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Some Formulas and Recurrences of Certain Orthogonal Polynomials Generalizing Chebyshev Polynomials of the Third-Kind

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Abstract: This paper investigates certain Jacobi polynomials that involve one parameter and generalize the well-known orthogonal polynomials called Chebyshev polynomials of the third-kind. Some new formulas are developed for these polynomials. We will show that some of the previous results in the literature can be considered special ones of our derived formulas. The derivatives of the moments of these polynomials are derived. Hence, two important formulas that explicitly give the derivatives and the moments of these polynomials in terms of their original ones can be deduced as special cases. Some new expressions for the derivatives of different symmetric and non-symmetric polynomials are expressed as combinations of the generalized third-kind Chebyshev polynomials. Some new linearization formulas are also given using different approaches. Some of the appearing coefficients in derivatives and linearization formulas are given in terms of different hypergeometric functions. Furthermore, in several cases, the existing hypergeometric functions can be summed using some standard formulas in the literature or through the employment of suitable symbolic algebra, in particular, Zeilberger's algorithm.

Keywords: orthogonal polynomials; Chebyshev polynomials; recurrence relations; connection and linearization formulas; generalized hypergeometric functions

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1. Introduction

Orthogonal polynomials in general and Jacobi polynomials, in particular, occupy distinguished places due to their great use in applied mathematics (see, for instance, [1–4]). It is commonly known that symmetric classes and non-symmetric classes of polynomials are included in the Jacobi polynomial class. The most well-known used classes of polynomials are the classes of Gegenbauer, Legendre and the four classes of Chebyshev polynomials. Chebyshev polynomials of the third- and fourth-kinds are the two non-symmetric classes of polynomials, whereas the other four classes of polynomials are symmetric. Recently, Abd-Elhameed and Alkenedri in [5] investigated a non-symmetric class of polynomials that generalizes the class of Chebyshev polynomials of the third-kind. On the basis of the use of the spectral Galerkin method, they additionally utilized this class of polynomials to handle particular linear and non-linear ordinary differential equations.

Chebyshev polynomials are pivotal in many branches. The well-known kinds of Chebyshev polynomials are special kinds of the Jacobi polynomials, whereas the other two kinds of Chebyshev polynomials, namely Chebyshev polynomials of the fifth and sixth kinds are special types of the so-called ultraspherical polynomials (see, [6,7]). Each kind of the six kinds of Chebyshev polynomials has its role in different fields, in particular in the scope of numerical analysis and approximation theory. For example, the author in [8] obtained numerical solutions of integral and integro-differential equations employing the third-kind Chebyshev polynomials. In [9], the authors employed a certain shifted second-kind Chebyshev operational matrix of fractional integration to treat some types of fractional differential equations. The authors in [10] developed certain tau and Galerkin operational matrices of derivatives for handling Emden–Fowler third-order-type equations based on using Chebyshev polynomials of the second-kind and a type of their modified polynomials. Some other contributions regarding different kinds of Chebyshev polynomials can be found in [11–14].

Linearization and connection formulas between the different special functions in general and orthogonal polynomials, in particular, are crucial. One can be referred for some applications to these formulas to [15,16]. There are several considerable old and recent contributions concerning these formulas. In this regard, and for some old contributions, one can be referred to [17–19]. Other important contributions can be found in [20–24]. Regarding some recent articles that deal with the linearization formulas of Jacobi polynomials and their different classes, one can be referred to the papers of Abd-Elhameed [25,26].

Hypergeometric functions and their generalized functions are crucial in mathematical analysis and its applications. Almost all important functions and polynomials may be represented in terms of them. For example, the linearization and connection coefficients between different polynomials can be expressed in terms of generalized hypergeometric functions of certain arguments. As an important example, the connection coefficients between two different parameters Jacobi polynomials can be expressed in terms of a certain terminating hypergeometric function of the type ${}_3F_2(1)$ that can be reduced in some specific choices of the Jacobi polynomials parameters. In addition, the authors in [27] derived some new linearization formulas of Jacobi polynomials based on reducing some types of hypergeometric functions. Moreover, the high-order derivatives of some celebrated polynomials can be linked with their original polynomials by coefficients that involve hypergeometric functions. For instance, the authors in [28] found new formulas that express the derivatives of the fifth-kind Chebyshev polynomials in terms of their original ones. The linking coefficients involve terminating hypergeometric functions of the type ${}_4F_3(1)$.

It is interesting to investigate various polynomial sequences in general and orthogonal polynomials in particular. The authors in [29] investigated two types of generalized Fibonacci and generalized Lucas polynomials, and they developed some connection formulas between them. The authors in [30] investigated general odd and even central factorial polynomial sequences. The same authors suggested an approach for investigating orthogonal polynomials sequence based on matrix calculus in [31]. The authors in [32] developed recurrence relations and determinant forms for general polynomial sequences. A matrix approach for the semiclassical and coherent orthogonal polynomials is developed in [33]. The establishment of the different formulas related to special functions is crucial in numerical analysis. For example, the derivatives expressions of different polynomials in terms of their original ones serve in obtaining numerical solutions to different types of differential equations. The authors found new formulas for the derivatives of the third- and fourth-kinds of Chebyshev polynomials in [34]. In addition, they employed them to deal with some types of even-order boundary value problems. The author in [35] found new formulas for the derivatives of Chebyshev polynomials of the sixth-kind. Furthermore, these expressions served to obtain a numerical solution to the non-linear one-dimensional Burgers' equation.

The main aim of this article is to theoretically investigate the generalized third-kind Chebyshev polynomials. Several formulas concerning these polynomials are established. Some of the well-known formulas in the literature are obtained as special cases of our developed formulas. Some connections between these polynomials with some celebrated polynomials are given. We think that most of the formulas presented in this paper are new and useful in a wide range of applications.

The paper is organized as follows. The next section presents some properties of Jacobi polynomials in general and some special classes of Jacobi polynomials in particular. In addition, some properties for some other polynomials are stated. Section 3 derives the formula that expresses the derivatives of the moments of the generalized third-kind polynomials. Some important specific formulas are deduced as special cases. Section 4 derives other expressions for the derivatives of the generalized third-kind Chebyshev polynomials but in terms of different symmetric and non-symmetric polynomials. Some connection formulas are presented as special cases of the derivatives formulas. Some linearization formulas involving the generalized third-kind Chebyshev polynomials are given in Section 5. A few final remarks are presented in Section 6.

2. Preliminaries and Some Essential Formulas

This section is devoted to displaying some properties and formulas of the classical Jacobi classical polynomials. Some formulas concerning a class of polynomials that generalizes the third-kind Chebyshev class are also given. An overview of some other polynomials is also presented.

2.1. Some Fundamental Properties and Connection Formulas of Jacobi Polynomials

This section focuses on presenting some fundamental properties of Jacobi polynomials. Some connection formulas between different classes of Jacobi polynomials are presented. Some formulas concerned with the generalized third-kind generalized polynomials that will be useful throughout the paper are also displayed.

It is well-known that the Jacobi polynomials can be represented as

$$P_r^{(\gamma,\delta)}(x) = \frac{(\gamma+1)_r}{r!}\,{}_2F_1\left(\begin{array}{c} -r, r+\gamma+\delta+1 \\ \gamma+1 \end{array}\middle|\frac{1-x}{2}\right).$$

From now on, we will use the following normalized Jacobi polynomials that are used in [25]:

$$V_r^{(\gamma,\delta)}(x) = \,{}_2F_1\left(\begin{array}{c} -r, r+\gamma+\delta+1 \\ \gamma+1 \end{array}\middle|\frac{1-x}{2}\right).$$

The above polynomials satisfy the following property:

$$V_r^{(\gamma,\delta)}(1) = 1, \quad r = 0,1,2,\ldots.$$

Among the main advantages of Jacobi polynomials is that they include four celebrated classes of Chebyshev polynomials. In fact, we have

$$\begin{aligned} T_r(x) &= V_r^{(-\frac{1}{2},-\frac{1}{2})}(x), & U_r(x) &= (r+1)\,V_r^{(\frac{1}{2},\frac{1}{2})}(x), \\ V_r(x) &= V_r^{(-\frac{1}{2},\frac{1}{2})}(x), & W_r(x) &= (2r+1)\,V_r^{(\frac{1}{2},-\frac{1}{2})}(x), \end{aligned}$$

where $T_r(x)$, $U_r(x)$, $V_r(x)$, and $W_r(x)$ represent respectively, the first-, second-, third-, and fourth- kinds Chebyshev polynomials.

Furthermore, the two symmetric polynomials, namely Legendre and ultraspherical polynomials can be deduced as special cases of the polynomial $V_r^{(\gamma,\delta)}(x)$. We have

$$U_r^{(\lambda)}(x) = V_r^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x), \qquad P_r(x) = V_r^{(0,0)}(x),$$

while $U_r^{(\lambda)}(x)$, and $P_r(x)$ denote respectively, the ultraspherical and Legendre polynomials.

The four different types of Chebyshev polynomials have the following trigonometric representations, which we comment on here (see [36]):

$$T_r(x) = \cos(r\,\theta), \qquad U_r(x) = \frac{\sin((r+1)\,\theta)}{\sin\theta},$$
$$V_r(x) = \frac{\cos\left(\left(r+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{\theta}{2}\right)}, \qquad W_r(x) = \frac{\sin\left(\left(r+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)},$$

where $\theta = \cos^{-1}(x)$.

The trigonometric representations of the four kinds of Chebyshev polynomials imply that each kind of them can be defined for negative subscripts. We refer here to the following identities ([16]):

$$T_{-r}(x) = T_r(x), \quad U_{-r}(x) = -U_{r-2}(x),$$
$$V_{-r}(x) = V_{r-1}(x), \quad W_{-r}(x) = -W_{r-1}(x).$$

One can consult the important books [36,37] for properties of Jacobi polynomials and their special classes.

2.2. Connection Formulas Between Different Jacobi Polynomials

The connection problems between different classes of Jacobi polynomials are important. The following theorem, which connects two different parameters of Jacobi polynomials, will be useful to derive our results in the upcoming sections.

