delta_{j-2p,0} + \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \bar{G}_{p,j}\, \delta_{j-2p-1,0}, \tag{46}$$

where $\delta_{i,j}$ is the well-known Kronecker delta function.

It is evident from (46) that the following integral formula can be obtained:

$$\int_0^1 \phi_j(x)\, dx = \begin{cases} G_{\frac{j}{2},j}, & j \text{ even}, \\ \bar{G}_{\frac{j-1}{2},j}, & j \text{ odd}. \end{cases}$$

□

Now, and as an application of Theorem 7, several new formulas for definite integrals to some symmetric polynomials can be deduced. The following corollaries provide some of these results.

Corollary 13. *The following integral formula holds for every non-negative integer j:*

$$\int_0^1 C_j^{(\lambda)}(x)\, dx = \begin{cases} \dfrac{2\lambda - 1}{(j+1)\,(j+2\lambda-1)}, & j \text{ even}, \\ \dfrac{2\lambda - 1}{(j+1)(j+2\lambda-1)} + \dfrac{(-1)^{\frac{j-1}{2}}\, 4^{\lambda-1}\, j!\, \Gamma\left(\lambda + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}(j-1) + \lambda\right)}{\sqrt{\pi}\, \left(\frac{j+1}{2}\right)!\, \Gamma(j+2\lambda)}, & j \text{ odd}. \end{cases} \tag{47}$$

Proof. Formula (47) can be obtained by the application to Theorem 7 together with the connection Formula (23). □

The three formulas that follow are particular ones of Formula (47).

Corollary 14. *Let j be a non-negative integer. The following identities hold:*

$$\int_0^1 P_j(x)\,dx = \begin{cases} 0, & j \text{ even}, \\ \dfrac{(-1)^{\frac{j+3}{2}}\,\Gamma\left(\frac{j}{2}\right)}{2\sqrt{\pi}\left(\frac{j+1}{2}\right)!}, & j \text{ odd}, \end{cases} \quad j \geq 1, \tag{48}$$

$$\int_0^1 T_j(x)\,dx = \frac{1}{(1-j)(1+j)} \begin{cases} 1, & j \text{ even}, \\ 1 + j(-1)^{\frac{j+1}{2}}, & j \text{ odd}, \end{cases} \quad j \geq 0,\ j \neq 1,$$

$$\int_0^1 U_j(x)\,dx = \frac{1}{j+1} \begin{cases} 1, & j \text{ even}, \\ 1 + (-1)^{\frac{j+3}{2}}, & j \text{ odd}, \end{cases} \quad j \geq 0. \tag{49}$$

Proof. Formulas (48) and (49) are direct special cases of Formula (47) setting, respectively $\lambda = \frac{1}{2}, 0, 1$. □

Corollary 15. *The following integral formulas hold for every non-negative integer j:*

$$\int_0^1 F_j^{a,b}(x)\,dx = \begin{cases} \dfrac{a^j\ {}_2F_1\left(\begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ -j \end{array} \middle| -\frac{4\,b}{a^2}\right)}{j+1}, & j \text{ even}, \\ -\dfrac{2\,b^{\frac{j+1}{2}}}{a\,(j+1)} + \dfrac{a^j\ {}_2F_1\left(\begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ -j \end{array} \middle| -\frac{4\,b}{a^2}\right)}{j+1}, & j \text{ odd}, \end{cases} \tag{50}$$

$$\int_0^1 L_j^{c,d}(x)\,dx = \begin{cases} \dfrac{c}{2}, & j = 1, \\ \dfrac{c^j\ {}_2F_1\left(\begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ 1-j \end{array} \middle| -\frac{4\,d}{c^2}\right)}{j+1}, & j \text{ even}, \\ \dfrac{-4\,d^{\frac{1+j}{2}}\,j + c^{j+1}\,(j-1)\ {}_2F_1\left(\begin{array}{c} -\frac{1}{2}-\frac{j}{2}, -\frac{j}{2} \\ 1-j \end{array} \middle| -\frac{4\,d}{c^2}\right)}{r(j^2-1)}, & j \text{ odd},\ j \neq 1, \end{cases}$$

$$\int_0^1 H_j(x)\,dx = \begin{cases} \dfrac{2^j\,U\left(-\frac{j}{2}, \frac{3}{2}, 1\right)}{j+1}, & j \text{ even}, \\ \dfrac{(-1)^{\frac{j+1}{2}}\,j!\left(-1 + {}_1F_1\left(-\frac{1}{2}-\frac{j}{2}; \frac{1}{2}; 1\right)\right)}{2\left(\frac{j+1}{2}\right)!}, & j \text{ odd}. \end{cases} \tag{51}$$

