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The Arcsine Kumaraswamy-Generalized Family: Bayesian and Classical Estimates and Application

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Abstract: In this paper, by including a trigonometric function, we propose a family of heavy-tailed distribution called the arcsine Kumaraswamy generalized-X family of distributions. Based on the proposed approach, a four-parameter extension of the Lomax distribution called the arcsine Kumaraswamy generalized Lomax (ASKUG-LOMAX) distribution is discussed in detail. Maximum likelihood, bootstrap, and Bayesian estimation are used to estimate the model parameters. A simulation study is used to evaluate ASKUG-LOMAX performance. The flexibility and usefulness of the proposed ASKUG-LOMAX distribution to predict unique symmetric and asymmetric patterns is demonstrated by analyzing real data. The results show that the ASKUG-LOMAX model is a good candidate for analyzing claims based on heavy-tailed data.

Keywords: Lomax distribution; family of heavy-tailed distributions; maximum likelihood estimation; Bayesian estimation; statistical modeling

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1. Introduction

In the applied sciences, distributions that are heavy-tailed play an important role in modeling data, especially in the fields of industry, economics, finance, banking, and risk management. However, the quality of these methods depends primarily on the assumed probability model of the phenomenon under consideration. In the applied fields, industrial datasets are generally positive [1], right-skewed [2], unimodal [3], and with strong outliers [4]. Right-skewed data can be adequately modeled by skewed distributions [5]. Therefore, a set of unimodal positively-skewed parametric distributions can be used to model such datasets [6,7]. When the CDF $F(u)$ is an underlined distribution, the heavy-tailed models are those with probabilities at the right tail that are heavier than the exponential one (see Beirlant et al. [8] and Resnick [9]), satisfying

$$\lim_{u\to\infty} \frac{e^{(-\lambda u)}-1}{1-F(u)} = 0, \quad \lambda > 0. \tag{1}$$

Dutta and Perry [10] conducted an empirical analysis of loss distributions to estimate risk using various approaches. They rejected the use of the exponential, gamma, and Weibull models because of their poor results, and concluded that the best choice would be to use a model that is flexible enough in its structure. These results encouraged researchers to propose new flexible models that provide greater accuracy in data fitting. To overcome the problems associated with modeling based on classical distributions, new families of distributions (see [11–17]) have been introduced.

In this context, Mudholkar and Srivastava [18] have discussed the exponential family via the composition of a shape parameter to obtain a more adaptive extension of the basic model. The CDF of the random variable X over the exponential family is provided by

$$F(u;\zeta,\varphi) = [G(u;\theta)]^{\zeta}, \qquad \zeta \geq 0, \theta, u \in \mathbb{R}, \tag{2}$$

where $G(u;\varphi)$ is the CDF of the underline distribution, φ is the parameter vector, and $\zeta > 0$ is a shape parameter. Moreover, the Kumaraswamy generalised (KUG) family is a more adaptive family proposed by Cordeiro and de Castro [19] via the composition of two shape parameters. The CDF of the random variable X over the KUG family is provided by

$$F(u;\delta_1,\delta_2,\varphi) = 1 - \left(1 - [G(u;\varphi)]^{\delta_1}\right)^{\delta_2}, \qquad \delta_1,\delta_2 \geq 0, \varphi, u \in \mathbb{R}, \tag{3}$$

The arcsine exponentiated X family of distributions appears here as a special case when $\delta_2 = 1$. It is based on Equation (3); for a contribution, see Mead and Afify [20]. With respect to Equation (3), the KUG expansion of the current distributions has been discussed in the literature; see Mansour et al. [21] and Ahmad et al. [22]. The arcsine KUG-X family is denoted by (ASKUG-X). The CDF of the random variable X over ASKUG is

$$F(x;\delta_1,\delta_2,\varphi) = \frac{2}{\pi} arcsin\left(1 - \left(1 - [G(x;\varphi)]^{\delta_1}\right)^{\delta_2}\right), \qquad \delta_1,\delta_2 \geq 0, \varphi, x \in \mathbb{R}, \tag{4}$$

where δ_1 and δ_2 are two additional shape parameters. The main motivations for using the ASKUG-X family in practice are the following:

(i) To develop the flexibility and properties of basic models.
(ii) A suitable procedure for adding two extra parameters in expanded models with potent outliers, which is very useful when modeling industrial data (see Section 5).
(iii) To introduce the extended version of a basic model with closed forms for the cdf and hazard rate function, the special submodels of this family can be used in the analysis of censored data sets.
(iv) Compared to existing competing models, the special cases of the ASKUG-X approach are able to model data sets with high-tailed content in factorial habits.

The respective PDF, survival function (SF), and hazard rate function (HRF) of the random variable X via ASKUG are provided as follows:

$$f(x;\delta_1,\delta_2,\varphi) = \frac{2\delta_1\delta_2 g(x;\varphi)[G(x;\varphi)]^{\delta_1-1}\left(1 - \left(1 - [G(x;\varphi)]^{\delta_1}\right)^{\delta_2-1}\right)}{\pi \sqrt{1 - \left(1 - \left(1 - [G(x;\varphi)]^{\delta_1}\right)^{\delta_2}\right)^2}}, \tag{5}$$

$$S = 1 - \frac{2}{\pi}\text{arcSin}\left[1 - \left(1 - G(x;\varphi)^{\delta_1}\right)^{\delta_2}\right] \tag{6}$$

and

$$h = \frac{2\delta_1\delta_2 g(x;\varphi)G(x;\varphi)^{\delta_1-1}\left(1 - \left(1 - G(x;\varphi)^{\delta_1}\right)^{\delta_2-1}\right)}{\sqrt{1 - \left(1 - \left(1 - G(x;\varphi)^{\delta_1}\right)^{\delta_2}\right)^2}\left(\pi - 2\text{arcSin}\left[1 - \left(1 - G(x;\varphi)^{\delta_1}\right)^{\delta_2}\right]\right)}, \tag{7}$$

where $\delta_1,\delta_2 > 0, \varphi \in \mathbb{R}$ and $x \in \mathbb{R}$. Via the new CDF of ASKUG-X, many new heavy-tailed flexible models can be obtained. A number of newly contributed models based on the ASKUG-X approach are presented in Table 1.

The rest of this paper is outlined as follows. Section 2 defines ASKUG-LOMAX. Section 3 provides the maximum likelihood estimators (MLEs). Section 4 provides the Bayesian estimators. Section 5 presents a discussion of our simulations and analyzes two examples of real data to illustrate the proposed ASKUG-LOMAX potentiality. Section 6 provides a brief conclusion.

Table 1. New submodules via the ASKUG-X family.