Theorem 1 ([5])**.** *For every non-negative integer m, the following connection formula holds:*

$$V_m^{(\alpha,\beta)}(x) = \sum_{r=0}^{m} \xi_{r,m}\, V_{m-r}^{(\lambda,\mu)}(x), \tag{1}$$

where the linearization coefficients $\xi_{r,m}$ are given by

$$\xi_{r,m} = \frac{m!\,\Gamma(\alpha+1)\,\Gamma(2m-r+\alpha+\beta+1)\,\Gamma(m-r+\lambda)\,\Gamma(m-r+\lambda+\mu+1)}{(m-r)!\,r!\,\Gamma(m-r+\alpha+1)\,\Gamma(m+\alpha+\beta+1)\Gamma(\lambda+1)\Gamma(2m-2r+\lambda+\mu+1)} \times$$
$${}_3F_2\left(\begin{matrix} -r, 1-r+m+\lambda, 1-r+2m+\alpha+\beta \\ 2-2r+2m+\lambda+\mu, 1-r+m+\alpha \end{matrix}\,\middle|\, 1\right). \tag{2}$$

Remark 1. *It is worth noting here that the terminating hypergeometric function that appears in* (2) *cannot be summed in general, but for some particular choices of the involved parameters, it can be summed. In the following two corollaries, we give two important specific connection formulas that will be useful in the sequel.*

Corollary 1. *For every non-negative integer m, the following connection formula holds:*

$$V_m^{(\alpha,\alpha)}(x) = \frac{m+2\alpha+1}{2m+2\alpha+1}\, V_m^{(\alpha,\alpha+1)}(x) + \frac{m}{2m+2\alpha+1}\, V_{m-1}^{(\alpha,\alpha+1)}(x). \tag{3}$$

Proof. If we start with the connection Formula (1) for the following choices: $\beta = \lambda = \alpha$, and $\mu = \alpha + 1$, then the connection coefficients $\xi_{r,m}$ in such case are given by

$$\xi_{r,m} = \frac{m!\,\Gamma(m-r+2\alpha+2)\Gamma(2m-r+2\alpha+1)}{(m-r)!\,r!\,\Gamma(m+2\alpha+1)\Gamma(2m-2r+2\alpha+2)} \times$$
$${}_3F_2\left(\begin{matrix} -r, m-r+\alpha+1, 2m-r+2\alpha+1 \\ 2m-2r+2\alpha+3, m-r+\alpha+1 \end{matrix}\,\middle|\, 1\right).$$

It is not difficult to see that

$$
{}_3F_2\left(\begin{array}{c} -r, m-r+\alpha+1, 2m-r+2\alpha+1 \\ 2m-2r+2\alpha+3, m-r+\alpha+1 \end{array}\bigg| 1\right) = \begin{cases} 1, & r=0, \\ \dfrac{1}{2m+2\,\alpha+1}, & r=1, \\ 0, & \text{otherwise}, \end{cases}
$$

and therefore the coefficients $\xi_{r,m}$ reduce to

$$
\xi_{r,m} = \begin{cases} \dfrac{m+2\alpha+1}{2m+2\,\alpha+1}, & r=0, \\ \dfrac{m}{2m+2\,\alpha+1}, & r=1, \\ 0, & \text{otherwise}. \end{cases}
$$

□

Corollary 2. *For every non-negative integer m, the following connection formula holds:*

$$
V_m^{(\alpha,\alpha+1)}(x) = \sum_{r=0}^{m} \frac{(-1)^r\,(m-r+1)_r\,(2m+2\alpha-2r+1)}{(m+2\alpha-r+1)_{r+1}}\, V_{m-r}^{(\alpha,\alpha)}(x). \tag{4}
$$

Proof. Setting $\beta = \alpha+1$, $\lambda = \mu = \alpha$ in (1) produces the formula shown below:

$$
\begin{aligned}
V_m^{(\alpha,\alpha+1)}(x) =& \frac{m!}{\Gamma(m+2\alpha+2)} \sum_{r=0}^{m} \frac{\Gamma(1-r+m+2\alpha)\,\Gamma(-r+2(m+\alpha+1))}{r!\,(m-r)!\,\Gamma(1-2r+2m+2\alpha)} \times \\
& {}_2F_1\left(\begin{array}{c} -r, -r+2(m+\alpha+1) \\ 2(1-r+m+\alpha \end{array}\bigg| 1\right) V_{m-r}^{(\alpha,\alpha+1)}(x).
\end{aligned}
$$

Chu–Vandemond's identity enables one to compute the last ${}_2F_1(1)$ in the form

$$
{}_2F_1\left(\begin{array}{c} -r, -r+2(m+\alpha+1) \\ 2(1-r+m+\alpha \end{array}\bigg| 1\right) = \frac{(-1)^r\,r!\,\Gamma(2(1-r+m+\alpha))}{\Gamma(-r+2(m+\alpha+1))},
$$

and thus the connection Formula (4) can be obtained. □

2.3. Some Fundamental Properties of the Generalized Third-Kind Chebyshev Polynomials

Among the classes of Jacobi polynomials that were recently investigated in [5] is the class of the polynomials $V_j^{(\alpha,\alpha+1)}(x)$, $j \geq 0$. This class is of interest since it generalizes the third-kind Chebyshev polynomials class. The following three lemmas concerning these polynomials are of fundamental importance to derive some of our proposed results in what follows.

Lemma 1 ([5])**.** *Let j be a non-negative integer. The polynomials $V_j^{(\alpha,\alpha+1)}(x)$ has the following representation:*

$$
V_j^{(\alpha,\alpha+1)}(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} A_{r,j}\, x^{j-2r} + \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} B_{r,j}\, x^{j-2r-1}, \tag{5}
$$

where

$$
\begin{aligned}
A_{r,j} &= \frac{(-1)^r\,2^{j-2r+2\alpha+1} j!\;\Gamma(\alpha+1)\,\Gamma\left(j-r+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\,r!\;(j-2r)!\,\Gamma(j+2\alpha+2)}, \\
B_{r,j} &= \frac{(-1)^{r+1}\,j!\;2^{j-2r+2\alpha}\,\Gamma(\alpha+1)\,\Gamma\left(j-r+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\,r!\;(j-2r-1)!\;\Gamma(j+2\alpha+2)},
\end{aligned}
$$

where $\lfloor z \rfloor$ *denotes the well-known floor function.*

Lemma 2 ([5])**.** *For every non-negative integer j, the following inversion formula holds:*

$$x^j = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} F_{i,j}\, V_{j-2i}^{(\alpha,\alpha+1)}(x) + \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} G_{i,j}\, V_{j-2i-1}^{(\alpha,\alpha+1)}(x), \tag{6}$$

where

$$F_{i,j} = \frac{2^{-1-j-2\alpha}\sqrt{\pi}\, j!\ \Gamma(2-2i+j+2\alpha)}{i!\ (j-2i)!\ \Gamma(\alpha+1)\,\Gamma\left(\frac{3}{2}-i+j+\alpha\right)}, \tag{7}$$

$$G_{i,j} = \frac{2^{-1-j-2\alpha}\sqrt{\pi}\, j!\ \Gamma(1-2i+j+2\alpha)}{i!\ (j-2i-1)!\ \Gamma(\alpha+1)\,\Gamma\left(\frac{3}{2}-i+j+\alpha\right)}. \tag{8}$$

2.4. An Overview on a Generalized Class of Fibonacci Polynomials

This section presents an overview of some other polynomials. Among the important symmetric classes of polynomials are the classes of Fibonacci polynomials and their generalizations. Recently, Abd-Elhameed et al. in [38] investigated a type of generalized Fibonacci polynomials. This class of polynomials can be generated by the following recurrence relation:

$$F_k^{A,B}(x) = A\,x\,F_{k-1}^{A,B}(x) + B\,F_{k-2}^{A,B}(x), \quad F_0^{A,B}(x) = 1,\ F_1^{A,B}(x) = A\,x, \quad k \geq 2. \tag{9}$$

It is to be noted that several celebrated classes of polynomials can be obtained as special cases of the generalized class $F_k^{A,B}(x)$ (see, [38]). For example, the Fibonacci polynomials $F_{k+1}(x)$ are a special case of $F_k^{A,B}(x)$. In fact, we have:

$$F_{k+1}(x) = F_k^{1,1}(x).$$

Among the important properties of the polynomials $F_k^{A,B}(x)$ is the moment formula for these polynomials. Rewriting the recurrence relation (9) in the form:

$$x\,F_k^{A,B}(x) = \frac{1}{A}\,F_{k+1}^{A,B}(x) - \frac{B}{A}\,F_{k-1}^{A,B}(x),$$

then it is easy to obtain the moment formula for the generalized class $F_k^{A,B}(x)$. The moment formula for these polynomials is stated in the following lemma.

Lemma 3. *Let r and k be any two non-negative integers. The following moment formula applies:*

$$x^r\,F_k^{A,B}(x) = \sum_{m=0}^{r} \binom{r}{m} A^{-r}(-B)^m\,F_{k+r-2m}^{A,B}(x). \tag{10}$$

Proof. The proof can be easily calculated by induction based on the application of the recurrence relation (9). □

3. Derivatives of the Moments of the Generalized Third-Kind Chebyshev Polynomials

This section is devoted to deriving expressions for the derivatives of the moments of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$. In fact, we will state and prove two important theorems. The first theorem expresses the derivatives of the moments of $V_k^{(\alpha,\alpha+1)}(x)$ in terms of their original polynomials. We comment here that the following two expressions can be obtained as special cases:

1. The moments formula for $V_k^{(\alpha,\alpha+1)}(x)$.

2. The derivatives expression for the polynomials $V_k^{(\alpha,\alpha+1)}(x)$.

The second theorem gives the formula for the derivatives of the moments of $V_k^{(\alpha,\alpha+1)}(x)$ in terms of the ultraspherical polynomials.

Theorem 2. *Let k, r, and q be non-negative integers with $k + r \geq q$. The following formula is valid:*

$$D^q\Big(x^r\, V_k^{(\alpha,\alpha+1)}(x)\Big) = \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-q) \rfloor} H_{p,k,q,r}\, V_{k+r-q-2p}^{(\alpha,\alpha+1)}(x) + \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-q-1) \rfloor} \bar{H}_{p,k,q,r}\, V_{k+r-q-2p-1}^{(\alpha,\alpha+1)}(x), \tag{11}$$

where the coefficients $H_{p,k,q,r}$ and $\bar{H}_{p,k,q,r}$ are given respectively by the following formulas:

$$\begin{aligned} H_{p,k,q,r} =& \frac{2^{q-r} k!\, \Gamma(k-2p-q+r+2\alpha+2)}{(k-2p-q+r)!\, \Gamma(k+2\alpha+2)} \times \\ & \sum_{\ell=0}^{p} \frac{(-1)^\ell\, (k-2\ell+r-1)!\, \Gamma\Big(k-\ell+\alpha+\frac{1}{2}\Big)}{(k-2\ell)!\, \ell!\, (p-\ell)!\, \Gamma\big(k-\ell-p-q+r+\alpha+\frac{3}{2}\big)} \times \\ & \bigg((k-2\ell)(\ell-p) + \frac{1}{2}(k-2\ell+r)(2k-2\ell+2\alpha+1)\bigg), \end{aligned} \tag{12}$$

$$\begin{aligned} \bar{H}_{p,k,q,r} =& \frac{2^{-1-r+q}\, \Gamma(k+r-2p-q+2\alpha+1)}{(k+r-2p-q-1)!\, \Gamma(k+2\alpha+2)} \times \\ & \sum_{\ell=0}^{p} \frac{(-1)^\ell\, k!\, (k-2\ell+r-1)!\, \Gamma\Big(k-\ell+\alpha+\frac{1}{2}\Big)}{\ell!\, (p-\ell)!\, (k-2\ell)!\, \Gamma\big(k-\ell+r-p-q+\alpha+\frac{3}{2})\big)} \\ & \times (2(k-2\ell)(p+q) + r(2\ell+2\alpha+1)). \end{aligned} \tag{13}$$

