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Positive Solutions for a High-Order Riemann-Liouville Type Fractional Integral Boundary Value Problem Involving Fractional Derivatives

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Abstract: In this paper, under some super- and sub-linear growth conditions, we study the existence of positive solutions for a high-order Riemann–Liouville type fractional integral boundary value problem involving fractional derivatives. Our analysis methods are based on the fixed point index and nonsymmetric property of the Green function. Additionally, we provide some valid examples to illustrate our main results.

Keywords: Riemann–Liouville fractional differential equations; integral boundary value problems; positive solutions; fixed point index

MSC: 34B18; 34B10; 34B15

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1. Introduction

In this paper, we investigate the existence of positive solutions for the following high-order Riemann–Liouville type fractional integral boundary value problem involving fractional derivatives:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f\Big(t,u(t),D_{0+}^{\beta_1}u(t),\dots,D_{0+}^{\beta_{n-1}}u(t)\Big)=0,\ 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ D_{0+}^{\beta}u(1)=\displaystyle\int_0^1 D_{0+}^{\beta_{n-1}}u(t)dA(t), \end{cases} \tag{1}$$

where $n-1<\alpha\le n, i-1<\beta_i\le i(i=1,2,\dots,n-1), \alpha-\beta_{n-1}>\alpha-\beta>1$ and f, A satisfy the conditions:

Hypothesis 1 (H1). $f:[0,1]\times\mathbb{R}_+^n\to\mathbb{R}$ *is continuous, and there is a* $M>0$*, such that*

$$f(t,x_n,x_{n-1},...,x_1)\ge -M, \forall t\in[0,1], x_i\in\mathbb{R}_+, i=1,2,...,n, \mathbb{R}_+:=[0,+\infty),$$

Hypothesis 2 (H2). $A:[0,1]\to\mathbb{R}_+$ *is a function of bounded variation and* $\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)\in\left[0,\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}\right)$.

Recently, useful properties of fractional calculus were discovered in many scientific engineering phenomena which has motivated researchers to use this theory to analyze and apply them in various fields. We refer the reader to system modeling, controller design, and biomedical and signal processing fields. We also note that fractional differential equations have received much attention and there are many papers studying various kinds of fractional boundary value problems using methods in non-linear analysis, see, for example, Refs. [1–25] and the references cited therein. In [1], the authors used the method of mixed monotone operators to study unique positive solutions for the fractional differential system

$$\begin{cases} -D_{0+}^{\alpha}\chi_1(t) = f\Big(t,\chi_1(t),D_{0+}^{\beta}\chi_1(t),\chi_2(t)\Big),\ t\in(0,1), \\ -D_{0+}^{\gamma}\chi_2(t) = g(t,\chi_1(t)),\ t\in(0,1), \\ D_{0+}^{\beta}\chi_1(0)=0,\ D_{0+}^{\mu}\chi_1(1)=\sum_{j=1}^{p-2}a_jD_{0+}^{\mu}\chi_1(\xi_j), \\ \chi_2(0)=0,\ D_{0+}^{v}\chi_2(1)=\sum_{j=1}^{p-2}b_jD_{0+}^{v}\chi_2(\xi_j), \end{cases}$$

where $D_{0+}^{\sigma}(\sigma=\alpha,\beta,\gamma,\mu,v)$ is the Riemann–Liouville derivative and in [2], the authors studied the solvability for the fractional differential system

$$\begin{cases} D_{0+}^{\alpha}\chi_1(t)+\lambda f\Big(t,\chi_1(t),\ D_{0+}^{\beta}\chi_1(t),\chi_2(t)\Big)=0, \\ D_{0+}^{\gamma}\chi_2(t)+\lambda g(t,\chi_1(t))=0,\ 0<t<1, \\ D_{0+}^{\beta}\chi_1(0)=D_{0+}^{\beta+1}\chi_1(0)=0,\ D_{0+}^{\beta}\chi_1(1)=\int_0^1 D_{0+}^{\beta}\chi_1(s)dA(s), \\ \chi_2(0)=\chi_2'(0)=0,\ \chi_2(1)=\int_0^1\chi_2(s)dB(s), \end{cases}$$

where $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ are the Riemann-Liouville derivatives, $f:(0,1)\times\mathbb{R}_+^3\to\mathbb{R}$, $g:(0,1)\times\mathbb{R}_+\to\mathbb{R}$ are two semi-positone functions, and satisfy the following super-linear or sub-linear conditions:

(HZ)$_1$ There exists $M>0$ such that $\limsup_{\omega\to0^+}\max_{t\in[0,1]}\frac{g(t,\omega)}{\omega}<M$ (sub-linear growth condition);

(HZ)$_2$ There exists $[a,b]\subset(0,1)$, such that $\lim_{\omega\to+\infty}\min_{t\in[a,b]}\frac{g(t,\omega)}{\omega}=+\infty$, $\lim_{\omega_3\to+\infty}\min_{\substack{t\in[a,b]\\ \omega_1,\omega_2\geqslant0}}\frac{f(t,\omega_1,\omega_2,\omega_3)}{\omega_3}=+\infty$ (superlinear growth condition).