No.	Model	Distribution Function	Generated Model	Range
1	Beta	$\frac{2}{\pi} arcsin\left(1-\left(1-[I_x(a,b)]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Beta	$a<x<b, a, b, \delta_1, \delta_2>0$
2	Burr	$\frac{2}{\pi} arcsin\left(1-\left(1-\left[1-(1+x^a)^{-b}\right]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Burr	$x \geq 0, a, b, \delta_1, \delta_2>0$
3	Erlang	$\frac{2}{\pi} arcsin\left(1-\left(1-[(1/(a-1)!)\gamma(a,bx)]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Erlang	$x \geq 0, a, b, \delta_1, \delta_2>0$
4	Exponential	$\frac{2}{\pi} arcsin\left(1-\left(1-[1-\exp(-ax)]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Exponential	$x \geq 0, a, \delta_1, \delta_2>0$
5	Frechet	$\frac{2}{\pi} arcsin\left(1-\left(1-[\exp(-(x/b)^{-a})]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Frechet	$x \geq 0, a, b, \delta_1, \delta_2>0$
6	Gamma	$\frac{2}{\pi} arcsin\left(1-\left(1-[\gamma(a,bx)/Gamma(a)]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Gamma	$x \geq 0, a, b, \delta_1, \delta_2>0$
7	Gumbel	$\frac{2}{\pi} arcsin\left(1-\left(1-[\exp(-\exp(-(x-a)/b))]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Gumbel	$x, a \in R, b, \delta_1, \delta_2>0$
8	Half logistic	$\frac{2}{\pi} arcsin\left(1-\left(1-[(1-\exp(-x))/(1+\exp(-x))]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Half logistic	$x \geq 0, \delta_1, \delta_2>0$
9	Kumaraswamy	$\frac{2}{\pi} arcsin\left(1-\left(1-[1-(1-x^a)^b]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Kumaraswamy	$0<x<1, a, b, \delta_1, \delta_2>0$
10	Lindely	$\frac{2}{\pi} arcsin\left(1-\left(1-[1-((\exp(-ax)(1+a+ax))/(1+a))]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Lindely	$x \geq 0, a, \delta_1, \delta_2>0$
11	Linear failure rate	$\frac{2}{\pi} arcsin\left(1-\left(1-\left[1-\exp(-ax^b-cx)\right]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Linear failure rate	$x \geq 0, a, b, c, \delta_1, \delta_2>0$
12	Log logistics	$\frac{2}{\pi} arcsin\left(1-\left(1-[1/(1+(x/b)^{-a})]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Log logistics	$x \geq 0, a, b, \delta_1, \delta_2>0$
13	Lomax	$\frac{2}{\pi} arcsin\left(1-\left(1-\left[1-(1+ax)^{-b}\right]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Lomax	$x \geq 0, a, b, \delta_1, \delta_2>0$
14	Normal	$\frac{2}{\pi} arcsin\left(1-\left(1-[\phi((x-a)/b)]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Normal	$x, a \in R, b, \delta_1, \delta_2>0$
15	Pareto	$\frac{2}{\pi} arcsin\left(1-\left(1-[1-(x_m/x)^a]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Pareto	$x_m \leq x<\infty, a, x_m, \delta_1, \delta_2>0$
16	Power function	$\frac{2}{\pi} arcsin\left(1-\left(1-[(x/b)^a]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Power function	$0<x<b, a, b, \delta_1, \delta_2>0$
17	Rayleigh	$\frac{2}{\pi} arcsin\left(1-\left(1-\left[1-\exp(-ax^2)\right]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Rayleigh	$x \geq 0, a, \delta_1, \delta_2>0$
18	Topp Leone	$\frac{2}{\pi} arcsin\left(1-\left(1-[x^a(2-x^a)]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Topp Leone	$0<x<1, a, b, \delta_1, \delta_2>0$
19	Uniform	$\frac{2}{\pi} arcsin\left(1-\left(1-[x/a]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Uniform	$0<x<a, a, \delta_1, \delta_2>0$
20	Weibull	$\frac{2}{\pi} arcsin\left(1-\left(1-\left[1-\exp(-ax^b)\right]^{\delta_1}\right)^{\delta_2}\right)$	ASKUG-Weibull	$x \geq 0, a, b, \delta_2, \delta_2>0$

2. ASKUG-Lomax Distribution

In this section, we introduce the ASKUG-Lomax distribution and examine the attitude of its PDF and HRF. First, we assess the CDF of the Lomax distribution, $G(x;\kappa,\alpha) = 1-(1+\kappa x)^{-\alpha}, x \geq 0, \kappa, \alpha > 0$. Then, a random variable X is said to follow the ASKUG-Lomax distribution if its CDF takes the following form:

$$F(x;\kappa,\alpha,\delta_1,\delta_2) = \frac{2}{\pi}\text{arcSin}\left[1-\left(1-\left(1-(1+\kappa x)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right], \quad x \geq 0, \kappa, \alpha, \delta_1, \delta_2 > 0. \quad (8)$$

The respective PDF, SF, and HRF of the ASKUG-Lomax distribution are provided by

$$f(x;\kappa,\alpha,\delta_1,\delta_2) = \frac{2\kappa\alpha\delta_1\delta_2\left(1-(1+\kappa x)^{-\alpha}\right)^{\delta_1-1}\left(1-\left(1-(1+\kappa x)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2-1}}{\pi(1+\kappa x)^{\alpha+1}\sqrt{1-\left(1-\left(1-\left(1-(1+\kappa x)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right)^2}},$$
$$x \geq 0, \kappa, \alpha, \delta_1, \delta_2 > 0, \quad (9)$$

$$S = 1 - \frac{2\text{arcSin}\left[1-\left(1-(1-(1+x\kappa)^{-\alpha})^{\delta_1}\right)^{\delta_2}\right]}{\pi}, \quad (10)$$

and

$$h = \frac{2\kappa\alpha\delta_1\delta_2(1+x\kappa)^{-\alpha-1}(1-(1+\kappa x)^{-\alpha})^{\delta_1-1}\left(1-(1-(1+\kappa x)^{-\alpha})^{\delta_1}\right)^{\delta_2-1}}{\pi\sqrt{1-\left(1-\left(1-(1-(1+\kappa x)^{-\alpha})^{\delta_1}\right)^{\delta_2}\right)^2}\left(1-\frac{2\text{arcSin}\left[1-\left(1-(1-(1+\kappa x)^{-\alpha})^{\delta_1}\right)^{\delta_2}\right]}{\pi}\right)}. \quad (11)$$

The ASKUG-Lomax model reduces to the AS-exponentiate-Lomax distribution when $\delta_2 = 1$, and to the AS-Lomax distribution when $\delta_1 = \delta_2 = 1$.

2.1. Quantile Function

Let X be the ASKUG-Lomax random variable with the PDF from Equation (5); then, the quantile function of X, i.e., Q(u), reduces to

$$Q(u) = \frac{1}{k}\left(\left(1-\left(1-\left(1-\sin\frac{\pi u}{2}\right)^{\frac{1}{\delta_2}}\right)^{\frac{1}{\delta_1}}\right)^{\frac{-1}{\alpha}} - 1\right). \quad (12)$$

where u has a uniform distribution on the interval (0, 1). From the expression in Equation (12), it is evident that the ASKUG-Lomax family has a closed form solution of its quantile function.

2.2. Moments

Moments are very important in statistical analysis, and play an essential role. They help to capture important features and properties of the distribution (e.g., its central tendency, dispersion, skewness, and kurtosis). The r^{th} moment of the ASKUG-X family is

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x;\delta_1,\delta_2,\varphi)dx. \quad (13)$$

Substituting Equation (5) into Equation (12), we obtain

$$\mu_r^{'} = \int_{-\infty}^{\infty} x^r \frac{2\delta_1\delta_2 g(x;\varphi)[G(x;\varphi)]^{\delta_1-1}\left(1-\left(1-[G(x;\varphi)]^{\delta_1}\right)^{\delta_2-1}\right)}{\pi \quad \sqrt{1-\left(1-\left(1-[G(x;\varphi)]^{\delta_1}\right)^{\delta_2}\right)^2}} dx. \tag{14}$$

Using the binomial expansion, we have

$$\frac{1}{\sqrt{1-u^2}} = \sum_{t=0}^{\infty} \frac{1\times 3\times 5\times ...\times (2t-1)}{2^t \times t!} u^{2t}. \tag{15}$$

By replacing u with $\left(1-[G(x;\varphi)]^{\delta_1}\right)^{\delta_2}$ in Equation (14), we obtain

$$\mu_r^{'} = \frac{2\delta_1\delta_2}{\pi} \sum_{t=0}^{\infty} \frac{1\times 3\times 5\times ...\times (2t-1)}{2^t \times t!} \eta_{r,\delta_1,2t\delta_2}, \tag{16}$$

where $\eta_{r,\delta_1,2t\delta_2} = \int_{-\infty}^{\infty} x^r g(x;\varphi)[G(x;\varphi)]^{\delta_1-1}\left(1-[G(x;\varphi)]^{\delta_1}\right)^{2t\delta_2}\left(1-\left(1-[G(x;\varphi)]^{\delta_1}\right)^{\delta_2-1}\right)dx.$

The moment-generating function of the ASKUG-X class has the following form:

$$M_x(y) = \frac{2\delta_1\delta_2}{\pi} \sum_{t=0}^{\infty} \frac{1\times 3\times 5\times ...\times (2t-1)}{2^t \times t! \times r!} y^r \eta_{r,\delta_1,2t\delta_2} \tag{17}$$

3. Maximum Likelihood Estimation

Take the observed values $x_1, x_2, ..., x_n$ of $X_1, X_2, ..., X_n$, which is a random sample from the ASKUG-X model; then, the ASKUG-X log-likelihood is

$$\begin{aligned} L \quad = \quad & n\text{Log}\left[\frac{2}{\pi}\delta_1\delta_2\right] + \sum_{i=1}^{n} \text{Log}[g(x_i;\varphi)] + (\delta_1 - 1)\sum_{i=1}^{n} \text{Log}[G(x_i;\varphi)] \\ + \quad & \sum_{i=1}^{n} \text{Log}\left[1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2-1}\right] - \frac{1}{2}\sum_{i=1}^{n} \text{Log}\left[1-\left(1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2}\right)^2\right]. \end{aligned} \tag{18}$$

The MLE can be derived by maximizing Equation (12) (see Appendix A).