Proof. Making use of the analytic form of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ in (5) enables one to write the moments derivatives $D^q\Big(x^r\, V_k^{(\alpha,\alpha+1)}(x)\Big)$ in the form

$$D^q\Big(x^r\, V_k^{(\alpha,\alpha+1)}(x)\Big) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \bar{A}_{\ell,k,r,q}\, x^{k+r-2\ell-q} + \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \bar{B}_{\ell,k,r,q}\, x^{k+r-2\ell-q-1}, \tag{14}$$

where

$$\begin{aligned} \bar{A}_{\ell,k,r,q} =& \frac{(-1)^\ell\, 2^{k-2\ell+2\alpha+1} k!\, \Gamma(\alpha+1) \Gamma\big(k-\ell+\alpha+\frac{3}{2}\big) (k-2\ell-q+r+1)_q}{\sqrt{\pi}\, \ell!\, (k-2\ell)!\, \Gamma(k+2\alpha+2)}, \\ \bar{B}_{\ell,k,r,q} =& \frac{(-1)^{\ell+1}\, 2^{k-2\ell+2\alpha}\, k!\, \Gamma(\alpha+1)\, \Gamma\Big(k-\ell+\alpha+\frac{1}{2}\Big)\, (k-2\ell-q+r)_q}{\sqrt{\pi}\, \ell!\, (k-2\ell-1)!\, \Gamma(k+2\alpha+2)}. \end{aligned}$$

Now, the inversion Formula (6) converts Formula (14) into the following one:

$$\begin{aligned} D^q\Big(x^r\,V_k^{(\alpha,\alpha+1)}(x)\Big) = & \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} \bar{A}_{\ell,k,r,q} \left(\sum_{p=0}^{\lfloor \frac{1}{2}(k+r-2\ell-q) \rfloor} F_{p,k+r-2\ell-q}\, V_{k+r-2\ell-q-2p}^{(\alpha,\alpha+1)}(x) \right. \\ & \left. + \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-2\ell-q-1) \rfloor} G_{p,k+r-q-2\ell}\, V_{k+r-2\ell-2p-q-1}^{(\alpha,\alpha+1)}(x) \right) \\ & + \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \bar{B}_{\ell,k,r,q} \left(\sum_{p=0}^{\lfloor \frac{1}{2}(k+r-2\ell-q-1) \rfloor} F_{p,k+r-2\ell-q-1}\, V_{k+r-2\ell-2p-q-1}^{(\alpha,\alpha+1)}(x) \right. \\ & \left. + \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-2\ell-q-2) \rfloor} G_{p,k+r-2\ell-q-1}\, V_{k+r-2\ell-2p-q-2}^{(\alpha,\alpha+1)}(x) \right), \end{aligned}$$

where the coefficients $F_{i,k}$ and $G_{i,k}$ are, respectively, given in (7) and (8). Some lengthy manipulations lead to the following formula:

$$D^q\Big(x^r\,V_k^{(\alpha,\alpha+1)}(x)\Big) = \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-q) \rfloor} H_{p,k,q,r}\, V_{k+r-q-2p}^{(\alpha,\alpha+1)}(x) + \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-q-1) \rfloor} \bar{H}_{p,k,q,r}\, V_{k+r-2p-q-1}^{(\alpha,\alpha+1)}(x),$$

where the coefficients $H_{p,k,q,r}$ and $\bar{H}_{p,k,q,r}$ are given by (12) and (13). This proves Theorem 2. □

Remark 2. *Two important formulas can be obtained as two consequences of Formula (11). More precisely, the high-order derivatives formula of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ can be deduced by setting $r = 0$, while the moment's formula can be obtained from Formula (11) by setting $q = 0$. The following two corollaries exhibit these formulas.*

Corollary 3. *Let r and k be non-negative integers. One has the following moment formula:*

$$x^r\,V_k^{(\alpha,\alpha+1)}(x) = \sum_{p=0}^{\lfloor \frac{k+r}{2} \rfloor} W_{p,k,r}\, V_{k+r-2p}^{(\alpha,\alpha+1)}(x) + \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-1) \rfloor} \bar{W}_{p,k,r}\, V_{k+r-2p-1}^{(\alpha,\alpha+1)}(x), \tag{15}$$

where

$$\begin{aligned} W_{p,k,r} = & \frac{k!\;\Gamma(k+r-2p+2\alpha+2)}{2^r\,(k+r-2p)!\;\Gamma(k+2\alpha+2)} \times \\ & \sum_{\ell=0}^{p} \frac{(-1)^\ell\,(k-2\ell+r-1)!\,\Gamma\Big(k-\ell+\alpha+\frac{1}{2}\Big)}{\ell!\;(k-2\ell)!\;(p-\ell)!\;\Gamma\Big(k-\ell+r-p+\alpha+\frac{3}{2}\Big)} \times \\ & \left((k-2\ell)(\ell-p) + \frac{1}{2}(k-2\ell+r)\,(2k-2\ell+2\alpha+1)\right), \\ \bar{W}_{p,k,r} = & \frac{k!\;\Gamma(k+r-2p+2\alpha+1)}{2^{r+1}\,(k+r-2p-1)!\;\Gamma(k+2\alpha+2)} \times \\ & \sum_{\ell=0}^{p} \frac{(-1)^\ell\,(r+2\ell r+2kp-4\ell p+2r\alpha)\,(k-2\ell+r-1)!\;\Gamma\Big(k-\ell+\alpha+\frac{1}{2}\Big)}{\ell!\;(k-2\ell)!\;(p-\ell)!\;\Gamma\Big(k-\ell+r-p+\alpha+\frac{3}{2}\Big)}. \end{aligned}$$

Proof. If we set $q = 0$ in Formula (11), then Formula (15) can be obtained. □

Corollary 4. *Let k and q be non-negative integers such that $k \geq q$. The following derivatives formula holds*

$$\begin{aligned} D^q V_k^{(\alpha,\alpha+1)}(x) =& \frac{2^q\, k!}{(q-1)!\,\Gamma(k+2\alpha+2)} \times \\ & \left(\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(p+q-1)!\,\Gamma\left(k-p+\alpha+\frac{3}{2}\right)\Gamma(k-2p-q+2\alpha+2)}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)} V_{k-q-2p}^{(\alpha,\alpha+1)}(x) \right. \\ & \left. + \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{(p+q)!\,\Gamma\left(k-p+\alpha+\frac{1}{2}\right)\Gamma(k-2p-q+2\alpha+1)}{p!\,(k-2p-q-1)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)} V_{k-q-2p-1}^{(\alpha,\alpha+1)}(x) \right). \end{aligned} \tag{16}$$

Proof. Setting $r = 0$ in Formula (11) produces the formula shown below:

$$\begin{aligned} D^q V_k^{(\alpha,\alpha+1)}(x) =& \frac{2^{q-1}\, k!}{\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(2k-2p+2\alpha+1)\Gamma(k-2p-q+2\alpha+2)}{(k-2p-q)!} \times \\ & \sum_{\ell=0}^{p} \frac{(-1)^\ell\, \Gamma\left(k-\ell+\alpha+\frac{1}{2}\right)}{\ell!\,(p-\ell)!\,\Gamma\left(k-\ell-p-q+\alpha+\frac{3}{2}\right)} V_{k-q-2p}^{(\alpha,\alpha+1)}(x) \\ & + \frac{2^q k!}{\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{(p+q)\Gamma(k-2p-q+2\alpha+1)}{(k-2p-q-1)!} \times \\ & \sum_{\ell=0}^{p} \frac{(-1)^\ell\, \Gamma\left(k-\ell+\alpha+\frac{1}{2}\right)}{\ell!\,(p-\ell)!\,\Gamma\left(k-\ell-p-q+\alpha+\frac{3}{2}\right)} V_{k-q-2p-1}^{(\alpha,\alpha+1)}(x). \end{aligned} \tag{17}$$

Using the identity:

$$\begin{aligned} & \sum_{\ell=0}^{p} \frac{(-1)^\ell\, \Gamma\left(k-\ell+\alpha+\frac{1}{2}\right)}{\ell!\,(p-\ell)!\,\Gamma\left(k-\ell-p-q+\alpha\frac{3}{2}\right)} \\ & = \frac{\Gamma\left(k+\alpha+\frac{1}{2}\right)}{p!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)}\, {}_2F_1\left(\left. \begin{matrix} -p, -\frac{1}{2}-k+p+q-\alpha \\ \frac{1}{2}-k-\alpha \end{matrix} \right| 1 \right), \end{aligned}$$

along with the Chu–Vandermonde identity, serves to convert Formula (17) into the following form:

$$\begin{aligned} D^q V_k^{(\alpha,\alpha+1)}(x) =& \frac{2^q k!}{(q-1)!\,\Gamma(k+2\alpha+2)} \times \\ & \left(\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(p+q-1)!\,\Gamma\left(k-p+\alpha+\frac{3}{2}\right)\Gamma(k-2p-q+2\alpha+2)}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)} V_{k-q-2p}^{(\alpha,\alpha+1)}(x) \right. \\ & \left. + \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{(p+q)!\,\Gamma\left(k-p+\alpha+\frac{1}{2}\right)\Gamma(k-2p-q+2\alpha+1)}{p!\,(k-2p-q-1)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)} V_{k-q-2p-1}^{(\alpha,\alpha+1)}(x) \right). \end{aligned}$$

□

Remark 3. *It is to be noted here that the result in (16) fits with the same result obtained in [5].*

Now, we give the formula that expresses the derivatives of the moments of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ in terms of the ultraspherical polynomials. This formula generalizes important formulas that may be deduced as special cases.

Theorem 3. *Let k, r, and q be non-negative integers with $k + r \geq q$. The following formula is valid:*

$$
\begin{aligned}
D^q\Big(x^r\, V_k^{(\alpha,\alpha+1)}(x)\Big) =& \frac{2^{2-m+q+2\alpha-2\lambda}(k+m)!\,\Gamma(\alpha+1)\Gamma\big(k+\alpha+\frac{3}{2}\big)}{\Gamma(k+2\alpha+2)\Gamma\Big(\lambda+\frac{1}{2}\Big)}\times \\
& \sum_{p=0}^{\lfloor\frac{1}{2}(k+m-q)\rfloor} \frac{(k+m-2p-q+\lambda)\Gamma(k+m-2p-q+2\lambda)}{p!\,(k+m-2p-q)!\,\Gamma(k+m-p-q+\lambda+1)}\times \\
& {}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-\frac{k}{2}, -\frac{k}{2}, -k-m+p+q-\lambda \\ -\frac{k}{2}-\frac{m}{2}, \frac{1}{2}-\frac{k}{2}-\frac{m}{2}, -\frac{1}{2}-k-\alpha \end{matrix}\,\middle|\, 1\right) U_{k+m-q-2p}^{(\lambda)}(x) \\
& + \frac{2^{2-m+q+2\alpha-2\lambda}k(k+m-1)!\,\Gamma(\alpha+1)\Gamma\Big(k+\alpha+\frac{1}{2}\Big)}{\Gamma(k+2\alpha+2)\Gamma\Big(\lambda+\frac{1}{2}\Big)}\times \\
& \sum_{p=0}^{\lfloor\frac{1}{2}(k+m-q)\rfloor} \frac{(1-k-m+2p+q-\lambda)\Gamma(k+m-2p-q+2\lambda-1)}{p!\,(k+m-2p-q-1)!\,\Gamma(k+m-p-q+\lambda)}\times \\
& {}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-\frac{k}{2}, 1-\frac{k}{2}, 1-k-m+p+q-\lambda \\ \frac{1}{2}-\frac{k}{2}-\frac{m}{2}, 1-\frac{k}{2}-\frac{m}{2}, \frac{1}{2}-k-\alpha \end{matrix}\,\middle|\, 1\right) U_{k+m-q-2p-1}^{(\lambda)}(x).
\end{aligned}
\tag{18}
$$

Proof. The proof is similar to that given in the proof of Theorem 2. It is based on utilizing the power form representation of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ along with the utilization of the inversion formula of the ultraspherical polynomials given by [39]

$$
x^p = \frac{2^{1-p}\Gamma(\lambda+1)}{\Gamma(2\lambda+1)} \sum_{t=0}^{\lfloor\frac{p}{2}\rfloor} \frac{(p-2t+\lambda)p!\,\Gamma(p-2t+2\lambda)}{t!\ (p-2t)!\ \Gamma(p-t+\lambda+1)}\, U_{p-2t}^{(\lambda)}(x).
$$

□

As consequences of Theorem 3, the moments of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ are given in terms of the ultraspherical polynomials. In addition, the derivatives of these polynomials are expressed in terms of the ultraspherical polynomials. The following two corollaries exhibit these results.