In [3], the authors investigated the system of Riemann–Liouville fractional differential equations

$$\begin{cases} D_{0+}^{\alpha_1}\Big(D_{0+}^{\beta_1}\chi_1(t)\Big)+\lambda f(t,\chi_1(t),\chi_2(t))=0, t\in(0,1), \\ D_{0+}^{\alpha_2}\Big(D_{0+}^{\beta_2}\chi_2(t)\Big)+\mu g(t,\chi_1(t),\chi_2(t))=0, t\in(0,1), \\ \chi_1^{(j)}(0)=0, j=0,\dots,n-2; D_{0+}^{\beta_1}\chi_1(0)=0, D_{0+}^{\gamma_0}\chi_1(1)=\sum_{i=1}^{p}\int_0^1 D_{0+}^{\gamma_i}\chi_1(\tau)d\mathcal{H}_i(\tau), \\ \chi_2^{(j)}(0)=0, j=0,\dots,m-2; D_{0+}^{\beta_2}\chi_2(0)=0, D_{0+}^{\delta_0}\chi_2(1)=\sum_{i=1}^{q}\int_0^1 D_{0+}^{\delta_i}\chi_2(\tau)d\mathcal{K}_i(\tau), \end{cases}$$

where f,g are sign-changing singular non-linearities and satisfy the following growth condition:

(HZ)$_3$ There exist $0<\sigma_1<\sigma_2<1$, such that

$$\lim_{\omega_1+\omega_2\to+\infty}\min_{t\in[\sigma_1,\sigma_2]}\frac{f(t,\omega_1,\omega_2)}{\omega_1+\omega_2}=\infty \text{ or } \lim_{\omega_1+\omega_2\to+\infty}\min_{t\in[\sigma_1,\sigma_2]}\frac{g(t,\omega_1,\omega_2)}{\omega_1+\omega_2}=\infty.$$

It is widely known that certain conditions involving the eigenvalues of relevant linear operators play an important role in the study of fractional boundary value problems, see, for example, [4–9]. In [4], the authors used fixed point index theory to study positive solutions for the fractional integral boundary value problem

$$\begin{cases} D_{0+}^{\alpha}\chi(t)+h(t)f(t,\chi(t))=0,\ 0<t<1, \\ \chi(0)=\chi'(0)=\chi''(0)=0, \\ \chi(1)=\lambda\int_0^{\eta}\chi(s)ds, \end{cases} \quad (2)$$

where $f\in C([0,1]\times\mathbb{R}_+,\mathbb{R}_+)$ satisfies the following growth conditions:

(HZ)$_4$ $\liminf_{\zeta\to0+}\frac{f(t,\zeta)}{\zeta}>\lambda_1$ and $\limsup_{\zeta\to+\infty}\frac{f(t,\zeta)}{\zeta}<\lambda_1$, uniformly on $t\in[0,1]$,

(HZ)$_5$ $\limsup_{\zeta\to0+}\frac{f(t,\zeta)}{\zeta}<\lambda_1$ and $\liminf_{\zeta\to+\infty}\frac{f(t,\zeta)}{\zeta}>\lambda_1$, uniformly on $t\in[0,1]$, where $\lambda_1 > 0$ is the first eigenvalue of the operator L denoted by $(L\zeta)(t) = \int_0^1 G(t,s)h(s)\zeta(s)ds$, and G is the Green's function associated with (2). In [5], the authors generalized the methods in [4], and studied the following higher order fractional differential equation with integral boundary conditions

$$\begin{cases} -D_{0+}^{\alpha}\chi(t) = f\Big(t,\chi(t),\chi'(t),\ldots,\chi^{(n-2)}(t)\Big), \quad 0 < t < 1, \\ \chi(0) = \chi'(0) = \cdots = \chi^{(n-2)}(0) = 0, \quad \chi^{(n-2)}(1) = \lambda\Big[\chi^{(n-2)}\Big], \end{cases} \tag{3}$$

where $\lambda[\chi] = \int_0^1 \chi(t)dA(t)$, and $f \in C([0,1]\times(\mathbb{R}_+)^{n-1},\mathbb{R}_+)$ satisfies the following growth conditions:

(HZ)$_6$ $\liminf\limits_{\substack{\zeta_1\to 0\\ \cdots \\ \zeta_{n-1}\to 0}} \dfrac{f(t,\zeta_1,\ldots,\zeta_{n-1})}{\zeta_1+\cdots+\zeta_{n-1}} > \lambda_1, \quad \limsup\limits_{\substack{\zeta_1+\cdots+\zeta_{n-1}\to+\infty\\ \zeta_{n-1}\to+\infty}} \dfrac{f(t,\zeta_1,\ldots,\zeta_{n-1})}{\zeta_{n-1}} < \lambda_1$,
uniformly on $t \in [0,1]$,

(HZ)$_7$ $\liminf\limits_{\substack{\zeta_1\to 0\\ \cdots \\ \zeta_{n-1}\to 0}} \frac{f(t,\zeta_1,\ldots,\zeta_{n-1})}{\zeta_{n-1}} < \lambda_1, \quad \limsup\limits_{\zeta_1+\cdots+\zeta_{n-1}\to+\infty} \frac{f(t,\zeta_1,\ldots,\zeta_{n-1})}{\zeta_1+\cdots+\zeta_{n-1}} > \lambda_1$, uniformly on $t \in [0,1]$,

where $\lambda_1 > 0$ is the first eigenvalue of the operator L denoted by $(L\zeta)(t) = \int_0^1 H(t,s)\zeta(s)ds$, and H is the Green's function associated with (3).