Proof. Formulas (50) and (51) can be directly obtained by application to Theorem 7 using the three connection formulas, (29)–(31). □

5.2. Second Application to the Connection Formulas

In this section, we present another application for the connection formulas between the different polynomials with Bernoulli polynomials. Two theorems will be stated and proved in this regard.

Theorem 8. *Let $\psi_j(x)$ by any non-symmetric polynomial whose power form is as in (5), and let it have the connection formula with Bernoulli polynomials as in (39). Further, assume that j and m are positive integers. The following integral formula holds:*

$$\int_0^1 \phi_j(x)\,B_m(x)\,dx = m! \sum_{p=0}^{j-1} \frac{(-1)^{j-p+1}\,(j-p)!}{(j+m-p)!}\,B_{j+m-p}\,A_{p,j},$$

where $A_{p,j}$ are the connection coefficients in (39).

Proof. If we make use of the connection Formula (39), then:

$$\int_0^1 \phi_j(x)\, B_m(x)\, dx = \sum_{p=0}^{j} A_{p,j} \int_0^1 B_{j-p}(x) B_m(x)\, dx. \tag{52}$$

Based on the well-known identity:

$$\int_0^1 B_m(x)\, B_n(x)\, dx = \frac{(-1)^{n-1}\, n!\, m!}{(m+n)!}\, B_{m+n}, \quad m \geq 1, n \geq 1, \tag{53}$$

and the simple identity

$$\int_0^1 B_0(x)\, B_m(x)\, dx = 0, \quad m \geq 1, \tag{54}$$

Formula (52) turns into:

$$\int_0^1 \phi_j(x)\, B_m(x)\, dx = m! \sum_{p=0}^{j-1} \frac{(-1)^{j-p+1}\, (j-p)!}{(j+m-p)!}\, B_{j+m-p}\, A_{p,j}.$$

□

In the following, we show how to utilize Theorem 8 to obtain closed forms for some definite integrals.

Corollary 16. *For all positive integers j and m, the following integral formula holds:*

$$\begin{aligned}&\int_0^1 \tilde{R}_j^{(\alpha,\beta)}(x)\, B_m(x)\, dx = \frac{(-1)^{j+1}\, j!\, m!\, \Gamma(\alpha+1)}{\Gamma(j+\alpha+1)\, \Gamma(j+\alpha+1+\beta)} \times \\ &\sum_{p=0}^{j-1} \frac{\Gamma(2j-p+\alpha+\beta)\, (\Gamma(j-p+\alpha)\, \Gamma(j+\beta+1) + (-1)^p\, \Gamma(j+\alpha+1)\, \Gamma(j-p+\beta))}{(p+1)!\, (j+m-p)!\, \Gamma(j-p+\alpha)\, \Gamma(j-p+\beta)}\, B_{j+m-p}.\end{aligned} \tag{55}$$

Proof. Identity (55) can be obtained by application to Theorem 8 together with the connection Formula (32). □

Remark 3. *Since the shifted Jacobi polynomials have six well-known shifted polynomials, then six special integrals can be obtained as special cases of Formula (55).*