Corollary 5. *The moments of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ can be expressed in terms of the ultraspherical polynomials as*

$$
\begin{aligned}
x^r\, V_k^{(\alpha,\alpha+1)}(x) =& \frac{2^{2-r+2\alpha-2\lambda}(k+r)!\,\Gamma(\alpha+1)\Gamma\big(k+\alpha+\frac{3}{2}\big)}{\Gamma(k+2\alpha+2)\Gamma\Big(\lambda+\frac{1}{2}\Big)}\times \\
& \sum_{p=0}^{\lfloor\frac{k+r}{2}\rfloor} \frac{(k+r-2p+\lambda)\Gamma(k+r-2p+2\lambda)}{p!\,(k+r-2p)!\,\Gamma(k+r-p+\lambda+1)}\, {}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-\frac{k}{2}, -\frac{k}{2}, -k-r+p-\lambda \\ -\frac{k}{2}-\frac{r}{2}, \frac{1}{2}-\frac{k}{2}-\frac{r}{2}, -\frac{1}{2}-k-\alpha \end{matrix}\,\middle|\, 1\right) U_{k+r-2p}^{(\lambda)}(x) \\
& + \frac{2^{2-r+2\alpha-2\lambda}k(k+r-1)!\,\Gamma(\alpha+1)\Gamma\Big(k+\alpha+\frac{1}{2}\Big)}{\Gamma(k+2\alpha+2)\Gamma\Big(\lambda+\frac{1}{2}\Big)}\times \\
& \sum_{p=0}^{\lfloor\frac{1}{2}(k+r-1)\rfloor} \frac{(1-k-r+2p-\lambda)\Gamma(k+r-2p+2\lambda-1)}{p!\,(k+r-2p-1)!\,\Gamma(k+r-p+\lambda)}\times \\
& {}_4F_3\left(\begin{matrix} -p, \frac{1}{2}-\frac{k}{2}, 1-\frac{k}{2}, 1-k-r+p-\lambda \\ \frac{1}{2}-\frac{k}{2}-\frac{r}{2}, 1-\frac{k}{2}-\frac{r}{2}, \frac{1}{2}-k-\alpha \end{matrix}\,\middle|\, 1\right) U_{k+r-2p-1}^{(\lambda)}(x).
\end{aligned}
\tag{19}
$$

Proof. Formula (19) can be directly obtained from Formula (18) only by setting $q = 0$. □

Remark 4. *It is worth noting that the two ${}_4F_3(1)$ hypergeometric functions that appear in (19) can be summed for specific choices of the included parameters. Hence, simple moment formulas can be deduced. The following corollary gives one of these formulas.*

Corollary 6. *For two non-negative integers r and k such that $k \geq r$, the following moment formula holds:*

$$x^r\, V_k(x) = 2^{-r} r! \sum_{p=0}^{\lfloor \frac{k+r}{2} \rfloor} \frac{1}{p!\,(r-p)!} \left(U_{k+r-2p}(x) - U_{k+r-2p-1}(x)\right).$$

Proof. The substitution by $\alpha = -\frac{1}{2}$ and $\lambda = 1$ in (19) produces the formula shown below:

$$\begin{aligned} x^r\, V_k(x) &= 2^{-r}(k+r)! \sum_{p=0}^{\lfloor \frac{k+r}{2} \rfloor} \frac{k+r-2p+1}{p!\,(k+r-p+1)!}\, {}_4F_3\left(\left. \begin{array}{c} -p, -1-k-r+p, \frac{1}{2}-\frac{k}{2}, -\frac{k}{2} \\ -k, -\frac{k}{2}-\frac{r}{2}, \frac{1}{2}-\frac{k}{2}-\frac{r}{2} \end{array} \right| 1 \right) U_{k+r-2p}(x) \\ &+ 2^{-r}(k+r-1)! \sum_{p=0}^{\lfloor \frac{1}{2}(k+r-1) \rfloor} \frac{2p-r-k}{p!\,(k+r-p)!}\, {}_4F_3\left(\left. \begin{array}{c} -p, -k-r+p, \frac{1}{2}-\frac{k}{2}, 1-\frac{k}{2}, \\ 1-k, \frac{1}{2}-\frac{k}{2}-\frac{r}{2}, 1-\frac{k}{2}-\frac{r}{2} \end{array} \right| 1 \right) U_{k+r-2p-1}(x). \end{aligned} \tag{20}$$

We can employ any suitable symbolic algorithm to reduce the two ${}_4F_3(1)$ that appear in (20). Now, set

$$M_{p,k,r} = {}_4F_3\left(\left. \begin{array}{c} -p, -1-k-r+p, \frac{1}{2}-\frac{k}{2}, -\frac{k}{2} \\ -k, -\frac{k}{2}-\frac{r}{2}, \frac{1}{2}-\frac{k}{2}-\frac{r}{2} \end{array} \right| 1 \right),$$

and

$$\bar{M}_{p,k,r} = {}_4F_3\left(\left. \begin{array}{c} -p, -k-r+p, \frac{1}{2}-\frac{k}{2}, 1-\frac{k}{2}, \\ 1-k, \frac{1}{2}-\frac{k}{2}-\frac{r}{2}, 1-\frac{k}{2}-\frac{r}{2} \end{array} \right| 1 \right).$$

Zeilberger's algorithm (see, [40]) enables one to obtain the following recurrence relations satisfied respectively by $M_{p,k,r}$ and $\bar{M}_{p,k,r}$:

$$\begin{aligned} &(p-1)(p-r-2)(k-2p+r+5)M_{p-2,k,r} + (-2p+k+r+3)\times \\ &\left(2kp - kr - 2p^2 + 2pr - 2k + 6p - 2r - 4\right) M_{p-1,k,r} \\ &+ (k-p+1)(k-p+r+2)(k-2p+r+1)M_{p,k,r} = 0, \quad M_{0,k,r} = 1, M_{1,k,r} = \frac{r}{k+r-1}, \end{aligned} \tag{21}$$

$$\begin{aligned} &(p-1)(p-r-2)(k-2p+r+4)\bar{M}_{p-2,k,r} + (r+2-2p+k)\times \\ &\left(2kp - kr - 2p^2 + 2pr - 2k + 4p - r - 2\right) \bar{M}_{p-1,k,r} \\ &+ (k-p+r+1)(k-p)(k+r-2p)\,\bar{M}_{p,k,r} = 0, \quad \bar{M}_{0,k,r} = 1,\ \bar{M}_{1,k,r} = \frac{r}{k+r-2}. \end{aligned} \tag{22}$$

The exact solutions of (21) and (22) are given respectively by

$$M_{p,k,r} = \frac{(r-p+1)_p}{(k+r-2p+1)(k+r-p+2)_{p-1}}, \tag{23}$$

$$\bar{M}_{p,k,r} = \frac{(r-p+1)_p}{(k+r-2p).(k+r-p+1)_{p-1}}. \tag{24}$$

The substitution by (23) and (24) into (20) yields the following simplified moment formula

$$x^r\, V_k(x) = 2^{-r} r! \sum_{p=0}^{\lfloor \frac{k+r}{2} \rfloor} \frac{1}{p!\,(r-p)!} \left(U_{k+r-2p}(x) - U_{k+r-2p-1}(x)\right).$$

This finalizes the proof of Corollary 6. □

Corollary 7. *Let k and q be non-negative integers such that $k \geq q$. The following derivatives formula holds*

$$D^q V_k^{(\alpha,\alpha+1)}(x) = \frac{2^{q+2\alpha-2\lambda+2} k!\,\Gamma(\alpha+1)\Gamma\left(k+\alpha+\frac{3}{2}\right)}{\Gamma(k+2\alpha+2)\Gamma\left(\lambda+\frac{1}{2}\right)} \times$$
$$\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(k-2p-q+\lambda)\Gamma(k-2p-q+2\lambda)\left(q+\alpha-\lambda+\frac{3}{2}\right)_p}{p!\,(k-2p-q)!\,\Gamma(k-p-q+\lambda+1)\left(k-p+\alpha+\frac{3}{2}\right)_p} \times \tag{25}$$
$$\left(U_{k-q-2p}^{(\lambda)}(x) - \frac{2(k-2p-q)(k-2p-q+\lambda-1)(k-p-q+\lambda)}{(2k-2p+2\alpha+1)(k-2p-q+\lambda)(k-2p-q+2\lambda-1)}\, U_{k-q-2p-1}^{(\lambda)}(x)\right).$$

Proof. Setting $r=0$ in Formula (18) produces the formula shown below:

$$D^q V_k^{(\alpha,\alpha+1)}(x) = \frac{2^{q+2\alpha-2\lambda+2}\, k!\,\Gamma(\alpha+1)\Gamma\left(k+\alpha+\frac{3}{2}\right)}{\Gamma(k+2\alpha+2)\,\Gamma\left(\lambda+\frac{1}{2}\right)} \times$$
$$\left(\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(k-2p-q+\lambda)\Gamma(k-2p-q+2\lambda)}{p!\,(k-2p-q)!\,\Gamma(k-p-q+\lambda+1)}\, {}_2F_1\left(\left. \begin{array}{c} -p, -k+p+q-\lambda \\ -\frac{1}{2}-k-\alpha \end{array} \right| 1 \right) U_{k-q-2p}^{(\lambda)}(x)\right.$$
$$+ \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{(1-k+2p+q-\lambda)\Gamma(k-2p-q+2\lambda-1)}{p!\,(k-2p-q-1)!\,\Gamma(k-p-q+\lambda)} \times$$
$$\left. {}_2F_1\left(\left. \begin{array}{c} -p, 1-k+p+q-\lambda \\ \frac{1}{2}-k-\alpha \end{array} \right| 1 \right) U_{k-q-2p-1}^{(\lambda)}(x)\right).$$