Comparing these results there seems to be no real improvement in the stategy for these problems. However, in this paper we consider a different linear operator (see the operator $B_{\mu_1,\mu_2,\ldots,\mu_n}$ in Section 2), and discuss the effects of integral boundary conditions on its eigenvalues involving non-symmetric Green function. Moreover, the problem considered here involves fractional derivatives and a semi-positone non-linearity. As a result, our methods and results are more general than those in the aforementioned works.

2. Preliminaries

In this section, we first provide some basic material for Riemann–Liouville fractional calculus, for details see [21,22].

Definition 1. *The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $\varphi : (0,+\infty) \to \mathbb{R}$ is given by*

$$D_{0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}\varphi(s)ds,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2. *The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $\varphi : (0,+\infty) \to \mathbb{R}$ is given by*

$$I_{0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 1. *Assume that $\varphi \in C(0,1)\cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1)\cap L^1(0,1)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}\varphi(t) = \varphi(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N},$$

for some $C_i \in \mathbb{R}(i = 1,2,\ldots,N)$, where $N = [\alpha]+1$.

Let $v(t) = D_{0^+}^{\beta_{n-1}}u(t)$ in (1). Then, we can obtain the following lemma.

Lemma 2 (see [15,16]). *Problem (1) can be transformed into the following boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}}v(t) + f\Big(t, I_{0^+}^{\beta_{n-1}}v(t), I_{0^+}^{\beta_{n-1}-\beta_1}v(t), \dots, v(t)\Big) = 0,\ t \in (0,1), \\ I_{0^+}^{\beta_{n-1}-n+2}v(0) = 0, \quad D_{0^+}^{\beta-\beta_{n-1}}v(1) = \int_0^1 v(t)dA(t). \end{cases} \tag{4}$$

Furthermore, (4) is equivalent to the following integral equation

$$v(t) = \int_0^1 K(t,s) f\Big(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \dots, v(s)\Big)ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t),$$

where

$$K(t,s) = \frac{1}{\Gamma(\alpha-\beta_{n-1})}\begin{cases} t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta_{n-1}-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1. \end{cases} \tag{5}$$

Proof. Let $v(t) = D_{0^+}^{\beta_{n-1}}u(t)$. Then, from Lemma 4 of [15], Lemma 2.3 of [16], we have

$$u(t) = I_{0^+}^{\beta_{n-1}}v(t),\ u^{(n-2)}(t) = I_{0^+}^{\beta_{n-1}-n+2}v(t),\ D_{0^+}^{\alpha}u(t) = D_{0^+}^{\alpha-\beta_{n-1}}v(t),\ D_{0^+}^{\beta}u(t) = D_{0^+}^{\beta-\beta_{n-1}}v(t),$$

and

$$D_{0^+}^{\beta_i}u(t) = I_{0^+}^{\beta_{n-1}-\beta_i}v(t), i = 1, 2, \cdots, n-2.$$

Therefore, we easily obtain (4).

Let $y(s) = f\Big(s, I_{0^+}^{\beta_{n-1}}v(s), I_{0^+}^{\beta_{n-1}-\beta_1}v(s), \dots, v(s)\Big)$, and we have

$$v(t) = -I_{0^+}^{\alpha-\beta_{n-1}}y(t) + C_1 t^{\alpha-\beta_{n-1}-1}, \text{ and } D_{0^+}^{\beta-\beta_{n-1}}v(t) = -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}y(s)ds + C_1\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1},$$

where $C_1 \in \mathbb{R}$. Hence, we obtain

$$D_{0^+}^{\beta-\beta_{n-1}}v(1) = -\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}y(s)ds + C_1\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} = \int_0^1 v(t)dA(t).$$

Solving this equation, we have

$$C_1 = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 v(t)dA(t) + \frac{1}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 (1-s)^{\alpha-\beta-1}y(s)ds$$

and then

$$\begin{aligned} v(t) &= \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t) + \frac{1}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}y(s)ds \\ &\quad - \frac{1}{\Gamma(\alpha-\beta_{n-1})}\int_0^t (t-s)^{\alpha-\beta_{n-1}-1}y(s)ds \\ &= \int_0^1 K(t,s)y(s)ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t). \end{aligned} \tag{6}$$

This completes the proof. □

Remark 1. *Integrate (6) over $[0,1]$ and use (H2) to obtain*

$$\int_0^1 v(t)dA(t) = \int_0^1\int_0^1 K(t,s)y(s)dsdA(t) + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)\int_0^1 v(t)dA(t),$$

and

$$\int_0^1 v(t)dA(t) = \frac{1}{1 - \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)}\int_0^1\int_0^1 K(t,s)y(s)dsdA(t).$$

Therefore, we have

$$v(t) = \int_0^1 K(t,s)y(s)ds + \frac{t^{\alpha-\beta_{n-1}-1}\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1}) - \Gamma(\alpha-\beta)\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)} \int_0^1 \int_0^1 K(t,s)y(s)dsdA(t)$$
$$= \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = K(t,s) + \frac{t^{\alpha-\beta_{n-1}-1}\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1}) - \Gamma(\alpha-\beta)\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)} \int_0^1 K(t,s)dA(t), t,s \in [0,1].$$

Note that the function G appears in Lemma 2.3 of [16] ($l(t) \equiv 1$).

Lemma 3 (see [15,16])**.** *The function K has the properties:*

(i) $K \in C([0,1]\times[0,1],\mathbb{R}_+)$ *and* $K(t,s) > 0$ *for* $t,s \in (0,1)$*,*

(ii) $t^{\alpha-\beta_{n-1}-1}K(1,s) \le K(t,s) \le K(1,s),\ t,s \in [0,1]$.