The asymptotic CIs of κ, α, δ_1 and δ_2 can be computed. The variance–covariance matrix $V(\hat{\kappa}, \hat{\alpha}, \hat{\delta_1}, \hat{\delta_2})$ is provided by

$$V(\kappa,\alpha,\delta_1,\delta_2) = -\begin{bmatrix} \frac{\partial^2 L}{\partial\kappa^2} & \frac{\partial^2 L}{\partial\kappa\partial\alpha} & \frac{\partial^2 L}{\partial\kappa\partial\delta_1} & \frac{\partial^2 L}{\partial\kappa\partial\delta_2} \\ \frac{\partial^2 L}{\partial\alpha\partial\kappa} & \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\delta_1} & \frac{\partial^2 L}{\partial\alpha\partial\delta_2} \\ \frac{\partial^2 L}{\partial\delta_1\partial\kappa} & \frac{\partial^2 L}{\partial\delta_1\partial\alpha} & \frac{\partial^2 L}{\partial\delta_1^2} & \frac{\partial^2 L}{\partial\delta_1\partial\delta_2} \\ \frac{\partial^2 L}{\partial\delta_2\partial\kappa} & \frac{\partial^2 L}{\partial\delta_2\partial\alpha} & \frac{\partial^2 L}{\partial\delta_2\partial\delta_1} & \frac{\partial^2 L}{\partial\delta_2^2} \end{bmatrix}^{-1}, \tag{19}$$

The respective $100(1-\varepsilon)\%$ two-sided approximate CIs for κ, α, δ_1 and δ_2 are provided by

$$\widehat{\kappa} \pm z_{\kappa/2}\sqrt{V(\widehat{\kappa})}, \widehat{\alpha} \pm z_{\kappa/2}\sqrt{V(\widehat{\alpha})}, \widehat{\delta_1} \pm z_{\kappa/2}\sqrt{V(\widehat{\delta_1})}, \text{ and } \widehat{\delta_2} \pm z_{\kappa/2}\sqrt{V(\widehat{\delta_2})}, \tag{20}$$

where $V(\widehat{\kappa}), V(\widehat{\alpha}), V(\widehat{\delta_1})$ and $V(\widehat{\delta_2})$ are provided by the diagonal elements of $V(\hat{\kappa}, \hat{\alpha}, \hat{\delta_1}, \hat{\delta_2})$, and $z_{\varepsilon/2}$ is the upper $\left(\frac{\varepsilon}{2}\right)$ percentile of a standard normal distribution.

Next, to obtain the bootstrap CI boot-p for the unknown parameters $\phi = (\kappa, \alpha, \delta_1, \delta_2)$, we apply the following algorithm, Algorithm 1;

Algorithm 1 Boot-p interval algorithm:

step-1: Simulate $x_{1:n}, x_{2:n}, ..., x_{n:n}$ from the ASKUG-Lomax distribution and obtain an estimate $\hat{\phi}$ of ϕ.

step-2: Simulate another sample $x^*_{1:n}, x^*_{2:n}, ..., x^*_{n:n}$ via $\hat{\phi}$. Then, obtain the updated bootstrap estimate $\hat{\phi}^*$ of ϕ.

step-3: Iterate step 2 a previously fixed number of iterations B.

step-4: Via $\hat{F}(x) = P(\hat{\phi}^* \leq x)$, that is, the CDF of $\hat{\phi}^*$, the $100(1-\varepsilon)\%$ CI of ϕ is provided by

$$\left(\hat{\phi}_{Boot-p}(\frac{\varepsilon}{2}), \hat{\phi}_{Boot-p}(1-\frac{\varepsilon}{2})\right),$$

where $\hat{\phi}_{Boot-p}(x) = \hat{F}^{-1}(x)$ and x is previously fixed.

4. Bayesian Estimation

Suppose that κ, α, δ_1, and δ_2 are random variables that follow the prior PDFs $\text{Gamma}(\kappa; a_1, b_1)$, $\text{Gamma}(\alpha; a_2, b_2)$, $\text{Gamma}(\delta_1; a_3, b_3)$, and $\text{Gamma}(\delta_2; a_4, b_4)$, respectively, where a_i and b_i are positive constants and $i = 1, 2, 3, 4$. The posterior DF of $\kappa, \alpha, \delta_1, \delta_2$ and the data under the Gamma priors can take the forms

$$\pi_1(\kappa) = \frac{b_1^{a_1}}{\Gamma(a_1)}\kappa^{a_1-1}\exp[-b_1\kappa], \quad \kappa, a_1, b_1 > 0, \tag{21}$$

$$\pi_2(\alpha) = \frac{b_2^{a_2}}{\Gamma(a_2)}\alpha^{a_2-1}\exp[-b_2\alpha], \quad \alpha, a_2, b_2 > 0, \tag{22}$$

$$\pi_3(\delta_1) = \frac{b_3^{a_3}}{\Gamma(a_3)}\delta_1^{a_3-1}\exp[-b_3\delta_1], \quad \delta_1, a_3, b_3 > 0, \tag{23}$$

and

$$\pi_4(\delta_2) = \frac{b_4^{a_4}}{\Gamma(a_4)}\delta_2^{a_4-1}\exp[-b_4\delta_2], \quad \delta_2, a_4, b_4 > 0. \tag{24}$$

Then, the posterior density of $\kappa, \alpha, \delta_1, \delta_2$ and the data can be extracted as

$$\pi^*(\kappa, \alpha, \delta_1, \delta_2|\mathbf{x}) \propto \pi(\kappa, \alpha, \delta_1, \delta_2) \prod_{i=1}^{n} f(x_i; \kappa, \alpha, \delta_1, \delta_2),$$

$$= J^{-1} \kappa^{n+a_1-1} \alpha^{n+a_2-1} \delta_1^{n+a_3-1} \delta_2^{n+a_4-1} e^{-(b_1\kappa+b_2\alpha+b_3\delta_1+b_4\delta_2)}$$

$$\prod_{i=1}^{n} \frac{\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1-1}\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2-1}}{(1+\kappa x_i)^{\alpha+1}\sqrt{1-\left(1-\left(1-\left(1-(1+\kappa x)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right)^2}}, \tag{25}$$

where $x_i \geq 0, \kappa, \alpha, \delta_1, \delta_2, a_i, b_i > 0, i = 1, 2, 3, 4$ and J is the normalizing constant.

MCMC Method

We use the Metropolis Hastings (M-H) procedure as follows:

1. Set initial values $\kappa^{(0)}$, $\alpha^{(0)}$, $\delta_1^{(0)}$, and $\delta_2^{(0)}$. Then, simulate a sample of size n from ASKUG-Lomax, next set $l = 1$.
2. Simulate $\kappa^{(*)}$, $\alpha^{(*)}$, $\delta_1^{(*)}$, and $\delta_2^{(*)}$ using the proposal distributions $N(\kappa^{(l-1)}, V(\hat{\kappa}))$, $N(\alpha^{(l-1)}, V(\hat{\alpha}))$, $N(\delta_1^{(l-1)}, V(\hat{\delta_1}))$, and $N(\delta_2^{(l-1)}, V(\hat{\delta_2}))$.
3. Obtain the probability $r = min\left(\frac{\pi^*(\kappa^{(*)},\alpha^{(*)},\delta_1^{(*)},\delta_2^{(*)})}{\pi^*(\kappa^{(l-1)},\alpha^{(l-1)},\delta_1^{(l-1)},\delta_2^{(l-1)})}, 1\right)$.
4. Simulate U from Uniform (0, 1).
5. If $U < r$, then $\left(\kappa^{(l)}, \alpha^{(l)}, \delta_1^{(l)}, \delta_2^{(l)}\right) = \left(\kappa^{(*)}, \alpha^{(*)}, \delta_1^{(*)}, \delta_2^{(*)}\right)$.
   If $U \geq r$, then $\left(\kappa^{(l-1)}, \alpha^{(l-1)}, \delta_1^{(l-1)}, \delta_2^{(l-1)}\right) = \left(\kappa^{(*)}, \alpha^{(*)}, \delta_1^{(*)}, \delta_2^{(*)}\right)$.
6. Set $l = l + 1$.
7. Iterate Steps 2–6, M repetitions, and obtain $\kappa^{(l)}, \alpha^{(l)}, \delta_1^{(l)}$ and $\delta_2^{(l)}$ for $l = 1, ..., M$.