Based on the Chu–Vandermonde identity, it is easy to see the following two identities

$${}_2F_1\left(\left. \begin{array}{c} -p, -k+p+q-\lambda \\ -\frac{1}{2}-k-\alpha \end{array} \right| 1 \right) = \frac{\left(\alpha-\lambda+q+\frac{3}{2}\right)_p}{\left(k+\alpha+\frac{3}{2}-p\right)_p},$$
$${}_2F_1\left(\left. \begin{array}{c} -p, 1-k+p+q-\lambda \\ \frac{1}{2}-k-\alpha \end{array} \right| 1 \right) = \frac{\left(\alpha-\lambda+q+\frac{3}{2}\right)_p}{\left(k+\alpha+\frac{1}{2}-p\right)_p},$$

and therefore, Formula (25) can be obtained. □

Corollary 8. *Assume that k and q are non-negative integers such that $k \geq q$. The following derivative formulas hold:*

$$D^q V_k^{(\alpha,\alpha+1)}(x) = \frac{2^{q+2\alpha+1} k!\,\Gamma(\alpha+1)\Gamma\left(k+\alpha+\frac{3}{2}\right)}{\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\left(k-2p-q+\frac{1}{2}\right)(q+\alpha+1)_p}{p!\,\Gamma\left(k-p-q+\frac{3}{2}\right)\left(k-p+\alpha+\frac{3}{2}\right)_p} \times \tag{26}$$
$$\left(P_{k-q-2p}(x) - \frac{(2k-4p-2q-1)\,(2k-2p-2q+1)}{(2k-4p-2q+1)(2k-2p+2\alpha+1)}\, P_{k-q-2p-1}(x)\right),$$

$$D^q V_k^{(\alpha,\alpha+1)}(x) = \frac{2^{q+2\alpha+1} k!\, \Gamma(\alpha+1)\Gamma\big(k+\alpha+\frac{3}{2}\big)}{\sqrt{\pi}\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma\big(k-p+\alpha+\frac{3}{2}\big)\,\Gamma\big(p+q+\alpha+\frac{3}{2}\big)}{p!\,(k-p-q)!\,\Gamma\big(k+\alpha+\frac{3}{2}\big)\Gamma\big(q+\alpha+\frac{3}{2}\big)} \times \left(c_{k-q-2p}\, T_{k-q-2p}(x) + \frac{2\, c_{k-q-2p-1}\,(-k+p+q))}{2k-2p+2\alpha+1}\, T_{k-q-2p-1}(x)\right), \tag{27}$$

$$D^q V_k^{(\alpha,\alpha+1)}(x) = \frac{2^{q+2\alpha+1} k!\, \Gamma(\alpha+1)\Gamma\big(k+\alpha+\frac{3}{2}\big)}{\sqrt{\pi}\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma\big(k-p+\alpha+\frac{3}{2}\big)\Gamma\Big(p+q+\alpha+\frac{1}{2}\Big)}{p!\,(k-p-q+1)!\,\Gamma\big(k+\alpha+\frac{3}{2}\big)\Gamma\Big(q+\alpha+\frac{1}{2}\Big)} \times \left((k-2p-q+1)\, U_{k-2p-q}(x) + \frac{2(k-p-q+1)(-k+2p+q)}{2k-2p+2\alpha+1}\, U_{k-2p-q-1}(x)\right), \tag{28}$$

where c_k is defined as

$$c_k = \begin{cases} \frac{1}{2}, & k = 0, \\ 1, & k > 0. \end{cases} \tag{29}$$

Proof. Setting $\lambda = \frac{1}{2}, 0, 1,$ in (25) yields, respectively, (26), (27), and (28). □

The following corollary gives the derivatives of the Chebyshev polynomials of the third-kind in terms of the Legendre and Chebyshev polynomials of the first- and second-kinds.

Corollary 9. *Assume that k and q are non-negative integers such that $k \geq q$. The following derivative formulas hold:*

$$D^q V_k(x) = 2^q \sqrt{\pi}\, k! \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Big(k-2p-q+\frac{1}{2}\Big)\Big(q+\frac{1}{2}\Big)_p}{p!\,\Gamma\big(k-p-q+\frac{3}{2}\big)(k-p+1)_p} \times \left(P_{k-2p-q}(x) - \frac{(2k-4p-2q-1)(2k-2p-2q+1)}{2(k-p)(2k-4p-2q+1)}\, P_{k-2p-q-1}(x)\right), \tag{30}$$

$$D^q V_k(x) = 2^{q+1} k! \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(p+q)!\,(k-p)!}{p!\,q!\,(k-p-q)!\,k!} \times \left(c_{k-2p-q}\, T_{k-2p-q}(x) + \frac{c_{k-2p-q-1}\,(-k+p+q)}{k-p}\, T_{k-2p-q-1}(x)\right), \tag{31}$$

$$D^q V_k(x) = 2^q\, q! \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(p+q-1)!\,(k-p)!}{p!\,(q-1)!\,(k-p-q+1)!\,k!} \times \left((k-2p-q+1)\, U_{k-2p-q}(x) + \frac{2(k-p-q+1)(-k+2p+q)}{2k-2p+2\alpha+1}\, U_{k-2p-q-1}(x)\right), \tag{32}$$

where c_k is defined in (29).

Proof. Setting $\alpha = -\frac{1}{2}$ in (26), (27), and (28) yields, respectively, (30), (31), and (32). □

4. Some New Expressions for the Derivatives of Different Polynomials

This section is dedicated to developing new derivative formulas for various polynomials in terms of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$. Inversion formulas for these formulas will also be obtained.

4.1. Expressions of the Derivatives of Some Polynomials in Terms of $V_k^{(\alpha,\alpha+1)}(x)$

Theorem 4. *Assume that k and q are non-negative integers such that $k \geq q$. The following derivative formula for Hermite polynomials holds:*

$$
\begin{aligned}
D^q H_k(x) = {} & \frac{2^{q-2\alpha-1}\sqrt{\pi}\,k!}{\Gamma(\alpha+1)} \times \\
& \left(\sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(-1)^p\,\Gamma(k-2p-q+2\alpha+2)\,{}_1F_1\left(-p;k-2p-q+\alpha+\frac{3}{2};1\right)}{p!\,(k-2p-q)!\,\Gamma\left(k-2p-q+\alpha+\frac{3}{2}\right)}\, V_{k-q-2p}^{(\alpha,\alpha+1)}(x) \right. \\
& \left. + \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{(-1)^p\,(k-2p-q)\,\Gamma(k-2p-q+2\alpha+2)\,{}_1F_1\left(-p;k-2p-q+\alpha+\frac{3}{2};1\right)}{(k-2p-q+2\alpha+1)\,p!\,(k-2p-q)!\,\Gamma\left(k-2p-q+\alpha+\frac{3}{2}\right)}\, V_{k-q-2p-1}^{(\alpha,\alpha+1)}(x) \right).
\end{aligned}
\tag{33}
$$

Proof. The proof can be calculated using the power form representation of the Hermite polynomials given by [39]

$$
H_k(x) = k! \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^m 2^{k-2m}}{m!\,(k-2m)!} x^{k-2m},
$$

along with the inversion Formula (6). □

Theorem 5. *Assume that k and q are non-negative integers such that $k \geq q$. The following derivative formula for the generalized Fibonacci polynomials $F_k^{A,B}(x)$ that is defined in (9) holds:*

$$
\begin{aligned}
D^q F_k^{A,B}(x) = {} & \frac{2^{-1-k+q-2\alpha} A^k \sqrt{\pi} k!}{\Gamma(\alpha+1)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+2)}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)} \times \\
& {}_2F_1\left(\left. \begin{matrix} -p, -\frac{1}{2}-k+p+q-\alpha \\ -k \end{matrix} \right| \frac{-4\,B}{A^2} \right) \times \\
& \left(V_{k-q-2p}^{(\alpha,\alpha+1)}(x) + \frac{(k-2p-q)}{k-2p-q+2\alpha+1}\, V_{k-q-2p-1}^{(\alpha,\alpha+1)}(x) \right).
\end{aligned}
$$

Proof. Similar to the proof of Theorem 4. □

Theorem 6. *Assume that k and q are non-negative integers such that $k \geq q$. The following derivative formula for the ultraspherical polynomials holds*

$$
\begin{aligned}
D^q U_k^{(\lambda)}(x) = {} & \frac{2^{q-2\alpha+2\lambda-2} k!\,\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(k+\lambda)}{\Gamma(\alpha+1)\Gamma(k+2\lambda)} \times \\
& \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+2)\left(\frac{3}{2}-p-q+\alpha-\lambda\right)_p}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)(1-k-\lambda)_p} \times \\
& \left(V_{k-q-2p}^{(\alpha,\alpha+1)}(x) + \frac{(k-2p-q)}{k-2p-q+2\alpha+1}\, V_{k-q-2p-1}^{(\alpha,\alpha+1)}(x) \right).
\end{aligned}
\tag{34}
$$

Proof. Similar to the proof of Theorem 4. □

Corollary 10. *The following are, respectively, the expressions of the derivatives of the Legendre, Chebyshev polynomials of the first- and second-kinds in terms of the polynomials* $V_k^{(\alpha,\alpha+1)}(x)$ *as*

$$
\begin{aligned}
D^q P_k(x) =& \frac{2^{q-2\alpha-1}\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(\alpha+1)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+2)(1-p-q+\alpha)_p}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)\left(\frac{1}{2}-k\right)_p} \times \\
& \left(V_{k-2p-q}^{\alpha,\alpha+1}(x) + \frac{(k-2p-q)}{k-2p-q+2\alpha+1}\, V_{k-2p-q-1}^{\alpha,\alpha+1}(x)\right), \quad k \geq q,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
D^q T_k(x) =& \frac{2^{q-2\alpha-2}\sqrt{\pi}k!}{\Gamma(\alpha+1)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+2)\left(\frac{3}{2}-p-q+\alpha\right)_p}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)(1-k)_p} \times \\
& \left(V_{k-2p-q}^{(\alpha,\alpha+1)}(x) + \frac{(k-2p-q)}{k-2p-q+2\alpha+1}\, V_{k-2p-q-1}^{(\alpha,\alpha+1)}(x)\right), \quad k \geq q,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
D^q U_k(x) =& \frac{2^{q-2\alpha-1}\sqrt{\pi}k!}{\Gamma(\alpha+1)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\Gamma(k-2p-q+2\alpha+2)\left(\frac{1}{2}-p-q+\alpha\right)_p}{p!\,(k-2p-q)!\,\Gamma\left(k-p-q+\alpha+\frac{3}{2}\right)(-k)_p} \times \\
& \left(V_{k-2p-q}^{(\alpha,\alpha+1)}(x) + \frac{(k-2p-q)}{k-2p-q+2\alpha+1}\, V_{k-2p-q-1}^{(\alpha,\alpha+1)}(x)\right), \quad k \geq q.
\end{aligned} \tag{37}
$$

Proof. Formulas (35), (36), and (37) can be obtained directly as special cases of (34) setting, respectively, $\lambda = \frac{1}{2}, 0$, and 1. □

Corollary 11. *The following are, respectively, the expressions of the derivatives of the Legendre, Chebyshev polynomials of the first- and second-kinds in terms of the Chebyshev polynomials of the third-kind Chebyshev polynomials* $V_k(x)$*.*

$$
\begin{aligned}
D^q P_k(x) =& \frac{2^q\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{\left(q+\frac{1}{2}\right)_p}{p!\,(k-p-q)!\left(k-p+\frac{1}{2}\right)_p} \Big(V_{k-2p-q}(x) + V_{k-2p-q-1}(x)\Big), \\
& k \geq q,
\end{aligned} \tag{38}
$$

$$
D^q T_k(x) = 2^{q-1}k! \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(q)_p}{p!\,(k-p-q)!\,(k-p)_p} \Big(V_{k-2p-q}(x) + V_{k-2p-q-1}(x)\Big),\ k \geq q, \tag{39}
$$

$$
D^q U_k(x) = 2^q k! \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(q+1)_p}{p!\,(k-p-q)!\,(k-p+1)_p} \Big(V_{k-2p-q}(x) + V_{k-2p-q-1}(x)\Big), \quad k \geq q. \tag{40}
$$

Proof. Formulas (38), (39), and (40) are special ones of, respectively, (35), (36), and (37) only setting $\alpha = -\frac{1}{2}$. □

4.2. Expressions for the Derivatives of $V_k^{(\alpha,\alpha+1)}(x)$ *in Terms of Some Other Polynomials*

In this section, some of the inversion formulas to the derivative formulas given in Section 4.1 can be introduced using similar techniques. Some of these results are displayed without proof.