Let $E = C[0,1]$ with the norm $\|\varphi\| = \max_{0\leqslant t\leqslant 1}|\varphi(t)|$. Define a cone P by $P = \{\varphi \in E : \varphi(t) \ge 0, t \in [0,1]\}$. Then, E is a Banach space, and P a closed cone on E. From ([26], p. 188), we know that the conjugate space of E, denoted by E^*, is $V = V[0,1]$, i.e., $E^* = V$, where $V := \{z : z$ has bounded variation on $[0,1]\}$. Moreover, the bounded linear functional on E can be given by the Riemann–Stieltjes integral

$$z(\varphi) := \int_0^1 \varphi(t)dz(t),\ \varphi \in E, z \in E^*. \tag{7}$$

By ([27], p. 125), we have

$$P^* := \{z \in E^* : z(\varphi) \ge 0, \varphi \in P\}$$

is the dual cone of P. From (7), we have

$$z(\varphi) = \int_0^1 \varphi(t)dz(t) = \lim_{\lambda\to 0}\sum_{i=1}^{n}\varphi(\xi_i)[z(t_i) - z(t_{i-1})] \ge 0, \tag{8}$$

where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, $\lambda = \max_{1\le i\le n}(t_i - t_{i-1})$, $\forall \xi_i \in [t_{i-1}, t_i], i = 1,2,\cdots,n$. From $\varphi \in P(\varphi(\xi_i) \ge 0)$, for all division t_i (8) holds, we only need $z(t_i) - z(t_{i-1}) \ge 0$ for $i = 1,2,\cdots,n$. Therefore, the dual cone of P can also be expressed by

$$P^* := \{z \in E^* : z \text{ is non-decreasing on } [0,1]\}.$$

Let $\mu_i \ge 0 (i = 1,2,...,n)$ with $\sum_{i=1}^n \mu_i^2 \ne 0$, and

$$(L_{\mu_1,\mu_2,\ldots,\mu_n}v)(t) = \int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(t,\tau)v(\tau)d\tau,\ v \in E,$$

where $K_{\mu_1,\mu_2,\ldots,\mu_n}(t,\tau) = \mu_n K_n(t,\tau) + \mu_{n-1}K_{n-1}(t,\tau) + \cdots + \mu_2 K_2(t,\tau) + \mu_1 K(t,\tau), (t,\tau) \in [0,1]\times[0,1]$, and $K_i (i = 2,\ldots,n)$ are

$$\begin{bmatrix} K_2(t,\tau) \\ K_3(t,\tau) \\ \vdots \\ K_{n-1}(t,\tau) \\ K_n(t,\tau) \end{bmatrix} = \begin{bmatrix} \frac{1}{\Gamma(\beta_{n-1}-\beta_{n-2})}\int_\tau^1 K(t,s)(s-\tau)^{\beta_{n-1}-\beta_{n-2}-1}ds \\ \frac{1}{\Gamma(\beta_{n-1}-\beta_{n-3})}\int_\tau^1 K(t,s)(s-\tau)^{\beta_{n-1}-\beta_{n-3}-1}ds \\ \frac{1}{\Gamma(\beta_{n-1}-\beta_{1})}\int_\tau^1 K(t,s)(s-\tau)^{\beta_{n-1}-\beta_{1}-1}ds \\ \frac{1}{\Gamma(\beta_{n-1})}\int_\tau^1 K(t,s)(s-\tau)^{\beta_{n-1}-1}ds \end{bmatrix}. \tag{9}$$

Let $r\left(L_{\mu_1,\mu_2,\ldots,\mu_n}\right)$ *denote the spectral radius of* $L_{\mu_1,\mu_2,\ldots,\mu_n}$*, and we can obtain the following lemma.*

Lemma 4. $r\left(L_{\mu_1,\mu_2,\ldots,\mu_n}\right) > 0$.

Proof. From Lemma 3(ii) and (9), we have

$$t^{\alpha-\beta_{n-1}-1}K_i(1,\tau) \le K_i(t,\tau) \le K_i(1,\tau), \forall t,\tau \in [0,1], i = 2,3,\cdots,n. \tag{10}$$

This implies that

$$t^{\alpha-\beta_{n-1}-1}K_{\mu_1,\mu_2,\ldots,\mu_n}(1,\tau) \le K_{\mu_1,\mu_2,\ldots,\mu_n}(t,\tau) \le K_{\mu_1,\mu_2,\ldots,\mu_n}(1,\tau), \forall t,\tau \in [0,1].$$

Consequently, for all $m \in \mathbb{N}_+$, we have

$$\|L^m_{\mu_1,\mu_2,\ldots,\mu_n}\| \ge \max_{t\in[0,1]} \underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{m} K_{\mu_1,\mu_2,\ldots,\mu_n}(t,s_1)K_{\mu_1,\mu_2,\ldots,\mu_n}(s_1,s_2)\cdot\cdots\cdot K_{\mu_1,\mu_2,\ldots,\mu_n}(s_{m-1},s_m)ds_1ds_2\cdots ds_m$$

$$\ge \max_{t\in[0,1]} t^{\alpha-\beta_{n-1}-1}\underbrace{\int_0^1\int_0^1\cdots\int_0^1}_{m} K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s_1)s_1^{\alpha-\beta_{n-1}-1}K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s_2)\cdot\cdots\cdot s_{m-1}^{\alpha-\beta_{n-1}-1}K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s_m)ds_1ds_2\cdots ds_m$$

$$= \left[\int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s)s^{\alpha-\beta_{n-1}-1}ds\right]^{m-1}\int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s)ds.$$