Now, we use the squared error loss function provided by $L_{SE}(\vartheta, \hat{\vartheta}) = (\vartheta - \hat{\vartheta})^2$, where $\hat{\vartheta}$ is an estimate of the unknown parameter ϑ, which against the SE loss function is the posterior mean. Using the generated random samples from the above Gibbs sampling technique and with N the nburn, the Bayes estimator of ϑ, say, $\hat{\vartheta}_{SE}$, can be obtained as

$$\hat{\vartheta}_{SE} = E_{\vartheta}[\vartheta|\mathbf{x}] = \frac{1}{M-N} \sum_{l=N+1}^{M} \vartheta^{(l)}. \tag{26}$$

The second loss function is the LINEX loss function, provided by

$$L_{LE}(\vartheta, \hat{\vartheta}) = \exp\left[\rho(\vartheta - \hat{\vartheta})\right] - \rho(\vartheta - \hat{\vartheta}) - 1, \quad \rho \neq 0. \tag{27}$$

The approximate Bayes estimate of $\vartheta = \sigma, \delta, \gamma$ under the LE loss function based on the Gibbs sampling technique becomes

$$\hat{\vartheta}_{LE} = \frac{-1}{\rho} \log(E_{\vartheta}[\exp(-\rho\vartheta)|\mathbf{x}]) = \frac{-1}{\rho} \log\left(\frac{\sum_{l=N+1}^{M} exp\left(-\rho\vartheta^{(l)}\right)}{M-N}\right), \tag{28}$$

Finally, the general entropy (GE) loss function is provided by

$$L_{GE}(\vartheta, \hat{\vartheta}) = \left(\frac{\hat{\vartheta}}{\vartheta}\right)^{\varepsilon} - \varepsilon \log\left(\frac{\hat{\vartheta}}{\vartheta}\right) - 1. \tag{29}$$

The approximate Bayes estimate of the parameters is provided by

$$\hat{\vartheta}_{GE} = \left(E_{\vartheta}\left[\vartheta^{-\varepsilon}\middle|\mathbf{x}\right]\right)^{\frac{-1}{\varepsilon}} = \left(\frac{1}{M-N}\sum_{l=N+1}^{M}\left(\vartheta^{(l)}\right)^{-\varepsilon}\right)^{\frac{-1}{\varepsilon}}, \tag{30}$$

MCMC HPD-credible interval algorithm:

1. Sort $\kappa^{(*)}, \alpha^{(*)}, \delta_1^{(*)}$, and $\delta_2^{(*)}$ in rising values.
2. The lower bounds of κ, α, δ_1, and δ_2 in the rank $(M-N) * \varepsilon/2$.
3. The lower bounds of κ, α, δ_1, and δ_2 in the rank $(M-N) * (1-\varepsilon/2)$.
4. Iterate the previous steps M times. Obtain the average value of the lower and upper bounds of κ, α, δ_1, and δ_2.

5. Simulation Study

We generate M = 1000 samples of size n = 25, 30, 40, 50, 60, 70, 80, 90, 100 from the ASKUG-LOMAX model via the initial parameter values $\kappa^{(0)} = 0.5, \alpha^{(0)} = 0.9, \delta_1^{(0)} = 1.89$, and $\delta_2^{(0)} = 1.1$. Suppose that κ, α, δ_1, and δ_2 are random variables that follow the prior PDFs $\text{Gamma}(\kappa; 0.05, 0.68)$, $\text{Gamma}(\alpha; 0.6, 0.9)$, $\text{Gamma}(\delta_1; 0.8, 1)$, and $\text{Gamma}(\delta_2; 0.5, 0.69)$, respectively. In this simulation study, we empirically obtain the bias and expected risk (ER) of the MLEs and the Bayesian methods for different parameter combinations and each sample. The point estimations of the parameters are obtained using 200 burns MCMC methods. Two LINEX loss function are used, $LE1$ when $\rho = -0.3$ and $LE2$ when $\rho = 0.7$. The respective biases and ERs are provided by

$$Bias(\hat{\vartheta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\vartheta}_i - \vartheta),$$

and

$$ER(\hat{\vartheta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\vartheta}_i - \vartheta)^2,$$

Coverage probabilities (CPs) are calculated at the 95% and 90% HPD credible intervals. The simulation results are presented in Tables 2–5 for the parameters κ, α, δ_1, and δ_2, respectively. Based on the generated data, it can be seen that

1. The MLEs seem to behave as expected, i.e., the MSE values and the estimated biases decrease as n increases. Moreover, the mean values of the estimates tend to the true values as n increases, showing the consistency property of the MLEs.
2. It is well known that the Bayesian estimation method provides better results in practice than the classical one, especially when the sample size is relatively small, which is exactly what the results show. The standard deviation of the MLE is greater than the Bayesian estimate for $n \leq 90$.
3. The interval width of the MLE for a given confidence level is greater than the Bayesian estimate in most cases.
4. A General Entropy Loss Function is a suitable alternative to the Modified LINEX loss function. The approximate Bayes estimate of the parameters based on the general entropy loss function provides better results in most cases.

Application of the ASKUG-Lomax Model

We evaluate the usefulness of the ASKUG-Lomax model by analyzing two examples of lifetime data.

Example 1. *Industry lifetime data taken from [23]. The observations represent the reliability time of a coating machine: 1.00, 1.00, 5.00, 5.50, 12.50, 16.75, 17.75, 20.75, 22.50, 22.75, 25.00, 25.00, 27.25, 30.25, 43.75, 45.00, 48.00, 48.25, 97.50, 99.75, 136.75, 143.50, 207.75, 215.00, 225.50, 235.00, 283.50, 567.00, 970.50.*

Example 2. *Business data provided by Nigm and Hamdy [24] and Wong [25]: 1.01, 1.05, 1.08, 1.14, 1.28, 1.30, 1.33, 1.43, 1.59, 1.62.*

After analyzing the data, the estimate results are

Example 3. *The average per capita carbon dioxide emissions (in metric tons) in the Arab world, provided by [26]: 0.609268, 0.662618, 0.727117, 0.853116, 0.972381, 1.13867, 1.252, 1.31609, 1.45773, 1.76705, 1.79794, 1.99733, 2.11931, 2.19399, 2.2808, 2.40051, 2.5815, 2.64469, 2.71073, 2.75889, 2.80602, 2.86087, 2.87078, 2.91291, 2.9245, 2.95929, 2.97, 3.04479, 3.08222, 3.0918, 3.1313, 3.16119, 3.16314, 3.1633, 3.16669, 3.18374, 3.19605, 3.21329, 3.2388, 3.27108, 3.2779, 3.28251, 3.34723, 3.36173, 3.47424, 3.7043, 3.80165, 3.88314, 4.09354, 4.19327, 4.30874, 4.322, 4.43872, 4.49519, 4.51219, 4.52835, 4.57031, 4.60019, 4.61796.*

The data are presented in Tables 6–8 for the parameters κ, α, δ_1, and δ_2, respectively. Tables 9–11 compare the ASKUG-Lomax distribution via several recognition criterion: the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), and consistent Akaike information Criterion (CAIC). The goodness-of-fit results of the ASKUG-Lomax model are compared with several other models in Table 1, including Arcsine exponentiated Lomax (ASEXG-Lomax), Arcsine Lomax (AS-Lomax), Exponentiated Weibull (EX-Weibull), and Weibull distribution.

The results in Table 9 indicate that the ASKUG-Lomax distribution provides better fits than the alternative models, and represents an adequate model for analyzing heavy-tailed industry claims data. In addition, the results in Table 10 indicate that the ASKUG-Lomax distribution provides better fits than other competing models, and is adequate for analyzing business data. The results in Table 11 again indicate that the ASKUG-Lomax distribution provides better fits than the other competing models, and is adequate for analyzing carbon dioxide emissions data.

Table 2. Point and Interval estimation of the κ parameter when $\kappa = 0.5$.