Theorem 7. *Assume that k and q are non-negative integers such that $k \geq q$. The following formula is valid for the derivatives of $V_k^{(\alpha,\alpha+1)}(x)$:*

$$\begin{aligned} D^q V_k^{(\alpha,\alpha+1)}(x) = & \frac{2^{q+2\alpha+1} k!\, \Gamma(\alpha+1) \Gamma\left(k+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{{}_1F_1\left(-p; -\frac{1}{2}-k-\alpha; -1\right)}{p!\,(k-2p-q)!} H_{k-q-2p}(x) \\ - & \frac{2^{q+2\alpha+1} k!\, \Gamma(\alpha+1) \Gamma\left(k+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{{}_1F_1\left(-p; \frac{1}{2}-k-\alpha; -1\right)}{p!\,(k-2p-q-1)!} H_{k-q-2p-1}(x). \end{aligned} \tag{41}$$

Theorem 8. *Assume that k and q are non-negative integers such that $k \geq q$. The following formula is valid for the derivatives of $V_k^{(\alpha,\alpha+1)}(x)$:*

$$\begin{aligned} D^q V_k^{(\alpha,\alpha+1)}(x) = & \frac{2^{k+2\alpha+1} A^{-k+q} k!\, \Gamma(\alpha+1) \Gamma\left(k+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(k+2\alpha+2)} \times \\ & \sum_{p=0}^{\left\lfloor \frac{k-q}{2} \right\rfloor} \frac{(-1)^{p+1} B^p (-1-k+2p+q)}{p!\,(k-p-q+1)!} \, {}_2F_1\left(\begin{matrix} -p, -1-k+p+q \\ -\frac{1}{2}-k-\alpha \end{matrix} \,\middle|\, \frac{-A^2}{4\,B} \right) F_{k-q-2p}^{A,B}(x) \\ + & \frac{2^{k+2\alpha}\, A^{1-k+q} k!\, \Gamma(\alpha+1) \Gamma\left(k+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(k+2\alpha+2)} \times \\ & \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-q-1) \right\rfloor} \frac{(-1)^p\, B^p\, (-k+2p+q)}{p!\,(k-p-q)!} \, {}_2F_1\left(\begin{matrix} -p, -k+p+q \\ \frac{1}{2}-k-\alpha \end{matrix} \,\middle|\, \frac{-A^2}{4\,B} \right) F_{k-q-2p-1}^{A,B}(x). \end{aligned}$$

4.3. Some Connection Formulas

Since all the results in Sections 4.1 and 4.2 are valid for $q = 0$, for every derivatives formula, we can easily deduce a connection formula. In this section, we will present two of these formulas.

Corollary 12. *The Hermite-generalized third-kind Chebyshev and the generalized third-kind Hermite connection formulas are:*

$$\begin{aligned} H_k(x) = & \frac{2^{-1-2\alpha}\sqrt{\pi} k!}{\Gamma(\alpha+1)} \left(\sum_{p=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^p\, \Gamma(k-2p+2\alpha+2)\, {}_1F_1\left(-p; k-2p+\alpha+\frac{3}{2}; 1\right)}{p!\,(k-2p)!\, \Gamma\left(k-2p+\alpha+\frac{3}{2}\right)} V_{k-2p}^{(\alpha,\alpha+1)}(x) \right. \\ & \left. + \sum_{p=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^p (k-2p) \Gamma(k-2p+2\alpha+2)\, {}_1F_1\left(-p; k-2p+\alpha+\frac{3}{2}; 1\right)}{p!(k-2p)!(k-2p+2\alpha+1)\, \Gamma\left(k-2p+\alpha+\frac{3}{2}\right)} V_{k-2p-1}^{(\alpha,\alpha+1)}(x) \right), \end{aligned} \tag{42}$$

$$\begin{aligned} V_k^{(\alpha,\alpha+1)}(x) = & \frac{2^{2\alpha+1}\, k!\, \Gamma(\alpha+1) \Gamma\left(k+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\, \Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{{}_1F_1\left(-p; -\frac{1}{2}-k-\alpha; -1\right)}{p!\,(k-2p)!} H_{k-2p}(x) \\ - & \frac{2^{2\alpha+1} k!\, \Gamma(\alpha+1)\, \Gamma\left(k+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma(k+2\alpha+2)} \sum_{p=0}^{\left\lfloor \frac{1}{2}(k-1) \right\rfloor} \frac{{}_1F_1\left(-p; \frac{1}{2}-k-\alpha; -1\right)}{p!\,(k-2p-1)!} H_{k-2p-1}(x). \end{aligned} \tag{43}$$

Corollary 13. *Formula (42) can be immediately obtained from Formula (33) by setting $q = 0$, while Formula (43) can be immediately obtained from Formula (41) by setting $q = 0$.*

5. Some New Linearization Formulas Involving $V_k^{(\alpha,\alpha+1)}(x)$

This section is confined to presenting some linearization formulas involving the polynomials $V_k^{(\alpha,\alpha+1)}(x)$.

Theorem 9. *Let i and j be any non-negative integer. The following linearization formula applies*

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \frac{4^\alpha i!\, j!\, \Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+2\alpha+3)\Gamma(j+2\alpha+1)} \times$$
$$\left(\sum_{p=0}^{\min(i,j)} \frac{\Gamma(i-p+\alpha+\frac{3}{2})\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma\left(p+\alpha+\frac{1}{2}\right)\Gamma(i+j-p+2\alpha+2)}{p!\,(i-p)!\,(j-p)!\,\Gamma(i+j-p+\alpha+\frac{3}{2})}\, V_{j+i-2p}^{(\alpha,\alpha+1)}(x) \right. \tag{44}$$
$$\left. + \sum_{p=0}^{\min(i,j)} \frac{\Gamma\left(i-p+\alpha+\frac{1}{2}\right)\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma(p+\alpha+\frac{3}{2})\Gamma(i+j-p+2\alpha+1)}{p!\,(i-p)!\,(j-p-1)!\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+j-p+\alpha)}\, V_{j+i-2p-1}^{(\alpha,\alpha+1)}(x) \right).$$

Proof. If we make use of the power form representation of the $V_j^{(\alpha,\alpha)}(x)$, then we can write

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} H_{r,j}\, x^{j-2r} V_i^{(\alpha,\alpha+1)}(x), \tag{45}$$

where $H_{r,j}$ is given by

$$H_{r,j} = \frac{(-1)^r 2^{j-2r-1} j!\, \Gamma\left(j-r+\alpha+\frac{1}{2}\right)\Gamma(2\alpha+2)}{(j-2r)!\, r!\, \Gamma(\alpha+\frac{3}{2})\Gamma(j+2\alpha+1)}.$$

The moment formula that is given in Equation (15) enables one to convert (45) into the following one:

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} H_{r,j} \left(\sum_{p=0}^{\lfloor \frac{1}{2}(i+j-2r) \rfloor} U_{p,i,j-2r}\, V_{i+j-2r-2p}^{(\alpha,\alpha+1)}(x) \right.$$
$$\left. + \sum_{p=0}^{\lfloor \frac{1}{2}(i+j-2r-1) \rfloor} \bar{U}_{p,i,j-2r}\, V_{i+j-2r-2p-1}^{(\alpha,\alpha)}(x) \right), \tag{46}$$

where the coefficients $U_{p,j,m}$ and $\bar{U}_{p,j,m}$ are as given by the following formulas:

$$U_{p,j,m} = \sum_{\ell=0}^{p} \frac{(-1)^\ell\, 2^{-m}\, j!\, (j-2\ell+m-1)!\, \Gamma\left(j-\ell+\alpha+\frac{1}{2}\right)\Gamma(j+m-2p+2\alpha+2)}{\ell!\,(j-2\ell)!\,(p-\ell)!\,(j+m-2p)!\,\Gamma(j-\ell+m-p+\alpha+\frac{3}{2})\Gamma(j+2\alpha+2)} \times$$
$$\left((j-2\ell)(\ell-p) + \frac{1}{2}(j-2\ell+m)(2j-2\ell+2\alpha+1)\right),$$
$$\bar{U}_{p,j,m} = \sum_{\ell=0}^{p} \frac{(-1)^\ell\, 2^{-1-m}(m+2\ell m+2jp-4\ell p+2m\alpha) j!(j-2\ell+m-1)!}{\ell!(j-2\ell)!(j+m-2p-1)!(p-\ell)!\Gamma(j-\ell+m-p+\alpha+\frac{3}{2})\Gamma(j+2\alpha+2)} \times$$
$$\Gamma\left(j-\ell+\alpha+\frac{1}{2}\right)\Gamma(j+m-2p+2\alpha+1).$$

After some lengthy algebraic computations, Formula (46) can be transformed into:

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \sum_{p=0}^{\lfloor \frac{i+j}{2} \rfloor} M_{p,i,j}\, V_{i+j-2p}^{(\alpha,\alpha+1)}(x) + \sum_{p=0}^{\lfloor \frac{1}{2}(i+j-1) \rfloor} \bar{M}_{p,i,j}\, V_{i+j-2p-1}^{(\alpha,\alpha+1)}(x),$$

where the coefficients $M_{p,i,j}$ and $\bar{M}_{p,i,j}$

$$M_{p,i,j} = \sum_{\ell=0}^{p} H_{\ell,j}\, U_{p-\ell,i,j-2\ell},$$

and

$$\bar{M}_{p,i,j} = \sum_{\ell=0}^{p} H_{\ell,j}\, \bar{U}_{p-\ell,i,j-2\ell}.$$

Making use of any suitable symbolic algorithm, and in particular, Zeilberger's algorithm [40], it can be shown that $M_{p,i,j}$ and $\bar{M}_{p,i,j}$ satisfy, respectively, the two following recurrence relations of order one:

$$\begin{aligned}&(p+1)(2i-2p+2\alpha+1)(2j-2p+2\alpha-1)(i+j-p+2\alpha+1)M_{p+1,i,j}\\ &-(i-p)(j-p)(2i+2j-2p+2\alpha+1)(2p+2\alpha+1)M_{p,i,j}=0,\end{aligned} \tag{47}$$

with the initial value:

$$M_{0,i,j} = \frac{4^{\alpha}\,\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{3}{2}\right)\Gamma\left(j+\alpha+\frac{1}{2}\right)\Gamma(i+j+2\alpha+2)}{\sqrt{\pi}\,\Gamma\left(i+j+\alpha+\frac{3}{2}\right)\Gamma(i+2\alpha+2)\Gamma(j+2\alpha+1)},$$

and

$$\begin{aligned}&(p+1)(1-2j+2p-2\alpha)(2i-2p+2\alpha-1)(i+j-p+2\alpha)\,\bar{M}_{p+1,i,j}\\ &-(i-p)(1-j+p)(2i+2j-2p+2\alpha+1)(2p+2\alpha+3)\,\bar{M}_{p,i,j},\end{aligned} \tag{48}$$

with the initial value:

$$\bar{M}_{0,i,j} = \frac{2^{2\alpha-1}\,j\,(2\alpha+1)\,\Gamma(\alpha+1)\Gamma\left(i+\alpha+\frac{1}{2}\right)\Gamma\left(j+\alpha+\frac{1}{2}\right)\Gamma(i+j+2\alpha+1)}{\sqrt{\pi}\,\Gamma\left(i+j+\alpha+\frac{3}{2}\right)\Gamma(i+2\alpha+2)\Gamma(j+2\alpha+1)}.$$