By Gelfand's theorem, we have

$$r(L_{\mu_1,\mu_2,\ldots,\mu_n}) = \lim_{m\to\infty}\sqrt[m]{\|L^m_{\mu_1,\mu_2,\ldots,\mu_n}\|} \ge \int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s)s^{\alpha-\beta_{n-1}-1}ds > 0.$$

This completes the proof. □

We denote an operator $B_{\mu_1,\mu_2,\ldots,\mu_n}$ as follows

$\left(B_{\mu_1,\mu_2,\ldots,\mu_n}v\right)(t) = \int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(t,\tau)v(\tau)d\tau + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t),\ v \in E.$

Now, $B_{\mu_1,\mu_2,\ldots,\mu_n} : P \to P$ is a completely continuous, linear, positive operator. Note that the spectral radius $r(B_{\mu_1,\mu_2,\ldots,\mu_n}) \ge r(L_{\mu_1,\mu_2,\ldots,\mu_n}) > 0$. Now, the well-known Krein–Rutman theorem [28] guarantees that there exist two functions $\varphi_{\mu_1,\mu_2,\ldots,\mu_n} \in P\backslash\{0\}$ and $\psi_{\mu_1,\mu_2,\ldots,\mu_n} \in P^*\backslash\{0\}$ with $\psi_{\mu_1,\mu_2,\ldots,\mu_n}(1) = 1$ and

$$B_{\mu_1,\mu_2,\ldots,\mu_n}\varphi_{\mu_1,\mu_2,\ldots,\mu_n} = r\left(B_{\mu_1,\mu_2,\ldots,\mu_n}\right)\varphi_{\mu_1,\mu_2,\ldots,\mu_n}, \quad B^*_{\mu_1,\mu_2,\ldots,\mu_n}\psi_{\mu_1,\mu_2,\ldots,\mu_n} = r\left(B_{\mu_1,\mu_2,\ldots,\mu_n}\right)\psi_{\mu_1,\mu_2,\ldots,\mu_n}, \tag{11}$$

where $B^*_{\mu_1,\mu_2,\ldots,\mu_n} : E^* \to E^*$ is the conjugate operator of $B_{\mu_1,\mu_2,\ldots,\mu_n}$, denoted by

$$\left(B^*_{\xi}v\right)(t) := \int_0^t ds\int_0^1 K\mu_1,\mu_2,\ldots,\mu_n(\tau,s)dv(\tau) + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}A(t)\int_0^1 \tau^{\alpha-\beta_{n-1}-1}dv(\tau).$$

Remark 2. *From Lemma 4 and the definition of operator* $B_{\mu_1,\mu_2,\ldots,\mu_n}$*, we have*

$$\int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s)s^{\alpha-\beta_{n-1}-1}ds \le r(L_{\mu_1,\mu_2,\ldots,\mu_n}) \le r(B_{\mu_1,\mu_2,\ldots,\mu_n}) \le \int_0^1 K_{\mu_1,\mu_2,\ldots,\mu_n}(1,s)ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 dA(t).$$

Define a modified function $[\cdot]^*$ for any $z \in C[0,1]$ by

$$[z(t)]^* = \begin{cases} z(t), & z(t) \ge 0, \\ 0, & z(t) < 0, \end{cases}$$

and consider the following boundary value problem

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}}v(t) + f\Big(t, I_{0^+}^{\beta_{n-1}}[v(t)-\omega(t)]^*, I_{0^+}^{\beta_{n-1}-\beta_1}[v(t)-\omega(t)]^*, \ldots, [v(t)-\omega(t)]^*\Big) + M = 0,\ t \in (0,1), \\ I_{0^+}^{\beta_{n-1}-n+2}v(0) = 0, \quad D_{0^+}^{\beta-\beta_{n-1}}v(1) = \int_0^1 v(t)dA(t), \end{cases} \tag{12}$$

where

$$\omega(t) = M\int_0^1 G(t,s)ds. \tag{13}$$

From Lemma 2.6 in [13], we have the following lemma.

Lemma 5. *Suppose that v is a solution of (12) with $v(t) \ge \omega(t), t \in [0,1]$. Then, $v-\omega$ is a positive solution of (4). Consequently, $u(t) = I_{0^+}^{\beta_{n-1}}[v(t)-w(t)]$ is also a positive solution of (1).*

Proof. Since v is a solution of (12) with $v(t) \ge \omega(t), t \in [0,1]$, then we have

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}}v(t) + f\Big(t, I_{0^+}^{\beta_{n-1}}[v(t)-\omega(t)], I_{0^+}^{\beta_{n-1}-\beta_1}[v(t)-\omega(t)], \ldots, [v(t)-\omega(t)]\Big) + M = 0,\ t \in (0,1), \\ I_{0^+}^{\beta_{n-1}-n+2}v(0) = 0, \quad D_{0^+}^{\beta-\beta_{n-1}}v(1) = \int_0^1 v(t)dA(t). \end{cases} \tag{14}$$

From (13), Lemma 2 and Remark 1 we have

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}}\omega(t) + M = 0,\ t \in (0,1), \\ I_{0^+}^{\beta_{n-1}-n+2}\omega(0) = 0, \quad D_{0^+}^{\beta-\beta_{n-1}}\omega(1) = \int_0^1 \omega(t)dA(t). \end{cases} \tag{15}$$