n	Point					Intrval					
	ML	SE	$LE1$	$LE2$	GE	ML	$Bootstrap$	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
25	0.6661	0.6435	0.6605	0.6062	0.5018	0.2393 1.0928	0.06 1.861	0.467 0.822	0.4798 0.8348	0.4375 0.7678	0.3198 0.6758
	0.1521	0.1295	0.1465	0.0922	−0.0121	0.8535	1.801	0.355	0.355	0.3304	0.3561
	0.0474	0.026	0.0312	0.0168	0.0088	0.3068 1.0253	0.11 1.598	0.492 0.798	0.4958 0.8232	0.4663 0.7601	0.3605 0.6499
						0.7185	1.488	0.306	0.3274	0.2938	0.2894
30	0.6428	0.6435	0.6605	0.6062	0.5018	−2.4957 3.7812	0.086 1.689	0.467 0.822	0.4798 0.8348	0.4375 0.7678	0.3198 0.6758
	0.1288	0.1295	0.1465	0.0922	−0.0121	6.2769	1.603	0.355	0.355	0.3304	0.3561
	2.564	0.026	0.0312	0.0168	0.0088	−1.9993 3.2848	0.137 1.526	0.492 0.798	0.4958 0.8232	0.4663 0.7601	0.3605 0.6499
						5.2841	1.389	0.306	0.3274	0.2938	0.2894
40	0.6476	0.6435	0.6605	0.6062	0.5018	0.0818 1.2134	0.12 1.942	0.467 0.822	0.4798 0.8348	0.4375 0.7678	0.3198 0.6758
	0.1336	0.1295	0.1465	0.0922	−0.0121	1.1316	1.822	0.355	0.355	0.3304	0.3561
	0.0833	0.026	0.0312	0.0168	0.0088	0.1713 1.1239	0.153 1.554	0.492 0.798	0.4958 0.8232	0.4663 0.7601	0.3605 0.6499
						0.9527	1.401	0.306	0.3274	0.2938	0.2894
50	0.6152	0.6435	0.6605	0.6062	0.5018	0.2347 0.9958	0.117 1.696	0.467 0.822	0.4798 0.8348	0.4375 0.7678	0.3198 0.6758
	0.1012	0.1295	0.1465	0.0922	−0.0121	0.7611	1.579	0.355	0.355	0.3304	0.3561
	0.0377	0.026	0.0312	0.0168	0.0088	0.2949 0.9356	0.162 1.435	0.492 0.798	0.4958 0.8232	0.4663 0.7601	0.3605 0.6499
						0.6407	1.273	0.306	0.3274	0.2938	0.2894
60	0.6152	0.6435	0.6605	0.6062	0.5018	0.1762 1.0542	0.117 1.696	0.467 0.822	0.4798 0.8348	0.4375 0.7678	0.3198 0.6758
	0.1012	0.1295	0.1465	0.0922	−0.0121	0.878	1.579	0.355	0.355	0.3304	0.3561
	0.0502	0.026	0.0312	0.0168	0.0088	0.2456 0.9848	0.162 1.435	0.492 0.798	0.4958 0.8232	0.4663 0.7601	0.3605 0.6499
						0.7391	1.273	0.306	0.3274	0.2938	0.2894
70	0.6429	0.6435	0.6605	0.6062	0.5018	0.2584 1.0275	0.151 1.699	0.467 0.822	0.4798 0.8348	0.4375 0.7678	0.3198 0.6758
	0.1289	0.1295	0.1465	0.0922	−0.0121	0.7691	1.548	0.355	0.355	0.3304	0.3561
	0.0385	0.026	0.0312	0.0168	0.0088	0.3192 0.9667	0.197 1.492	0.492 0.798	0.4958 0.8232	0.4663 0.7601	0.3605 0.6499
						0.6475	1.295	0.306	0.3274	0.2938	0.2894
80	0.6356	0.6035	0.6116	0.5853	0.5352	0.3024 0.9688	0.161 1.755	0.283 0.999	0.2847 1.0067	0.278 0.9798	0.2518 0.9417
	0.1216	0.0895	0.0976	0.0713	0.0213	0.6664	1.594	0.716	0.722	0.7017	0.6899
	0.0272	0.0407	0.0431	0.0358	0.031	0.3551 0.93	0.211 1.48	0.336 0.923	0.3387 0.9399	0.3313 0.8851	0.2889 0.8189
						0.561	1.269	0.587	0.6012	0.5537	0.53
90	0.6526	0.5869	0.5939	0.5708	0.5248	0.0865 1.2187	0.188 1.743	0.304 1.029	0.3067 1.0402	0.2988 0.9913	0.2691 0.9165
	0.1386	0.0729	0.08	0.0568	0.0109	1.1322	1.555	0.725	0.7335	0.6925	0.6474
	0.0834	0.041	0.0431	0.0365	0.033	0.176 1.1291	0.231 1.475	0.341 0.963	0.3438 0.976	0.3346 0.9338	0.2922 0.8904
						0.9531	1.244	0.622	0.6322	0.5992	0.5982
100	0.6526	0.6449	0.6533	0.626	0.5788	0.33 0.9752	0.188 1.743	0.32 0.97	0.3234 0.9821	0.3127 0.9425	0.2904 0.911
	0.1386	0.1309	0.1393	0.112	0.0648	0.6453	1.555	0.65	0.6587	0.6298	0.6206
	0.0271	0.0475	0.0505	0.0412	0.0336	0.381 0.9242	0.231 1.475	0.376 0.933	0.3796 0.9537	0.3681 0.9029	0.3214 0.8833
						0.5432	1.244	0.557	0.574	0.5347	0.5619

Point Estimation: The first row represents the Estimate, the second row represents the Bias, and the third row represents the ER. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 3. Point and Interval estimation of the α parameter when $\alpha = 0.9$.

n	*Point*					*Intrval*					
	ML	SE	$LE1$	$LE2$	GE	ML	$Bootstrap$	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
25	1.0657	1.0174	1.0342	0.9794	0.9315	0.8151 1.3164	0.369 1.953	0.664 1.332	0.6703 1.3465	0.6513 1.2946	0.6241 1.2583
	0.1456	0.0973	0.1141	0.0592	0.0114	0.5013	1.584	0.668	0.6762	0.6433	0.6343
	0.0164	0.0332	0.0374	0.0259	0.0236	0.8548 1.2767	0.414 1.826	0.789 1.28	0.8029 1.2926	0.7589 1.2242	0.7142 1.1888
						0.422	1.412	0.491	0.4897	0.4654	0.4746
30	1.0601	1.0174	1.0342	0.9794	0.9315	0.3908 1.7294	0.355 1.971	0.664 1.332	0.6703 1.3465	0.6513 1.2946	0.6241 1.2583
	0.1399	0.0973	0.1141	0.0592	0.0114	1.3387	1.616	0.668	0.6762	0.6433	0.6343
	0.1166	0.0332	0.0374	0.0259	0.0236	0.4966 1.6236	0.401 1.844	0.789 1.28	0.8029 1.2926	0.7589 1.2242	0.7142 1.1888
						1.1269	1.443	0.491	0.4897	0.4654	0.4746
40	1.0603	1.0174	1.0342	0.9794	0.9315	0.8153 1.3054	0.389 1.899	0.664 1.332	0.6703 1.3465	0.6513 1.2946	0.6241 1.2583
	0.1402	0.0973	0.1141	0.0592	0.0114	0.4901	1.51	0.668	0.6762	0.6433	0.6343
	0.0156	0.0332	0.0374	0.0259	0.0236	0.854 1.2666	0.432 1.789	0.789 1.28	0.8029 1.2926	0.7589 1.2242	0.7142 1.1888
						0.4126	1.357	0.491	0.4897	0.4654	0.4746
50	1.0966	1.0174	1.0342	0.9794	0.9315	0.8639 1.3293	0.393 1.951	0.664 1.332	0.6703 1.3465	0.6513 1.2946	0.6241 1.2583
	0.1764	0.0973	0.1141	0.0592	0.0114	0.4654	1.558	0.668	0.6762	0.6433	0.6343
	0.0141	0.0332	0.0374	0.0259	0.0236	0.9007 1.2925	0.433 1.8	0.789 1.28	0.8029 1.2926	0.7589 1.2242	0.7142 1.1888
						0.3918	1.367	0.491	0.4897	0.4654	0.4746
60	1.0966	1.0174	1.0342	0.9794	0.9315	0.7381 1.4552	0.393 1.951	0.664 1.332	0.6703 1.3465	0.6513 1.2946	0.6241 1.2583
	0.1764	0.0973	0.1141	0.0592	0.0114	0.7171	1.558	0.668	0.6762	0.6433	0.6343
	0.0335	0.0332	0.0374	0.0259	0.0236	0.7948 1.3985	0.433 1.8	0.789 1.28	0.8029 1.2926	0.7589 1.2242	0.7142 1.1888
						0.6037	1.367	0.491	0.4897	0.4654	0.4746
70	1.031	1.0174	1.0342	0.9794	0.9315	0.8339 1.228	0.4 1.851	0.664 1.332	0.6703 1.3465	0.6513 1.2946	0.6241 1.2583
	0.1108	0.0973	0.1141	0.0592	0.0114	0.3941	1.451	0.668	0.6762	0.6433	0.6343
	0.0101	0.0332	0.0374	0.0259	0.0236	0.8651 1.1969	0.434 1.747	0.789 1.28	0.8029 1.2926	0.7589 1.2242	0.7142 1.1888
						0.3318	1.313	0.491	0.4897	0.4654	0.4746
80	1.0798	1.0193	1.0242	1.0074	0.9942	0.892 1.2675	0.395 1.87	0.584 1.546	0.585 1.5535	0.579 1.528	0.5677 1.5169
	0.1596	0.0991	0.1041	0.0873	0.0741	0.3755	1.475	0.962	0.9686	0.9489	0.9492
	0.0092	0.0829	0.0848	0.0786	0.0772	0.9217 1.2378	0.442 1.763	0.67 1.522	0.6717 1.5324	0.6654 1.4936	0.6554 1.4919
						0.3161	1.321	0.852	0.8608	0.8282	0.8365
90	1.0516	1.0025	1.0058	0.9948	0.9856	0.8093 1.2939	0.395 1.823	0.597 1.495	0.5993 1.5019	0.5916 1.4786	0.5786 1.4707
	0.1314	0.0824	0.0857	0.0746	0.0655	0.4846	1.428	0.898	0.9026	0.887	0.8921
	0.0153	0.0633	0.0643	0.0612	0.0609	0.8476 1.2556	0.435 1.739	0.669 1.364	0.6716 1.3652	0.6619 1.3609	0.649 1.3591
						0.408	1.304	0.695	0.6935	0.699	0.7101
100	1.0516	0.9737	0.9789	0.9617	0.9475	0.8846 1.2185	0.395 1.823	0.594 1.518	0.5945 1.5236	0.5914 1.5051	0.5859 1.4979
	0.1314	0.0535	0.0587	0.0415	0.0273	0.3338	1.428	0.924	0.9291	0.9136	0.912
	0.0073	0.0774	0.0789	0.0743	0.0741	0.911 1.1921	0.435 1.739	0.621 1.457	0.6223 1.4597	0.6188 1.4514	0.6131 1.4485
						0.281	1.304	0.836	0.8374	0.8326	0.8354