Corollary 17. *For all positive integers j and m, the following integral formulas hold:*

$$\begin{aligned}&\int_0^1 \tilde{C}_j^{(\alpha)}(x)\, B_m(x)\, dx = \frac{2\, (-1)^{j+1}\, j!\, m!\, \Gamma\left(\frac{1}{2}+\alpha\right)}{\Gamma(j+2\alpha)} \times \\ &\sum_{p=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{\Gamma(2j-2p+2\alpha-1)}{(2p+1)!\, (j+m-2p)!\, \Gamma\left(-\frac{1}{2}+j-2p+\alpha\right)}\, B_{j+m-2p},\end{aligned}$$

$$\int_0^1 \tilde{P}_j(x)\, B_m(x)\, dx = 2\, (-1)^{j+1}\, m! \sum_{p=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{(2j-2p-1)!}{(2p+1)!\, (j-2p-1)!\, (j+m-2p)!}\, B_{j+m-2p},$$

$$\int_0^1 \tilde{T}_j(x)\, B_m(x)\, dx = 2\, (-1)^{j+1}\, j\, \sqrt{\pi}\, m! \sum_{p=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{(2j-2p-2)!}{(2p+1)!\, \Gamma\left(-\frac{1}{2}+j-2p\right) (j+m-2p)!}\, B_{j+m-2p},,$$

$$\int_0^1 \tilde{U}_j(x)\, B_m(x)\, dx = (-1)^{j+1}\sqrt{\pi}\, m! \sum_{p=0}^{\left\lfloor \frac{j-1}{2} \right\rfloor} \frac{(2j-2p)!}{(2p+1)!\,\Gamma\left(\frac{1}{2}+j-2p\right)(j+m-2p)!}\, B_{j+m-2p},$$

$$\int_0^1 \tilde{V}_j(x)\, B_m(x)\, dx = \frac{1}{2}(-1)^{j+1}\sqrt{\pi}\, m!\times$$
$$\sum_{p=0}^{j-1} \frac{(2j-p-1)!\,(1+2j+(-1)^p(2j-2p-1))}{(p+1)!\,\Gamma\left(\frac{1}{2}+j-p\right)(j+m-p)!}\, B_{j+m-p},$$

$$\int_0^1 \tilde{W}_j(x)\, B_m(x)\, dx = \frac{1}{2}(-1)^{j+1}\sqrt{\pi}\, m!\times$$
$$\sum_{p=0}^{j-1} \frac{(2j-p-1)!\,(-1+(-1)^p+2(1+(-1)^p)j-2p)}{(p+1)!\,\Gamma\left(\frac{1}{2}+j-p\right)(j+m-p)!}\, B_{j+m-p}.$$

Proof. The results of Corollary 17 are specific cases of the result of Corollary 16 taking into consideration the suitable choices of the two parameters α and β. □

Corollary 18. *For all positive integers j and m, the following integral formulas hold:*

$$\int_0^1 L_j^{(\alpha)}(x) B_m(x)\, dx = m!\,\Gamma(j+\alpha+1) \sum_{p=0}^{j-1} \frac{(-1+{}_1F_1(-p-1;j-p+\alpha;1))}{(p+1)!\,(j+m-p)!\,\Gamma(j-p+\alpha)}\, B_{j+m-p}, \tag{56}$$

$$\int_0^1 S_j(x)\, B_m(x)\, dx = m! \sum_{p=0}^{j-1} \frac{(-1)^{j-p}((j+1)!\,(2j-p-1)! - (2j)!(j-p)!)}{(j+1)!\,(j-p)!\,(j+m-p)!\,(p+1)!}\times {}_2F_1\left(\begin{matrix} -p-1, -j-1 \\ -2j \end{matrix}\middle| -1\right) B_{j+m-p}. \tag{57}$$

Proof. Identities (56) and (57) can be obtained by the application of Theorem 8 together with the connection Formulas (33) and (34). □

Now, we are going to prove a corresponding result for Corollary 8, but for any symmetric polynomial. The next theorem exhibits this result.