Combining with (14)–(15), we have

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}}[v(t)-\omega(t)] + f\Big(t, I_{0^+}^{\beta_{n-1}}[v(t)-\omega(t)], I_{0^+}^{\beta_{n-1}-\beta_1}[v(t)-\omega(t)], \ldots, [v(t)-\omega(t)]\Big) = 0,\ t \in (0,1), \\ I_{0^+}^{\beta_{n-1}-n+2}[v(0)-\omega(0)] = 0, \quad D_{0^+}^{\beta-\beta_{n-1}}[v(1)-\omega(1)] = \int_0^1 [v(t)-\omega(t)]dA(t). \end{cases}$$

This implies that $v-w$ is a positive solution of (4). From the relation between (1) and (4), we obtain $u(t) = I_{0^+}^{\beta_{n-1}}[v(t)-w(t)]$ is a positive solution of (1). This completes the proof. □

From Lemma 5, we define an operator $T : P \to P$ as

$$\begin{aligned} (Tv)(t) :=& \int_0^1 K(t,s)\Big[f\Big(s, I_{0^+}^{\beta_{n-1}}[v(s)-\omega(s)]^*, I_{0^+}^{\beta_{n-1}-\beta_1}[v(s)-\omega(s)]^*, \ldots, [v(s)-\omega(s)]^*\Big) + M\Big]ds \\ &+ \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t) \\ =& \int_0^1 G(t,s)\Big[f\Big(s, I_{0^+}^{\beta_{n-1}}[v(s)-\omega(s)]^*, I_{0^+}^{\beta_{n-1}-\beta_1}[v(s)-\omega(s)]^*, \ldots, [v(s)-\omega(s)]^*\Big) + M\Big]ds. \end{aligned}$$

If there exists $v^* \in P\backslash\{0\}$, such that $Tv^* = v^*$ with $v^*(t) \geq \omega(t), t \in [0,1]$, then $v^* - \omega$ is a positive solution of (4), and $u^*(t) = I_{0^+}^{\beta_{n-1}}[v^*(t) - \omega(t)]$ is a positive solution for (1).

Lemma 6. *Let $P_0 = \{v \in P : v(t) \geq t^{\alpha-\beta_{n-1}-1}\|v\|, t \in [0,1]\}$. Then $T(P) \subset P_0$.*
From Lemma 3(ii), we can obtain this lemma, so we omit its proof.
Note if $Tv^* = v^*$ and
$v^* \in P_0$ then

$$v^*(t) - \omega(t) \geq t^{\alpha-\beta_{n-1}-1}\|v^*\| - M\int_0^1 G(t,s)ds$$
$$\geq t^{\alpha-\beta_{n-1}-1}\left[\|v^*\| - M\int_0^1 \left(\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta_{n-1})} + \frac{\int_0^1 K(t,s)dA(t)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1}) - \Gamma(\alpha-\beta)\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)}\right)ds\right].$$

Therefore, if $\|v^*\| \geq M\int_0^1 \left(\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta_{n-1})} + \frac{\int_0^1 K(t,s)dA(t)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})-\Gamma(\alpha-\beta)\int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)}\right)ds := \Theta_{M,K,A}$, we have $v^*(t) \geq \omega(t), t \in [0,1]$. As a result, we seek the fixed point of T, with the norm greater than $\Theta_{M,K,A}$.

Lemma 7 (see [29]). *Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega} \cap P \to P$ a continuous, compact operator. If there exists $u_0 \in P \setminus \{0\}$, such that $u - Au \neq \mu u_0$ for all $\mu \geq 0$ and $u \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.*

Lemma 8 (see [29]). *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose that $A : \overline{\Omega} \cap P \to P$ is a continuous, compact operator. If $u \neq \mu Au$ for all $u \in \partial\Omega \cap P$ and $0 \leq \mu \leq 1$, then $i(A, \Omega \cap P, P) = 1$.*

3. Main Results

In this section, we first list our assumptions.

Hypothesis 3 (H3). *There exist $\gamma_i \geq$ with $\sum_{i=1}^n \gamma_i^2 \neq 0$, such that $r(B_{\gamma_1,\gamma_2,\ldots,\gamma_n}) > 1$ and*

$$\liminf_{\gamma_1 x_1+\gamma_2 x_2+\cdots+\gamma_n x_n \to +\infty} \frac{f(t, x_n, x_{n-1}, \ldots, x_1) + M}{\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n} \geq 1, \text{ uniformly on } t \in [0,1].$$

Hypothesis 4 (H4). *There exist $Q : [0,1] \to \mathbb{R}_+$ with $\int_0^1 G(1,t)Q(t)dt < \Theta_{M,K,A}$, such that*

$$f(t, x_n, x_{n-1}, \ldots, x_1) + M \leqslant Q(t),\ t \in [0,1], x_i \in [0, \Theta_{M,K,A}], i = 1, 2, \ldots, n.$$

Hypothesis 5 (H5). *There exist $\delta_i \geq$ with $\sum_{i=1}^n \delta_i^2 \neq 0$, such that $r(B_{\delta_1,\delta_2,\ldots,\delta_n}) < 1$ and*

$$\limsup_{\delta_1 x_1+\delta_2 x_2+\cdots+\delta_n x_n \to +\infty} \frac{f(t, x_n, x_{n-1}, \ldots, x_1) + M}{\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n} \leq 1, \text{ uniformly on } t \in [0,1].$$