Point Estimation: The first row represents the Estimate, the second row represents the Bias, and the third row represents the ER. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 4. Point and Interval estimation of the δ_1 parameter when $\delta_1 = 1.89$.

n	*Point*					*Intrval*					
	ML	SE	$LE1$	$LE2$	GE	ML	$Bootstrap$	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
25	2.1144	1.8219	1.8586	1.7395	1.7187	1.1013 3.1275	0.876 3.69	1.498 2.105	1.5513 2.1416	1.4204 2.0183	1.3967 2.0081
	0.2245	−0.068	−0.0313	−0.1504	−0.1713	2.0262	2.814	0.607	0.5903	0.5978	0.6115
	0.2672	0.0272	0.0233	0.0453	0.0538	1.2616 2.9673	0.992 3.515	1.591 2.079	1.6242 2.104	1.4997 1.9998	1.4538 1.9863
						1.7057	2.523	0.488	0.4798	0.5001	0.5324
30	2.0915	1.8219	1.8586	1.7395	1.7187	−3.0784 7.2613	0.885 3.675	1.498 2.105	1.5513 2.1416	1.4204 2.0183	1.3967 2.0081
	0.2015	−0.068	−0.0313	−0.1504	−0.1713	10.3397	2.79	0.607	0.5903	0.5978	0.6115
	6.9574	0.0272	0.0233	0.0453	0.0538	−2.2607 6.4436	1.025 3.475	1.591 2.079	1.6242 2.104	1.4997 1.9998	1.4538 1.9863
						8.7043	2.45	0.488	0.4798	0.5001	0.5324
40	2.106	1.8219	1.8586	1.7395	1.7187	1.4116 2.8004	0.968 3.634	1.498 2.105	1.5513 2.1416	1.4204 2.0183	1.3967 2.0081
	0.216	−0.068	−0.0313	−0.1504	−0.1713	1.3888	2.666	0.607	0.5903	0.5978	0.6115
	0.1255	0.0272	0.0233	0.0453	0.0538	1.5214 2.6906	1.09 3.464	1.591 2.079	1.6242 2.104	1.4997 1.9998	1.4538 1.9863
						1.1692	2.374	0.488	0.4798	0.5001	0.5324
50	2.068	1.8219	1.8586	1.7395	1.7187	1.3319 2.8041	0.963 3.624	1.498 2.105	1.5513 2.1416	1.4204 2.0183	1.3967 2.0081
	0.1781	−0.068	−0.0313	−0.1504	−0.1713	1.4722	2.661	0.607	0.5903	0.5978	0.6115
	0.141	0.0272	0.0233	0.0453	0.0538	1.4484 2.6877	1.078 3.385	1.591 2.079	1.6242 2.104	1.4997 1.9998	1.4538 1.9863
						1.2393	2.307	0.488	0.4798	0.5001	0.5324
60	2.068	1.8219	1.8586	1.7395	1.7187	1.3888 2.7473	0.963 3.624	1.498 2.105	1.5513 2.1416	1.4204 2.0183	1.3967 2.0081
	0.1781	−0.068	−0.0313	−0.1504	−0.1713	1.3585	2.661	0.607	0.5903	0.5978	0.6115
	0.1201	0.0272	0.0233	0.0453	0.0538	1.4962 2.6398	1.078 3.385	1.591 2.079	1.6242 2.104	1.4997 1.9998	1.4538 1.9863
						1.1436	2.307	0.488	0.4798	0.5001	0.5324
70	2.0956	1.8219	1.8586	1.7395	1.7187	1.5005 2.6907	1.093 3.545	1.498 2.105	1.5513 2.1416	1.4204 2.0183	1.3967 2.0081
	0.2056	−0.068	−0.0313	−0.1504	−0.1713	1.1902	2.452	0.607	0.5903	0.5978	0.6115
	0.0922	0.0272	0.0233	0.0453	0.0538	1.5946 2.5966	1.166 3.291	1.591 2.079	1.6242 2.104	1.4997 1.9998	1.4538 1.9863
						1.0019	2.125	0.488	0.4798	0.5001	0.5324
80	2.1042	1.9246	1.9476	1.8714	1.8612	1.5096 2.6988	1.114 3.569	1.349 2.377	1.3577 2.389	1.316 2.3457	1.2991 2.3471
	0.2143	0.0346	0.0577	−0.0185	−0.0287	1.1892	2.455	1.028	1.0313	1.0297	1.048
	0.092	0.0797	0.0811	0.0798	0.0838	1.6037 2.6048	1.22 3.364	1.394 2.364	1.4065 2.3747	1.3686 2.3279	1.3563 2.3297
						1.0011	2.144	0.97	0.9683	0.9593	0.9734
90	2.1136	1.9146	1.9369	1.8635	1.853	1.7443 2.4829	1.157 3.548	1.38 2.403	1.3962 2.4239	1.3143 2.3759	1.2948 2.3782
	0.2236	0.0246	0.047	−0.0264	−0.0369	0.7385	2.391	1.023	1.0278	1.0617	1.0834
	0.0355	0.0781	0.0784	0.0807	0.0856	1.8027 2.4245	1.248 3.324	1.54 2.386	1.5627 2.4048	1.4904 2.3383	1.4683 2.3402
						0.6217	2.076	0.846	0.8421	0.8479	0.8719
100	2.1136	1.9762	1.9915	1.9416	1.9364	1.5508 2.6764	1.157 3.548	1.259 2.537	1.2657 2.5425	1.2445 2.5238	1.2361 2.5257
	0.2236	0.0863	0.1015	0.0516	0.0465	1.1256	2.391	1.278	1.2767	1.2793	1.2896
	0.0825	0.1188	0.1209	0.116	0.1191	1.6398 2.5874	1.248 3.324	1.406 2.491	1.4252 2.4986	1.3644 2.4742	1.3449 2.4765
						0.9476	2.076	1.085	1.0734	1.1098	1.1316

Point Estimation: The first row represents the Estimate, the second row represents the Bias, and the third row represents the ER. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 5. Point and Interval estimation of the δ_2 parameter when $\delta_2 = 1.1$.