Theorem 9. *Let $\phi_j(x)$ by any symmetric polynomial of degree j that can be represented as in (44), and let it have the connection formula with Bernoulli polynomials given as in (44). The following integral formula applies for every non-negative integer j and m:*

$$\int_0^1 \phi_j(x)\, B_m(x)\, dx = \begin{cases} \displaystyle\sum_{p=0}^{\frac{j}{2}-1} L_{p,j,m}, & j \text{ even}, \\ \displaystyle\sum_{p=0}^{\frac{j-1}{2}} Z_{p,j,m} + \sum_{p=0}^{\frac{j-3}{2}} \bar{Z}_{p,j,m}, & j \text{ odd}, \end{cases} \tag{58}$$

where the coefficients $L_{p,j,m}$, $Z_{p,j,m}$ and $\bar{Z}_{p,j,m}$ are respectively given by

$$L_{p,j,m} = \frac{(-1)^j\, m!\; (j-2p-1)!\; \left(-(j-2p)\, B_{j+m-2p} G_{p,j} + (j+m-2p)\, B_{j+m-2p-1}\, \bar{G}_{p,j}\right)}{(j+m-2p)!},$$

$$Z_{p,j,m} = \frac{(-1)^{j+1}\, m!\,(j-2p)!}{(j+m-2p)!}\, B_{j+m-2p}\, G_{p,j},$$

$$\bar{Z}_{p,m,j} = \frac{(-1)^j\, m!\,(j-2p-1)!}{(j+m-2p-1)!}\, B_{j+m-2p-1}\, \bar{G}_{p,j},$$

where the coefficients $G_{p,j}$ and $\bar{G}_{p,j}$ are the connection coefficients in (44).

Proof. Starting from the connection Formula (44), we integrate both sides from 0 to 1, to obtain:

$$\int_0^1 \phi_j(x)\, B_m(x)\, dx = \sum_{p=0}^{\lfloor \frac{j}{2} \rfloor} G_{p,j} \int_0^1 B_m(x)\, B_{j-2p}(x)\, dx + \sum_{p=0}^{\lfloor \frac{j-1}{2} \rfloor} \bar{G}_{p,j} \int_0^1 B_m(x)\, B_{j-2p-1}(x)\, dx.$$

Now, we consider the following two cases:
(i) The case corresponds to j odd. In this case, we have:

$$\int_0^1 \phi_{2j+1}(x)\, B_m(x)\, dx = \sum_{p=0}^{j} G_{p,2j+1} \int_0^1 B_m(x)\, B_{2j-2p+1}(x)\, dx + \sum_{p=0}^{j} \bar{G}_{p,2j+1} \int_0^1 B_m(x)\, B_{2j-2p}(x)\, dx. \tag{59}$$

Making use of the two Identities (53) and (54) turns Formula (59) into the following one:

$$\int_0^1 \phi_{2j+1}\, B_m(x)\, dx = \sum_{p=0}^{j} G_{p,2j+1}\, F_{m,2j-2p+1} + \sum_{p=0}^{j-1} \bar{G}_{p,2j+1} F_{m,2j-2p}, \tag{60}$$

where $F_{m,n}$ are given by

$$F_{m,n} = \frac{(-1)^{n-1}\, n!\, m!}{(m+n)!}\, B_{m+n}.$$

Formula (60) can be written as:

$$\int_0^1 \phi_{2j+1}(x)\, B_m(x)\, dx = \sum_{p=0}^{j} S_{p,j,m} + \sum_{p=0}^{j-1} \bar{S}_{p,j,m}, \tag{61}$$

with the coefficients $S_{p,j,m}$ and $\bar{S}_{p,j,m}$ are given as:

$$S_{p,j,m} = G_{p,2j+1}\, F_{m,2j-2p+1}, \quad \bar{S}_{p,j,m} = \bar{G}_{p,2j+1}\, F_{m,2j-2p}.$$

(ii) The case corresponds to j even. In such a case, we have:

$$\int_0^1 \phi_{2j}(x)\, B_m(x)\, dx = \sum_{p=0}^{j} G_{p,2j} \int_0^1 B_m(x) B_{2j-2p}(x)\, dx + \sum_{p=0}^{j-1} \bar{G}_{p,2j} \int_0^1 B_m(x) B_{2j-2p-1}(x)\, dx,$$

which gives after using identity (53), the following formula

$$\int_0^1 \phi_{2j}(x)\, B_m(x)\, dx = \sum_{p=0}^{j-1} G_{p,2j}\, F_{m,2j-2p} + \sum_{p=0}^{j-1} \bar{G}_{p,2j} F_{m,2j-2p-1},$$