The two recurrence relations in (47) and (48) can be solved to give:

$$M_{p,i,j} = \frac{4^{\alpha} i!\, j!\, \Gamma(\alpha+1)\Gamma\left(i-p+\alpha+\frac{3}{2}\right)\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma\left(p+\alpha+\frac{1}{2}\right)\Gamma(i+j-p+2\alpha+2)}{\sqrt{\pi}\,p!\,(i-p)!\,(j-p)!\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(i+j-p+\alpha+\frac{3}{2}\right)\Gamma(i+2\alpha+2)\Gamma(j+2\alpha+1)},$$

and

$$\bar{M}_{p,i,j} = \frac{4^{\alpha} i!\, j!\, \Gamma(\alpha+1)\Gamma\left(i-p+\alpha+\frac{1}{2}\right)\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma\left(p+\alpha+\frac{3}{2}\right)\Gamma(i+j-p+2\alpha+1)}{\sqrt{\pi}\,p!\,(i-p)!\,(j-p-1)!\,\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(i+j-p+\alpha+\frac{3}{2}\right)\Gamma(i+2\alpha+2)\Gamma(j+2\alpha+1)}.$$

Therefore, Formula (44) can be obtained. □

Remark 5. *Taking the limit as α tends to $-\frac{1}{2}$, a specific linearization formula of Formula (44) can be obtained. The following corollary exhibits this result.*

Corollary 14. *Let i and j be any non-negative integers. The following linearization formula holds:*

$$V_i(x)\,T_j(x) = \frac{1}{2}\left(V_{i+j}(x) + V_{j-i-1}(x)\right). \tag{49}$$

Remark 6. *The linearization Formula (49) can be translated into the following simple trigonometric identity:*

$$\cos\left(\left(j+i+\tfrac{1}{2}\right)\theta\right) + \cos\left(\left(j-i-\tfrac{1}{2}\right)\theta\right) = 2\cos\left(\left(i+\tfrac{1}{2}\right)\theta\right)\cos(j\theta).$$

Theorem 10. *Let i and j be any non-negative integer. The following linearization formula applies*

$$V_i^{(\alpha,\alpha)}(x)\,V_j^{(\alpha,\alpha)}(x) = \frac{2^{2\alpha-1} j!\,\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(i+2\alpha+2)\,\Gamma(j+2\alpha+1)}\times$$
$$\sum_{p=0}^{\min(i,j)} \frac{(i+j-2p)\Gamma\left(i-p+\alpha+\frac{1}{2}\right)\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma(i+j-p+2\alpha+1)(i-p+1)_p\left(\alpha+\frac{1}{2}\right)_p}{(j-p)!\,p!\,\Gamma\left(i+j-p+\alpha+\frac{3}{2}\right)}\times \tag{50}$$
$$\left(\frac{(i+j-2p+2\alpha+1)}{i+j-2p}\,V_{i+j-2p}^{(\alpha,\alpha+1)}(x) + V_{i+j-2p-1}^{(\alpha,\alpha+1)}(x)\right).$$

Proof. We make use of the following linearization formula (see [37])

$$V_i^{(\alpha,\alpha)}(x)\,V_j^{(\alpha,\alpha)}(x) = \frac{4^\alpha\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(i+2\alpha+1)\Gamma(j+2\alpha+1)}\times$$
$$\sum_{p=0}^{\min(i,j)} \frac{\Gamma\left(i-p+\alpha+\frac{1}{2}\right)\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma(i+j-p+2\alpha+1)(i-p+1)_p\,(j-p+1)_p\left(\alpha+\frac{1}{2}\right)_p}{p!\,\Gamma\left(i+j-2p+\alpha+\frac{1}{2}\right)\left(i+j-2p+\alpha+\frac{3}{2}\right)_p}\times \tag{51}$$
$$V_{i+j-2p}^{(\alpha,\alpha)}(x).$$

If we insert the connection Formula (3) into the last linearization formula, then the linearization Formula (50) can be obtained. □

As a special case of Theorem 10, the following linearization formula can be deduced by taking the limit as α tends to $-\frac{1}{2}$.

Corollary 15. *Let i and j be any non-negative integers. The following linearization formula holds:*

$$T_i(x)\,T_j(x) = \frac{1}{4}\left(V_{i+j}(x) + V_{i+j-1}(x) + V_{j-i}(x) + V_{j-i-1}(x)\right). \tag{52}$$

Remark 7. *The linearization Formula (52) can be translated into the following trigonometric identity:*

$$\cos\left(\left(i+j+\tfrac{1}{2}\right)\theta\right) + \cos\left(\left(i+j-\tfrac{1}{2}\right)\theta\right) + \cos\left(\left(j-i+\tfrac{1}{2}\right)\theta\right) + \cos\left(\left(j-i-\tfrac{1}{2}\right)\theta\right)$$
$$= 4\cos(i\,\theta)\cos(j\,\theta)\cos\left(\frac{\theta}{2}\right).$$

Theorem 11. *Let i and j be any non-negative integer. The following linearization formula applies*

$$V_i^{(\alpha,\alpha+1)}(x)\,V_j^{(\alpha,\alpha)}(x) = \sum_{p=0}^{\min(i,j)} G_{p,i,j}\,V_{i+j-2p}^{(\alpha,\alpha)}(x) + \sum_{p=0}^{\min(i,j)} \bar{G}_{p,i,j}\,V_{i+j-2p-1}^{(\alpha,\alpha)}(x), \tag{53}$$

where the linearization coefficients $G_{p,i,j}$ and $\bar{G}_{p,i,j}$ are given by the following formulas:

$$G_{p,i,j} = \frac{4^\alpha\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(j+2\alpha+1)}\times$$
$$\sum_{\ell=0}^{p} \frac{(2i-4\ell+2\alpha+1)\,\Gamma\left(i-\ell-p+\alpha+\frac{1}{2}\right)\Gamma\left(j+\ell-p+\alpha+\frac{1}{2}\right)\Gamma(i+j-\ell-p+2\alpha+1)}{(p-\ell)!\,\Gamma\left(i+j-2p+\alpha+\frac{1}{2}\right)\Gamma(i-2\ell+2\alpha+1)}\times$$
$$\frac{(i-2\ell+1)_{2\ell}(i-\ell-p+1)_{p-\ell}(j+\ell-p+1)_{p-\ell}\left(\alpha+\frac{1}{2}\right)_{p-\ell}}{\left(i+j-2p+\alpha+\frac{3}{2}\right)_{p-\ell}(i-2\ell+2\alpha+1)_{2\ell+1}},$$

$$\bar{G}_{p,i,j} = \frac{2^{2\alpha-1} i!\, j!\, \Gamma(\alpha+1)}{\sqrt{\pi}\, \Gamma\left(\alpha+\frac{1}{2}\right) \Gamma(i+2\alpha+2)\, \Gamma(j+2\alpha+1)} \times$$
$$\sum_{\ell=0}^{p} \frac{(1-2i+4\ell-2\alpha)(2i+2j-4p+2\alpha)\Gamma\left(i-\ell-p+\alpha-\frac{1}{2}\right)\Gamma\left(j+\ell-p+\alpha+\frac{1}{2}\right))}{(i-\ell-p-1)!\,(j+\ell-p)!\,(p-\ell)!} \times$$
$$\frac{\Gamma\left(p-\ell+\alpha+\frac{1}{2}\right)\Gamma(i+j-\ell-p+2\alpha)}{\Gamma\left(i+j-\ell-p+\alpha+\frac{1}{2}\right)}.$$

Proof. Based on the connection Formula (4), we can write

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \sum_{k=0}^{i} M_{k,i} V_{i-k}^{(\alpha,\alpha)}(x)\, V_j^{(\alpha,\alpha)}(x),$$

where the coefficients $M_{k,i}$ are given by the following formula

$$M_{k,i} = \frac{(-1)^k\,(2i-2k+2\alpha+1)(i-k+1)_k}{(i-k+2\alpha+1)_{k+1}}.$$

Based on the linearization Formula (51), we have

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \sum_{k=0}^{i} M_{k,i} \sum_{\ell=0}^{\min(i-k,j)} B_{\ell,i-k,j}\, V_{i+j-k-2\ell}^{(\alpha,\alpha)}(x),$$

where the coefficients $B_{p,i,j}$ are given by

$$B_{p,i,j} = \frac{4^{\alpha}\Gamma(\alpha+1)\Gamma\left(i-p+\alpha+\frac{1}{2}\right)\Gamma\left(j-p+\alpha+\frac{1}{2}\right)\Gamma(i+j-p+2\alpha+1)}{\sqrt{\pi}\, p!\, \Gamma\left(i+j-2p+\alpha+\frac{1}{2}\right)\Gamma(i+2\alpha+1)\Gamma(j+2\alpha+1)} \times$$
$$\frac{(i-p+1)_p (j-p+1)_p \left(\alpha+\frac{1}{2}\right)_p}{\left(i+j-2p+\alpha+\frac{3}{2}\right)_p}.$$

Some algebraic computations lead to the following linearization formula

$$V_i^{(\alpha,\alpha+1)}(x)\, V_j^{(\alpha,\alpha)}(x) = \sum_{p=0}^{\min(i,j)} \left(\sum_{\ell=0}^{p} M_{2\ell,i}, B_{p-\ell,i-2\ell,j}\right) V_{i+j-2p}^{(\alpha,\alpha)}(x)$$
$$+ \sum_{p=0}^{\min(i,j)} \left(\sum_{\ell=0}^{p} M_{2\ell+1,i} B_{p-\ell,i-2\ell-1,j}\right) V_{i+j-2p-1}^{(\alpha,\alpha)}(x).$$

The last formula leads to the linearization Formula (53). □

Corollary 16. *Let i and j be any non-negative integers. The following linearization formula holds:*

$$V_i(x)\, T_j(x) = \sum_{p=0}^{2i} (-1)^p\, T_{i+j-p}(x). \tag{54}$$

Proof. Formula (54) can be obtained as a special case of Formula (53) taking the limit as α tends to $-\frac{1}{2}$. □

Remark 8. *The linearization Formula (54) can be translated into the following trigonometric identity:*

$$\sum_{p=0}^{2i}(-1)^p\cos((j+i-p)\theta)=\frac{\cos\left(\left(i+\frac{1}{2}\right)\theta\right)\cos(j\,\theta)}{\cos\left(\frac{\theta}{2}\right)}.$$

The following theorem exhibits the product formula of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ with the generalized Fibonacci polynomials that are defined in (9) in terms of the generalized Fibonacci polynomials.