n	*Point*					*Intrval*					
	ML	SE	$LE1$	$LE2$	GE	ML	*Bootstrap*	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
25	1.3447	1.0961	1.1214	1.0393	0.9731	0.8466 1.8428	0.431 2.904	0.718 1.415	0.7271 1.452	0.7003 1.3246	0.6627 1.2649
	0.2108	−0.0378	−0.0125	−0.0946	−0.1608	0.9962	2.473	0.697	0.725	0.6243	0.6022
	0.0646	0.0301	0.0302	0.0341	0.0507	0.9254 1.764	0.474 2.727	0.831 1.38	0.8531 1.4163	0.7842 1.3041	0.7133 1.2273
						0.8387	2.253	0.549	0.5632	0.5198	0.514
30	1.3636	1.0961	1.1214	1.0393	0.9731	−0.2178 2.945	0.444 2.91	0.718 1.415	0.7271 1.452	0.7003 1.3246	0.6627 1.2649
	0.2297	−0.0378	−0.0125	−0.0946	−0.1608	3.1627	2.466	0.697	0.725	0.6243	0.6022
	0.6509	0.0301	0.0302	0.0341	0.0507	0.0324 2.6948	0.486 2.771	0.831 1.38	0.8531 1.4163	0.7842 1.3041	0.7133 1.2273
						2.6625	2.285	0.549	0.5632	0.5198	0.514
40	1.3171	1.0961	1.1214	1.0393	0.9731	0.852 1.7822	0.46 2.85	0.718 1.415	0.7271 1.452	0.7003 1.3246	0.6627 1.2649
	0.1832	−0.0378	−0.0125	−0.0946	−0.1608	0.9302	2.39	0.697	0.725	0.6243	0.6022
	0.0563	0.0301	0.0302	0.0341	0.0507	0.9256 1.7086	0.512 2.713	0.831 1.38	0.8531 1.4163	0.7842 1.3041	0.7133 1.2273
						0.7831	2.201	0.549	0.5632	0.5198	0.514
50	1.2688	1.0961	1.1214	1.0393	0.9731	0.8648 1.6727	0.488 2.818	0.718 1.415	0.7271 1.452	0.7003 1.3246	0.6627 1.2649
	0.1348	−0.0378	−0.0125	−0.0946	−0.1608	0.8079	2.33	0.697	0.725	0.6243	0.6022
	0.0425	0.0301	0.0302	0.0341	0.0507	0.9287 1.6088	0.532 2.616	0.831 1.38	0.8531 1.4163	0.7842 1.3041	0.7133 1.2273
						0.6801	2.084	0.549	0.5632	0.5198	0.514
60	1.2688	1.0961	1.1214	1.0393	0.9731	0.6134 1.9241	0.488 2.818	0.718 1.415	0.7271 1.452	0.7003 1.3246	0.6627 1.2649
	0.1348	−0.0378	−0.0125	−0.0946	−0.1608	1.3107	2.33	0.697	0.725	0.6243	0.6022
	0.1118	0.0301	0.0302	0.0341	0.0507	0.7171 1.8205	0.532 2.616	0.831 1.38	0.8531 1.4163	0.7842 1.3041	0.7133 1.2273
						1.1034	2.084	0.549	0.5632	0.5198	0.514
70	1.314	1.0961	1.1214	1.0393	0.9731	0.9296 1.6985	0.499 2.682	0.718 1.415	0.7271 1.452	0.7003 1.3246	0.6627 1.2649
	0.1801	−0.0378	−0.0125	−0.0946	−0.1608	0.7689	2.183	0.697	0.725	0.6243	0.6022
	0.0385	0.0301	0.0302	0.0341	0.0507	0.9904 1.6377	0.541 2.55	0.831 1.38	0.8531 1.4163	0.7842 1.3041	0.7133 1.2273
						0.6473	2.009	0.549	0.5632	0.5198	0.514
80	1.2516	1.1065	1.1157	1.0853	1.0651	0.9197 1.5835	0.494 2.816	0.654 1.654	0.6615 1.6647	0.6362 1.6285	0.5971 1.6193
	0.1177	−0.0274	−0.0182	−0.0486	−0.0688	0.6638	2.322	1	1.0032	0.9923	1.0222
	0.0287	0.0823	0.0833	0.0805	0.0846	0.9722 1.531	0.522 2.527	0.705 1.616	0.7087 1.6247	0.694 1.5942	0.6705 1.5839
						0.5588	2.005	0.911	0.916	0.9002	0.9134
90	1.2983	1.1069	1.114	1.0905	1.0756	0.8266 1.77	0.503 2.799	0.651 1.62	0.6527 1.6371	0.6428 1.5905	0.6164 1.5803
	0.1644	−0.027	−0.0199	−0.0434	−0.0583	0.9434	2.296	0.969	0.9844	0.9477	0.9639
	0.0579	0.0771	0.0784	0.0742	0.076	0.9012 1.6954	0.546 2.59	0.713 1.575	0.7136 1.6028	0.7103 1.555	0.7055 1.5479
						0.7942	2.044	0.862	0.8892	0.8446	0.8425
100	1.2983	1.1627	1.1731	1.1388	1.1172	0.975 1.6216	0.503 2.799	0.647 1.667	0.653 1.6768	0.6344 1.6467	0.6087 1.6402
	0.1644	0.0288	0.0392	0.0049	−0.0168	0.6465	2.296	1.02	1.0238	1.0123	1.0315
	0.0272	0.0933	0.0955	0.0889	0.0917	1.0262 1.5704	0.546 2.59	0.687 1.662	0.6942 1.6747	0.6751 1.6249	0.6519 1.6131
						0.5443	2.044	0.975	0.9804	0.9497	0.9612

Point Estimation: The first row represents the Estimate, the second row represents the Bias, and the third row represents the ER. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 6. Point and Interval estimation of industry data.

Parameter	Point					Intrval				
	ML	SE	$LE1$	$LE2$	GE	ML	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
κ	0.6481	0.6112	0.6267	0.5775	0.4828	0.001 2.6472 2.6462 0.001 2.331 2.33	0.231 0.921 0.69 0.335 0.904 0.569	0.2375 0.9363 0.6988 0.3459 0.9261 0.5802	0.2184 0.8913 0.6729 0.3141 0.8488 0.5346	0.1926 0.8158 0.6232 0.2159 0.7785 0.5627
α	0.1389	0.137	0.1371	0.1368	0.1335	0.001 0.3055 0.3045 0.001 0.2791 0.2781	0.115 0.156 0.041 0.118 0.155 0.037	0.1147 0.1559 0.0413 0.1178 0.1553 0.0375	0.1144 0.1555 0.0411 0.1175 0.1549 0.0375	0.111 0.1519 0.0409 0.1146 0.1516 0.037
δ_1	3.06	2.8373	2.8674	2.7709	2.7863	0.2895 5.8305 5.5409 0.7277 5.3923 4.6645	2.213 3.169 0.956 2.379 3.159 0.78	2.2227 3.2024 0.9797 2.3859 3.1923 0.8064	2.1959 3.1157 0.9198 2.3496 3.0888 0.7392	2.1974 3.1235 0.926 2.3539 3.1063 0.7524
δ_2	27.2078	27.0577	27.098	26.969	27.0505	0.001 86.4802 86.4792 0.001 77.1055 77.1045	26.476 27.472 0.996 26.579 27.434 0.855	26.4907 27.5257 1.035 26.6082 27.4789 0.8707	26.4499 27.3939 0.9441 26.5259 27.3379 0.812	26.4738 27.4622 0.9884 26.5742 27.4268 0.8526

Point Estimation: The row represents the Estimate point. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 7. Point and Interval estimation of business data.

Parameter	Point					Intrval				
	ML	SE	$LE1$	$LE2$	GE	ML	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
κ	0.1627	0.1627	0.1627	0.1626	0.1619	0.0001 0.5561 0.556 0.0001 0.4939 0.4938	0.159 0.167 0.008 0.159 0.167 0.008	0.1589 0.1671 0.0082 0.1594 0.167 0.0076	0.1588 0.167 0.0081 0.1593 0.1669 0.0076	0.1583 0.1663 0.008 0.1587 0.1661 0.0074
α	11.0589	11.022	11.0591	10.9364	11.0051	9.0918 13.026 3.9341 9.403 12.7148 3.3119	10.796 11.156 0.36 10.824 11.153 0.329	10.8501 11.1942 0.3441 10.8665 11.1837 0.3172	10.7056 11.0895 0.3839 10.7253 11.0872 0.3619	10.7738 11.1408 0.367 10.8027 11.139 0.3363
δ_1	25.7072	25.613	25.6538	25.5249	25.6053	22.9367 28.4777 5.5409 23.3749 28.0395 4.6645	25.357 25.788 0.431 25.384 25.777 0.393	25.4052 25.8313 0.4261 25.4354 25.8177 0.3823	25.2631 25.6965 0.4334 25.2997 25.6864 0.3867	25.3485 25.7804 0.4319 25.3742 25.7704 0.3962
δ_2	36.4867	36.489	36.5309	36.3921	36.4832	30.4393 42.5341 12.0948 31.3958 41.5776 10.1818	36.196 36.67 0.474 36.23 36.662 0.432	36.2273 36.7122 0.4848 36.255 36.6998 0.4448	36.1252 36.5865 0.4612 36.1715 36.5771 0.4056	36.1917 36.6652 0.4735 36.2266 36.6573 0.4307

Point Estimation: The row represents the Estimate point. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 8. Point and Interval estimation of carbon dioxide emissions data.