which can be written as

$$\int_0^1 \phi_{2j}(x)\, B_m(x)\, dx = \sum_{p=0}^{j-1} W_{p,j,m}, \tag{62}$$

where $W_{p,j,m}$ are given by

$$W_{p,j,m} = G_{p,2j}\, F_{m,2j-2p} + \bar{G}_{p,2j}\, F_{m,2j-2p-1}.$$

Now, merging the two integrals in (62) and (61) yields the following identity:

$$\int_0^1 \phi_j(x)\, B_m(x)\, dx = \begin{cases} \displaystyle\sum_{p=0}^{\frac{j}{2}-1} L_{p,j,m}, & j \text{ even}, \\ \displaystyle\sum_{p=0}^{\frac{j-1}{2}} Z_{p,j,m} + \sum_{p=0}^{\frac{j-3}{2}} \bar{Z}_{p,j,m}, & j \text{ odd}, \end{cases}$$

where the coefficients $L_{p,j,m}$, $Z_{p,j,m}$ and $\bar{Z}_{p,j,m}$ are, respectively, given by

$$\begin{aligned}
L_{p,j,m} &= \frac{(-1)^j\, m!\, (j-2p-1)!\, \big(-(j-2p)\, B_{j+m-2p} G_{p,j} + (j+m-2p)\, B_{j+m-2p-1}\, \bar{G}_{p,j}\big)}{(j+m-2p)!},\\
Z_{p,j,m} &= \frac{(-1)^{j+1}\, m!\, (j-2p)!}{(j+m-2p)!}\, B_{j+m-2p}\, G_{p,j},\\
\bar{Z}_{p,m,j} &= \frac{(-1)^j\, m!\, (j-2p-1)!}{(j+m-2p-1)!}\, B_{j+m-2p-1}\, \bar{G}_{p,j},
\end{aligned}$$

where the coefficients $G_{p,j}$ and $\bar{G}_{p,j}$ are the connection coefficients in (44). □

Remark 4. *Formula* (58) *can be split into the following two formulas:*

$$\begin{aligned}
\int_0^1 \phi_{2j}(x)\, B_m(x)\, dx =& m!\Bigg(\sum_{p=0}^{j-1} \frac{(2j-2p-1)!}{(2j+m-2p)!}\times \\
& \big(2(p-j)\, G_{p,2j}\, B_{2j+m-2p} + (2j+m-2p)\, \bar{G}_{p,2j}\, B_{2j+m-2p-1}\big)\Bigg),
\end{aligned} \tag{63}$$

$$\begin{aligned}
\int_0^1 \phi_{2j+1}(x)\, B_m(x)\, dx =& m!\Bigg(\sum_{p=0}^{j} \frac{(2j-2p+1)!}{(2j+m-2p+1)!} G_{p,2j+1}\, B_{2j+m-2p+1} \\
& - \sum_{p=0}^{j-1} \frac{(2j-2p)!}{(2j+m-2p)!}\, \bar{G}_{p,2j+1}\, B_{2j+m-2p}\Bigg).
\end{aligned} \tag{64}$$

Corollary 19. *For all positive integers j and m, the following integral formulas hold:*