Hypothesis 6 (H6). *There exist $\widetilde{Q} : [0,1] \to \mathbb{R}_+$ with $\int_0^1 G(1,t)\widetilde{Q}(t)dt > \Theta_{M,K,A}$, such that*

$$f(t, x_n, x_{n-1}, \ldots, x_1) + M \geqslant \widetilde{Q}(t),\ t \in [0,1], x_i \in [0, \Theta_{M,K,A}], i = 1, 2, \ldots, n.$$

Theorem 1. *Suppose that **(H1)**–**(H4)** hold. Then, (1) has at least one positive solution.*

Proof. Step 1. There exists a sufficient large number $R_1 > \Theta_{M,K,A}$, such that

$$v - Tv \neq \varrho\sigma_0, \varrho \geq 0, v \in \partial B_{R_1} \cap P, \tag{16}$$

where σ_0 is a fixed element in P_0, $B_{R_1} = \{v \in P : \|v\| < R_1\}$. Suppose the contrary i.e., there exist $v_1 \in \partial B_{R_1} \cap P, \varrho_1 \geq 0$, such that

$$v_1 - Tv_1 = \varrho_1 \sigma_0. \tag{17}$$

Together with Lemma 6, this implies that

$$v_1 \in P_0. \tag{18}$$

From (**H3**) there exists $c_1 > 0$, such that

$$f(t, x_n, x_{n-1}, \ldots, x_1) + M \geq \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n - c_1, t \in [0,1], x_i \in \mathbb{R}_+, i = 1,2,...,n.$$

Consequently, note that $\|v_1\| = R_1 > \Theta_{M,K,A}$ and by (17) we have

$$\begin{aligned}
v_1(t) &\geq (Tv_1)(t)\\
&= \int_0^1 K(t,s)\Big[f\Big(s, I_{0^+}^{\beta_{n-1}}[v_1(s) - \omega(s)], I_{0^+}^{\beta_{n-1}-\beta_1}[v_1(s) - \omega(s)], \ldots, [v_1(s) - \omega(s)]\Big) + M\Big]ds\\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})} t^{\alpha-\beta_{n-1}-1} \int_0^1 v_1(t)dA(t)\\
&\geq \int_0^1 K(t,s)\Big(\gamma_n I_{0^+}^{\beta_{n-1}}[v_1(s) - \omega(s)] + \gamma_{n-1} I_{0^+}^{\beta_{n-1}-\beta_1}[v_1(s) - \omega(s)] + \cdots + \gamma_1[v_1(s) - \omega(s)] - c_1\Big)ds\\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})} t^{\alpha-\beta_{n-1}-1} \int_0^1 v_1(t)dA(t)\\
&\geq \int_0^1 [\gamma_n K_n(t,\tau) + \gamma_{n-1} K_{n-1}(t,\tau) + \cdots + \gamma_1 K(t,\tau)][v_1(\tau) - \omega(\tau)]d\tau + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})} t^{\alpha-\beta_{n-1}-1} \int_0^1 v_1(t)dA(t)\\
&\quad - c_1 \int_0^1 K(1,s)ds\\
&\geq \int_0^1 K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t,\tau)v_1(\tau)d\tau + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})} t^{\alpha-\beta_{n-1}-1} \int_0^1 v_1(t)dA(t) - c_1 \int_0^1 K(1,s)ds - \int_0^1 K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(1,\tau)\omega(\tau)d\tau.
\end{aligned}$$

This implies that

$$v_1(t) \geq (B_{\gamma_1,\gamma_2,\ldots,\gamma_n} v_1)(t) - c_2, \tag{19}$$

where $c_2 = c_1 \int_0^1 K(1,s)ds + \int_0^1 K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(1,\tau)\omega(\tau)d\tau$. Multiply both sides of (19) by $d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t)$ and integrate over $[0,1]$ and use (11) to obtain

$$\begin{aligned}
\int_0^1 v_1(t)d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t) &\geq \int_0^1 (B_{\gamma_1,\gamma_2,\ldots,\gamma_n} v_1)(t)d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t) - c_2\\
&= \int_0^1 d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t)\Big(\int_0^1 K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t,s)v_1(s)ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})} t^{\alpha-\beta_{n-1}-1} \int_0^1 v_1(t)dA(t)\Big) - c_2\\
&= \int_0^1 v_1(s)d\Big(\int_0^s d\tau \int_0^1 K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t,\tau)d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(\tau)\Big)\\
&\quad + \int_0^1 v_1(s)d\Big(A(\tau)\int_0^1 \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})} t^{\alpha-\beta_{n-1}-1} d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t)\Big) - c_2\\
&= \Big\langle B^*_{\gamma_1,\gamma_2,\ldots,\gamma_n}\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}, v_1\Big\rangle(t) - c_2\\
&= r(B_{\gamma_1,\gamma_2,\ldots,\gamma_n})\int_0^1 v_1(t)d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t) - c_2.
\end{aligned}$$

Solving this inequality, we have

$$\int_0^1 v_1(t)d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t) \leq \frac{c_2}{r(B_{\gamma_1,\gamma_2,\ldots,\gamma_n}) - 1}.$$

Using (18), we have

$$\|v_1\| \le \frac{c_2}{r(B_{\gamma_1,\gamma_2,\ldots,\gamma_n})-1}\left[\int_0^1 t^{\alpha-\beta_{n-1}-1}d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t)\right]^{-1}.$$