Theorem 12. *Let i and j be any non-negative integers, the following linearization formula holds:*

$$\begin{aligned}V_i^{(\alpha,\alpha+1)}(x)\,F_j^{A,B}(x)=&\frac{2^{i+2\alpha+1}\,A^{-i}\,\Gamma(\alpha+1)\,\Gamma\left(\frac{1}{2}(2i+2\alpha+3)\right)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)}\times\\&\sum_{p=0}^{i}(-B)^p\binom{i}{p}\,{}_2F_1\left(\left.\begin{array}{c}-p,-i+p\\-\frac{1}{2}-i-\alpha\end{array}\right|-\frac{A^2}{4\,B}\right)F_{j+i-2p}^{A,B}(x)\\&-\frac{2^{i+2\alpha}A^{1-i}\,i!\,\Gamma(\alpha+1)\Gamma\left(\frac{1}{2}(2i+2\alpha+1)\right)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)}\sum_{p=0}^{i-1}\frac{(-B)^p}{p!\,(i-p-1)!}\times\\&{}_2F_1\left(\left.\begin{array}{c}-p,1-i+p\\\frac{1}{2}-i-\alpha\end{array}\right|-\frac{A^2}{4B}\right)F_{j+i-2p-1}^{A,B}(x).\end{aligned}\tag{55}$$

Proof. The analytic form of the polynomials $V_k^{(\alpha,\alpha+1)}(x)$ allows us to write

$$\begin{aligned}V_i^{(\alpha,\alpha+1)}(x)\,F_j^{A,B}(x)=&\frac{i!\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)}\left(\sum_{r=0}^{\lfloor\frac{i}{2}\rfloor}\frac{(-1)^r\,2^{i-2r+2\alpha+1}\,\Gamma\left(i-r+\alpha+\frac{3}{2}\right)}{r!\,(i-2r)!}\,x^{i-2r}\,F_j^{A,B}(x)\right.\\&\left.+\sum_{r=0}^{\lfloor\frac{i-1}{2}\rfloor}\frac{(-1)^{r+1}\,2^{i-2r+2\alpha}\,\Gamma\left(i-r+\alpha+\frac{1}{2}\right)}{r!\,(i-2r-1)!}\,x^{i-2r-1}\,F_j^{A,B}(x)\right).\end{aligned}$$

The moment formula of the generalized Fibonacci polynomials in (10) turns the last formula into the following one:

$$\begin{aligned}V_i^{(\alpha,\alpha+1)}(x)\,F_j^{A,B}(x)=&\frac{i!\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)}\left(\sum_{r=0}^{\lfloor\frac{i}{2}\rfloor}\frac{(-1)^r\,A^{-i+2r}\,2^{i-2r+2\alpha+1}\,\Gamma\left(i-r+\alpha+\frac{3}{2}\right)}{r!\,(i-2r)!}\times\right.\\&\sum_{m=0}^{i-2r}(-B)^m\binom{i-2r}{m}F_{j+i-2r-2m}^{A,B}(x)+\sum_{r=0}^{\lfloor\frac{i-1}{2}\rfloor}\frac{(-1)^{r+1}A^{1-i+2r}2^{i-2r+2\alpha}\Gamma\left(i-r+\alpha+\frac{1}{2}\right)}{r!\,(i-2r-1)!}\times\\&\left.\sum_{m=0}^{i-2r-1}(-B)^m\binom{i-2r-1}{m}F_{j+i-2r-2m-1}^{A,B}(x)\right),\end{aligned}$$

which can be transformed again after some algebraic computations into the following form:

$$V_i^{(\alpha,\alpha+1)}(x)\, F_j^{A,B}(x) = \frac{i!\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)} \times$$
$$\left(\sum_{p=0}^{i} \sum_{\ell=0}^{p} \frac{(-1)^{\ell}\, 2^{i-2\ell+2\alpha+1}\, A^{-i+2\ell}\, (-B)^{p-\ell} \binom{i-2\ell}{p-\ell} \Gamma\left(i-\ell+\alpha+\frac{3}{2}\right)}{\ell!\,(i-2\ell)!} F_{j+i-2p}^{A,B}(x) \right.$$
$$\left. + \sum_{p=0}^{i-1} \sum_{\ell=0}^{p} \frac{(-1)^{\ell+1} 2^{i-2\ell+2\alpha} A^{1-i+2\ell} (-B)^{p-\ell} \Gamma\left(i-\ell+\alpha+\frac{1}{2}\right)}{\ell!\,(i-\ell-p-1)!\,(p-\ell)!} F_{j+i-2p-1}^{A,B}(x) \right).$$

Making use of the two following identities:

$$\sum_{\ell=0}^{p} \frac{(-1)^{\ell}\, 2^{i-2\ell+2\alpha+1}\, A^{-i+2\ell}\, (-B)^{p-\ell} \binom{i-2\ell}{p-\ell} \Gamma\left(i-\ell+\alpha+\frac{3}{2}\right)}{\ell!\,(i-2\ell)!}$$
$$= \frac{2^{i+2\alpha+1}\, A^{-i}\, (-B)^{p} \binom{i}{p} \Gamma\left(i+\alpha+\frac{3}{2}\right)}{i!} \, {}_2F_1\left(\begin{matrix} -p, -i+p \\ -\frac{1}{2}-i-\alpha \end{matrix} \middle| -\frac{A^2}{4\,B} \right),$$
$$\sum_{\ell=0}^{p} \frac{(-1)^{\ell+1}\, 2^{i-2\ell+2\alpha}\, A^{1-i+2\ell}\, (-B)^{p-\ell}\, \Gamma\left(i-\ell+\alpha+\frac{1}{2}\right)}{\ell!\,(i-\ell-p-1)!\,(p-\ell)!}$$
$$= -\frac{2^{i+2\alpha} A^{1-i} (-B)^{p} \Gamma\left(i+\alpha+\frac{1}{2}\right)}{p!\,(i-p-1)!} \, {}_2F_1\left(\begin{matrix} -p, 1-i+p \\ \frac{1}{2}-i-\alpha \end{matrix} \middle| -\frac{A^2}{4\,B} \right).$$

Therefore, the linearization Formula (55) can be obtained. □

Remark 9. *For the case corresponding to $B = \frac{-A^2}{4}$, the linearization Formula (55) reduces to a simple linearization formula due to the Chu–Vandermond identity. This case is treated in the following corollary.*

Corollary 17. *For all non-negative integers i and j, the following linearization formula holds:*

$$\begin{aligned} V_i^{(\alpha,\alpha+1)}(x)\, F_j^{A,\frac{-A^2}{4}}(x) = & \frac{\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)} \sum_{p=0}^{i} \frac{2^{i-2p+2\alpha+1}\, A^{-i+2p} \binom{i}{p} \left(\alpha+\frac{3}{2}\right)_p}{\left(i-p+\alpha+\frac{3}{2}\right)_p} F_{i+j-2p}^{A,\frac{-A^2}{4B}}(x) \\ & - \frac{i!\,\Gamma(\alpha+1)\,\Gamma\left(i+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(i+2\alpha+2)} \sum_{p=0}^{i-1} \frac{2^{i-2p+2\alpha}\, A^{1-i+2p} \left(\alpha+\frac{3}{2}\right)_p}{(i-p-1)!\,p!\left(i-p+\alpha+\frac{1}{2}\right)_p} F_{i+j-2p-1}^{A,\frac{-A^2}{4B}}(x). \end{aligned} \tag{56}$$

Proof. Setting $B = -\frac{A^2}{4}$ in Formula (55) yields

$$\begin{aligned} & V_i^{(\alpha,\alpha+1)}(x)\, F_j^{A,\frac{-A^2}{4}}(x) = \\ & \frac{\Gamma(\alpha+1)\Gamma\left(i+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(i+2\alpha+2)} \sum_{p=0}^{i} 2^{i-2p+2\alpha+1} A^{2p-i} \binom{i}{p} \, {}_2F_1\left(\begin{matrix} -p, -i+p \\ -\frac{1}{2}-i-\alpha \end{matrix} \middle| 1 \right) F_{i+j-2p}^{A,\frac{-A^2}{4B}}(x) \\ & - \frac{i!\Gamma(\alpha+1)\Gamma\left(i+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(i+2\alpha+2)} \sum_{p=0}^{i-1} \frac{2^{i-2p+2\alpha} A^{2p-i+1}}{p!(i-p-1)!} \, {}_2F_1\left(\begin{matrix} -p, 1-i+p \\ \frac{1}{2}-i-\alpha \end{matrix} \middle| 1 \right) F_{i+j-2p-1}^{A,\frac{-A^2}{4B}}(x). \end{aligned} \tag{57}$$

Based on Chu–Vandermond identity, the following two identities apply:

$$
{}_2F_1\left(\begin{matrix} -p, -i+p \\ -\frac{1}{2}-i-\alpha \end{matrix}\,\middle|\,1\right) = \frac{\left(\alpha+\frac{3}{2}\right)_p}{\left(\alpha+i-p+\frac{3}{2}\right)_p},
$$

$$
{}_2F_1\left(\begin{matrix} -p, 1-i+p \\ \frac{1}{2}-i-\alpha \end{matrix}\,\middle|\,1\right) = \frac{\left(\alpha+\frac{3}{2}\right)_p}{\left(\alpha+i-p+\frac{1}{2}\right)_p}.
$$

Inserting the last two identities into (57) leads to the following linearization formula:

$$
\begin{aligned}
V_i^{(\alpha,\alpha+1)}(x)\, F_j^{A,\frac{-A^2}{4}}(x) =& \frac{\Gamma(\alpha+1)\Gamma\left(i+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(i+2\alpha+2)} \sum_{p=0}^{i} \frac{2^{i-2p+2\alpha+1} A^{-i+2p} \binom{i}{p} \left(\alpha+\frac{3}{2}\right)_p}{\left(i-p+\alpha+\frac{3}{2}\right)_p} F_{i+j-2p}^{A,\frac{-A^2}{4B}}(x) \\
&- \frac{i!\Gamma(\alpha+1)\Gamma\left(i+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(i+2\alpha+2)} \sum_{p=0}^{i-1} \frac{2^{i-2p+2\alpha} A^{1-i+2p} \left(\alpha+\frac{3}{2}\right)_p}{(i-p-1)!p!\left(i-p+\alpha+\frac{1}{2}\right)_p} F_{i+j-2p-1}^{A,\frac{-A^2}{4B}}(x).
\end{aligned}
$$

□

Remark 10. *For $A=2$, and $B=1$, Formula (56) turns into:*

$$
\begin{aligned}
V_i^{(\alpha,\alpha+1)}(x)\, T_j(x) =& \frac{2^{2\alpha+1}\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{3}{2}\right)\Gamma(i+2\alpha+2)} \sum_{p=0}^{i} \binom{i}{p} \Gamma\left(i-p+\alpha+\frac{3}{2}\right) \Gamma\left(p+\alpha+\frac{3}{2}\right) T_{j+i-2p}(x) \\
&- \frac{2^{2\alpha+1} i!\Gamma(\alpha+1)\Gamma\left(i+\alpha+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(i+2\alpha+2)} \sum_{p=0}^{i-1} \frac{\left(\alpha+\frac{3}{2}\right)_p}{p!(i-p-1)!\left(i-p+\alpha+\frac{1}{2}\right)_p} T_{j+i-2p-1}(x).
\end{aligned}
$$

6. Conclusions

In this article, a class of Jacobi polynomials is investigated. This class generalizes the third-kind Chebyshev polynomials. Various interesting formulas concerned with these polynomials were developed. Expressions for the derivatives of these polynomials in terms of different polynomials were presented. Connections with some other polynomials were also deduced. Some linearization formulas involving the generalized third-kind Chebyshev polynomials were given. Symbolic algebra was employed in a variety of formulas to reduce the coefficients that involve hypergeometric functions. We do believe that other classes of polynomials of Jacobi polynomials can be investigated using similar approaches to those followed in this article.

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