$$\begin{aligned}
&\int_0^1 C_{2j}^{(\alpha)}(x)\, B_m(x)\, dx = \frac{(2j)!\, m!\, \Gamma\left(\frac{1}{2}+\alpha\right)}{\Gamma(2(j+\alpha))} \sum_{p=0}^{j-1} 2^{-2(j+p)-3} \times \\
&\Bigg(\Bigg(\frac{(-1)^p\, 4^{2j+\alpha}\, \Gamma(2j-p+\alpha-1)}{\sqrt{\pi}\, (p+1)!} + \frac{2^{4p+5}\, \Gamma(4j-2p+2\alpha-2)}{(2p+2)!\, \Gamma(-\frac{3}{2}+2j-2p+\alpha)}\Bigg) \frac{B_{2j+m-2p-1}}{(2j+m-2p-1)!} \\
&\quad - \frac{16^{p+1}\, \Gamma(4j-2p+2\alpha-1)}{(2j+m-2p)!\, (2p+1)!\, \Gamma\left(-\frac{1}{2}+2j-2p+\alpha\right)}\, B_{2j+m-2p}\Bigg),
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\int_0^1 C_{2j+1}^{(\alpha)}(x) B_m(x)\, dx = (2j+1)!\, \Gamma\left(\frac{1}{2}+\alpha\right) m! \times \\
&\Bigg(\sum_{p=0}^{j} \frac{2^{-2\,(j-p)}\ \Gamma(1+4j-2p+2\alpha)}{(2p+1)!\, (2j+m-2p+1)!\, \Gamma\left(\frac{1}{2}+2j-2p+\alpha\right)\ \Gamma(2j+2\alpha+1)}\, B_{2j+m-2p+1} \\
&- \sum_{p=0}^{j-1} \frac{2^{2(j-p+\alpha-1)}}{\sqrt{\pi}\, (2j+m-2p)!\, (p+1)!\, (2p+2)!\, \Gamma(2j+2\alpha+1)}\, B_{2j+m-2p} \times \\
&\Bigg((-1)^p\, (2p+2)!\, \Gamma(2j-p+\alpha) + \frac{4^{p+1}\, (p+1)!\, \Gamma(2j+\alpha+1)\, \left(-\frac{1}{2}+2j-2p+\alpha\right)_{p+1}}{(2j-p+\alpha)_{p+1}}\Bigg)\Bigg).
\end{aligned} \tag{66}$$

Proof. Formulas (65) and (66) can be obtained by the application of Formulas (63) and (64) making use of the connection Formula (23). □

Corollary 20. *For all positive integers j and m, the following integral formulas hold:*

$$\begin{aligned}
&\int_0^1 F_{2j}^{a,b}(x)\,B_m(x)\,dx = \\
&a^{2j}\,m!\sum_{p=0}^{j-1}\left(\left(-\frac{a^{-2(p+1)}\,b^{p+1}(2j-p-1)!}{(p+1)!}+\frac{(2j)!\,{}_2F_1\left(\begin{matrix}-p-1,-p-\frac{1}{2}\\ -2j\end{matrix}\,\middle|\,-\frac{4\,b}{a^2}\right)}{(2p+2)!}\right)\frac{B_{2j+m-2p-1}}{(2j+m-2p-1)!}\right. \\
&\qquad\left.-\frac{(2j)!\;{}_2F_1\left(\begin{matrix}-p,-p-\frac{1}{2}\\ -2j\end{matrix}\,\middle|\,-\frac{4\,b}{a^2}\right)}{(2p+1)!\,(2j+m-2p)!}\,B_{2j+m-2p}\right),
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\int_0^1 F_{2j+1}^{a,b}(x)\,B_m(x)\,dx = m!\,a^{2j+1}\times \\
&\left((2j+1)!\sum_{p=0}^{j}\frac{{}_2F_1\left(\begin{matrix}-p,-p-\frac{1}{2}\\ -2j-1\end{matrix}\,\middle|\,-\frac{4\,b}{a^2}\right)}{(2j+m-2p+1)!\,(2p+1)!}\,B_{2j+m-2p+1}\right. \\
&\left.-\sum_{p=0}^{j-1}\left(-\frac{a^{-2(p+1)}b^{p+1}(2j-p)!}{(p+1)!}+\frac{(2j+1)!\,{}_2F_1\left(\begin{matrix}-p-1,-p-\frac{1}{2}\\ -2j-1\end{matrix}\,\middle|\,-\frac{4\,b}{a^2}\right)}{(2p+2)!}\right)\frac{B_{2j+m-2p}}{(2j+m-2p)!}\right).
\end{aligned} \tag{68}$$

Proof. Formulas (67) and (68) can be obtained by the application to Formulas (63) and (64) making use of the connection Formula (29). □