Parameter	Point					Intrval				
	ML	SE	$LE1$	$LE2$	GE	ML	HPD_S	HPD_{LE1}	HPD_{LE2}	HPD_{GE}
κ	10.5	10.5257	10.5286	10.5189	10.5243	8.7048 12.2952	9.732 11.157	9.734 11.1576	9.7262 11.1571	9.7305 11.1574
						3.5905	1.425	1.4236	1.4309	1.4269
						8.9887 12.0113	9.914 11.093	9.9162 11.1155	9.908 11.0866	9.9127 11.0866
						3.0226	1.179	1.1993	1.1786	1.1739
α	2.9	2.2877	2.2938	2.2739	2.275	2.7406 3.0594	1.972 2.678	1.9726 2.6778	1.9717 2.6777	1.9717 2.6777
						0.3188	0.706	0.7052	0.706	0.7061
						2.7658 3.0342	2.016 2.637	2.0163 2.6591	2.0141 2.6272	2.014 2.629
						0.2684	0.621	0.6428	0.6131	0.615
δ_1	1.88	2.3156	2.3305	2.2785	2.2759	1.2522 2.5078	1.533 2.803	1.5448 2.8039	1.4496 2.801	1.4051 2.8014
						1.2556	1.27	1.2591	1.3514	1.3964
						1.3515 2.4085	1.702 2.778	1.7191 2.7777	1.6523 2.7773	1.6337 2.7773
						1.057	1.076	1.0586	1.125	1.1436
δ_2	1.13	0.2813	0.2818	0.2805	0.2749	1.06 1.2	0.192 0.373	0.1919 0.3727	0.1919 0.3726	0.1914 0.3722
						0.1399	0.181	0.1808	0.1808	0.1808
						1.0711 1.1889	0.216 0.358	0.2158 0.3585	0.2151 0.3568	0.2118 0.3515
						0.1178	0.142	0.1427	0.1417	0.1397

Point Estimation: The row represents the Estimate point. Interval Estimation: 95% and 90%, respectively. The first and second rows show the HPD credible interval and the corresponding width of the parameter, respectively.

Table 9. Relative quality of ASKUG-Lomax on the industry lifetime data vs. competing models.

Model	AIC	BIC	HQC	CAIC
ASKUG-Lomax	254.692	256.358	260.161	256.404
ASEXG-Lomax	272.857	271.897	268.755	271.573
AS-Lomax	266.395	265.934	263.661	265.539
EX-Weibull	338.517	339.477	342.619	339.802
Weibull	370.397	370.858	373.131	371.253

Table 10. Relative quality of ASKUG-Lomax on the business data vs. competing models.

Model	AIC	BIC	HQC	CAIC
ASKUG-Lomax	19.683	27.683	20.893	18.355
ASEXG-Lomax	31.750	35.750	32.658	30.754
AS-Lomax	38.695	40.410	39.301	38.032
EX-Weibull	82.178	86.178	83.085	81.182
Weibull	63.219	64.933	63.824	62.555

Table 11. Relative quality of ASKUG-Lomax on the carbon dioxide emissions data vs. competing models.

Model	AIC	BIC	HQC	CAIC
ASKUG-Lomax	244.007	244.748	252.317	247.251
ASEXG-Lomax	297.611	298.047	303.843	300.044
AS-Lomax	315.456	315.671	319.611	317.078
EX-Weibull	254.981	255.417	261.214	257.414
Weibull	303.874	304.088	308.029	305.496

6. Concluding Remarks

The use of the trigonometric arcsine function introduces a new family of heavy-tailed models, the Arcsine Kumaraswamy family of generalised X-distributions. The Arcsine Kumaraswamy-generalised X-distribution is very interesting, and provides a better fit for data with strong tails. A special submodel called the Arcsine Kumaraswamy-Lomax model is defined is this paper. We calculate the parameters of the Arcsine Kumaraswamy-generalised Lomax model with maximum likelihood, bootstrap, and Bayesian estimators. A simulation study and analysis of real industry data are provided to verify the performance of the Arcsine Kumaraswamy-Lomax model. The performance of the Bayesian estimators is better than that of the corresponding ML estimators. Based on our modeling of three real datasets, the results show that the Arcsine Kumaraswamy-generalised Lomax model provides a better fit than other competing models.

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Appendix A

$$\begin{aligned}
\frac{\partial L}{\partial \delta_1} &= \frac{n}{\delta_1} + \sum_{i=1}^{n} \mathrm{Log}[G(x_i;\varphi)] \\
&+ \sum_{i=1}^{n} \frac{(\delta_2-1)\mathrm{Log}[G(x_i;\varphi)]G(x_i;\varphi)^{\delta_1}\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2-2}}{1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2-1}} \\
&- \sum_{i=1}^{n} \frac{\delta_2\mathrm{Log}[G(x_i;\varphi)]G(x_i;\varphi)^{\delta_1}\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2-1}\left(\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2}-1\right)}{1-\left(1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2}\right)^2},
\end{aligned} \tag{A1}$$

$$\begin{aligned}
\frac{\partial L}{\partial \delta_2} &= \frac{n}{\delta_2} + \sum_{i=1}^{n} -\frac{\mathrm{Log}\left[1-G(x_i;\varphi)^{\delta_1}\right]\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2-1}}{1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{-1+\delta_2}} \\
&- \sum_{i=1}^{n} \frac{\mathrm{Log}\left[1-G(x_i;\varphi)^{\delta_1}\right]\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2}\left(1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2}\right)}{1-\left(1-\left(1-G(x_i;\varphi)^{\delta_1}\right)^{\delta_2}\right)^2}.
\end{aligned} \tag{A2}$$

The ASKUG-Lomax log-likelihood function is

$$\begin{aligned}
L &= n\mathrm{Log}\left[\frac{2}{\pi}\kappa\alpha\delta_1\delta_2\right] - (1+\alpha)\sum_{i=1}^{n}\mathrm{Log}[1+\kappa x_i] \\
&+ (\delta_1-1)\sum_{i=1}^{n}\mathrm{Log}\left[1-(1+\kappa x_i)^{-\alpha}\right] + (\delta_2-1)\sum_{i=1}^{n}\mathrm{Log}\left[1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right] \\
&- \frac{1}{2}\sum_{i=1}^{n}\mathrm{Log}\left[1-\left(1-\left(1-\left(1-(1+x_i\alpha)^{-\beta}\right)^{\delta_1}\right)^{\delta_2}\right)^2\right].
\end{aligned} \tag{A3}$$

Maximize Equation (15) to obtain the MLE estimates, as follows:

$$\begin{aligned}
\frac{\partial L}{\partial \kappa} &= \frac{n}{\kappa} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\kappa x_i} + \alpha(\delta_1-1)\sum_{i=1}^{n}\frac{x_i(1+\kappa x_i)^{-1-\alpha}}{1-(1+\kappa x_i)^{-\alpha}} \\
&- \alpha\delta_1(\delta_2-1)\sum_{i=1}^{n}\frac{x_i(1+\kappa x_i)^{-\alpha-1}\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1-1}}{1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}} \\
&+ \sum_{i=1}^{n}\frac{\alpha\delta_1\delta_2 x_i\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{-1+\delta_2}\left(1+2\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right)}{(1+\kappa x_i)^{\alpha+1}\left(1-(1+\kappa x_i)^{-\alpha}\right)^{1-\delta_1}\left(1-\left(1-\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right)^2\right)},
\end{aligned} \tag{A4}$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n} \mathrm{Log}[1+\kappa x_i] + (\delta_1 - 1)\sum_{i=1}^{n} \frac{(1+\kappa x_i)^{-\alpha}\mathrm{Log}[1+\kappa x_i]}{1-(1+\kappa x_i)^{-\alpha}} \\ &- \delta_1(\delta_2 - 1)\sum_{i=1}^{n} \frac{\mathrm{Log}[1+\kappa x_i](1+\kappa x_i)^{-\alpha}\left(1-(1+\kappa x_i)^{-\alpha}\right)^{-1+\delta_1}}{1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}} \\ &+ \sum_{i=1}^{n} \frac{\delta_1\delta_2\mathrm{Log}[1+\kappa x_i]\left(1-(1+\kappa x_i)^{-\alpha}\right)^{-1+\delta_1}\left(1+2\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right)}{(1+\kappa x_i)^{\alpha}\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{-\delta_2+1}\left(1-\left(1-\left(1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right)^{\delta_2}\right)^2\right)}, \end{aligned} \quad \text{(A5)}$$

$$\frac{\partial L}{\partial \delta_1} = \frac{n}{\delta_1} + \sum_{i=1}^{n} \mathrm{Log}\left[1-(1+\kappa x_i)^{-\alpha}\right] - (\delta_2 - 1)\sum_{i=1}^{n} \frac{\mathrm{Log}\left[1-(1+\kappa x_i)^{-\alpha}\right]\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}}{1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}}, \quad \text{(A6)}$$

and

$$\frac{\partial L}{\partial \delta_2} = \frac{n}{\delta_2} + \sum_{i=1}^{n} \mathrm{Log}\left[1-\left(1-(1+\kappa x_i)^{-\alpha}\right)^{\delta_1}\right]. \quad \text{(A7)}$$

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