Corollary 21. *For all positive integers j and m, the following integral formulas hold:*

$$\begin{aligned}
&\int_0^1 L_{2j}^{c,d}(x)\,B_m(x)\,dx = c^{2j}m!\times \\
&\sum_{p=0}^{j-1}\left(\left(-\frac{2\,j\,c^{-2(p+1)}\,d^{p+1}(2j-p-2)!}{(p+1)!}+\frac{(2j)!\;{}_2F_1\left(\begin{matrix}-p-1,-p-\frac{1}{2}\\ 1-2j\end{matrix}\,\middle|\,-\frac{4\,d}{c^2}\right)}{(2p+2)!}\right)\frac{B_{2j+m-2p-1}}{(2j+m-2p-1)!}\right. \\
&\qquad\left.-\frac{(2j)!\;{}_2F_1\left(\begin{matrix}-p,-p-\frac{1}{2}\\ 1-2j\end{matrix}\,\middle|\,-\frac{4\,d}{c^2}\right)}{(2p+1)!\,(2j+m-2p)!}\,B_{2j+m-2p}\right),
\end{aligned} \tag{69}$$

$$\begin{aligned}
&\int_0^1 L_{2j+1}^{c,d}(x)\,B_m(x)\,dx = m!\,c^{2j+1}\left((2j+1)!\sum_{p=0}^{j}\frac{{}_2F_1\left(\begin{matrix}-p,-p-\frac{1}{2}\\ -2j\end{matrix}\,\middle|\,-\frac{4\,d}{c^2}\right)}{(2p+1)!\,(2j+m-2p+1)!}\,B_{2j+m-2p+1}\right. \\
&\left.-(2j+1)\sum_{p=0}^{j-1}\frac{B_{2j+m-2p}}{(2j+m-2p)!}\left(-\frac{c^{-2(p+1)}\,d^{p+1}\,(2j-p-1)!}{(p+1)!}+\frac{(2j)!\,{}_2F_1\left(\begin{matrix}-p-1,-p-\frac{1}{2}\\ -2j\end{matrix}\,\middle|\,-\frac{4\,d}{c^2}\right)}{(2p+2)!}\right)\right).
\end{aligned} \tag{70}$$

Proof. Formulas (69) and (70) can be obtained by the application of Formulas (63) and (64), making use of the connection Formula (30). □

Corollary 22. *For all positive integers j and m, the following integral formulas hold:*

$$\begin{aligned}\int_0^1 H_{2j}(x)B_m(x)\,dx =& 4^{j-1}\,(2j)!\,m!\times\\ &\sum_{p=0}^{j-1}\left(\frac{(-1)^{p+1}\,4^{-p}\left(-1+{}_1F_1\left(-p-1;\frac{1}{2};1\right)\right)}{(p+1)!\,(2j+m-2p-1)!}\,B_{2j+m-2p-1}\right.\\ &\left.-\frac{4\,U\left(-p,\frac{3}{2},1\right)}{(2p+1)!\,(2j+m-2p)!}\,B_{2j+m-2p}\right),\end{aligned} \tag{71}$$

$$\begin{aligned}\int_0^1 H_{2j+1}(x)\,B_m(x)\,dx =& 2^{2j+1}\,m!\left(2^{2j+1}\sum_{p=0}^{j}\frac{U\left(-p,\frac{3}{2},1\right)}{(2p+1)!\,(2j+m-2p+1)!}\,B_{2j+m-2p+1}\right.\\ &\left.+\sum_{p=0}^{j-1}\frac{(-1)^p\,2^{2j-2p-1}\left(-1+{}_1F_1\left(-p-1;\frac{1}{2};1\right)\right)}{(p+1)!\,(2j+m-2p)!}\,B_{2j+m-2p}\right).\end{aligned} \tag{72}$$

Proof. Formulas (71) and (72) can be obtained by the application of Formulas (63) and (64) making use of the connection Formula (31). □

6. Concluding Remarks

This paper presented new identities involving Bernoulli polynomials. The associated identities of Bernoulli numbers can be deduced as direct special cases. As combinations of Bernoulli polynomials, we could express different polynomials. This yielded new formulas that connect different polynomials with Bernoulli polynomials. The inversion formulas to these formulas can be also obtained but the linking coefficients in them can involve Bernoulli numbers. Furthermore, we presented two applications to compute definite integrals for different polynomials and also for products of different polynomials with Bernoulli polynomials. As future work, the ideas in this article can be extended to obtain several identities for other types of polynomials and their associated numbers. Other important definite integrals can also be computed based on the formulas established in this paper.

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