Note that if we can take

$$R_1 > \max\left\{\Theta_{M,K,A}, \frac{c_2}{r(B_{\gamma_1,\gamma_2,\ldots,\gamma_n})-1}\left[\int_0^1 t^{\alpha-\beta_{n-1}-1}d\psi_{\gamma_1,\gamma_2,\ldots,\gamma_n}(t)\right]^{-1}\right\}$$

and when $v_1 \in \partial B_{R_1} \cap P$, (17) is not satisfied. Hence, we obtain (16), and Lemma 7 implies that

$$i(T, B_{R_1} \cap P, P) = 0. \tag{20}$$

Step 2. We prove that

$$v \ne \varrho Tv, v \in \partial B_{\Theta_{M,K,A}} \cap P, \varrho \in [0,1]. \tag{21}$$

Suppose the contrary, i.e., there exist $v_2 \in \partial B_{\Theta_{M,K,A}} \cap P, \varrho_2 \in [0,1]$, such that

$$v_2 = \varrho_2 Tv_2.$$

This implies that

$$\|v_2\| \le \|Tv_2\|. \tag{22}$$

Note that $\|v_2\| = \Theta_{M,K,A}$ and from (H4) we have

$$\begin{aligned}(Tv_2)(t) &= \int_0^1 G(t,s)\left[f\left(s, I_{0^+}^{\beta_{n-1}}[v_2(s)-\omega(s)], I_{0^+}^{\beta_{n-1}-\beta_1}[v_2(s)-\omega(s)],\ldots,[v_2(s)-\omega(s)]\right)+M\right]ds\\ &\le \int_0^1 G(1,s)Q(s)ds\\ &< \Theta_{M,K,A}.\end{aligned}$$

This contradicts with (22). Hence, (21) holds, and Lemma 8 implies that

$$i(T, B_{\Theta_{M,K,A}} \cap P, P) = 1. \tag{23}$$

Combining (20) and (23) we obtain

$$i(T,(B_{R_1}\backslash\overline{B_{\Theta_{M,K,A}}})\cap P,P) = i(T,B_{R_1}\cap P,P) - i(T,B_{\Theta_{M,K,A}}\cap P,P) = 0-1 = -1.$$

Then T has a fixed point in $(B_{R_1}\backslash\overline{B_{\Theta_{M,K,A}}})\cap P$, i.e., there exists $v^* \in (B_{R_1}\backslash\overline{B_{\Theta_{M,K,A}}})\cap P$, such that $Tv^* = v^*$, and then $u^*(t) = I_{0^+}^{\beta_{n-1}}[v^*(t)-\omega(t)]$ is a positive solution for (1). This completes the proof. □

Theorem 2. *Suppose that* ***(H1)–(H2)*** *and* ***(H5)–(H6)*** *hold. Then,* (1) *has at least one positive solution.*

Proof. Step 1. We claim that

$$v - Tv \ne \varrho\sigma_1, \varrho \ge 0, v \in \partial B_{\Theta_{M,K,A}} \cap P, \tag{24}$$

where $\sigma_1 \in P$ is a fixed element. Suppose the contrary, i.e., there exist $v_3 \in \partial B_{\Theta_{M,K,A}} \cap P$ and $\varrho_3 \ge 0$, such that

$$v_3 = Tv_3 + \varrho_3\sigma_1.$$

This implies that

$$\|v_3\| \ge \|Tv_3 + \varrho_3\sigma_1\| \ge \|Tv_3\| + \|\varrho_3\sigma_1\| \ge \|Tv_3\|. \tag{25}$$

Note that $\|v_3\| = \Theta_{M,K,A}$, and from (**H6**) we have

$$\begin{aligned}\|Tv_3\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\Big[f\Big(s, I_{0^+}^{\beta_{n-1}}[v_3(s)-\omega(s)], I_{0^+}^{\beta_{n-1}-\beta_1}[v_3(s)-\omega(s)],\dots,[v_3(s)-\omega(s)]\Big)+M\Big]ds \\ &\geq \max_{t\in[0,1]} t^{\alpha-\beta_{n-1}-1}\int_0^1 G(1,s)\widetilde{Q}(s)ds \\ &> \Theta_{M,K,A}.\end{aligned}$$

This contradicts with (25). Therefore, Lemma 7 implies that

$$i(T, B_{\Theta_{M,K,A}} \cap P, P) = 0. \tag{26}$$

Step 2. There exists a sufficient large number $R_2 > \Theta_{M,K,A}$ such that

$$v \neq \varrho Tv, v \in \partial B_{R_2} \cap P, \varrho \in [0,1]. \tag{27}$$

Suppose the contrary, i.e., there exist $v_4 \in \partial B_{R_2} \cap P$, $\varrho_4 \in [0,1]$, such that

$$v_4 = \varrho_4 Tv_4. \tag{28}$$

This, combined with Lemma 6, implies that

$$v_4 \in P_0.$$

By (**H5**) there exists $c_3 > 0$ such that

$$f(t,x_n,x_{n-1},\dots,x_1)+M \leq \delta_1x_1+\delta_2x_2+\cdots+\delta_nx_n+c_3, t\in[0,1], x_i\in\mathbb{R}_+, i=1,2,...,n.$$

Note that $\|v_4\| = R_2 > \Theta_{M,K,A}$, and from (28) we have

$$\begin{aligned}v_4(t) \leq (Tv_4)(t) &\leq \int_0^1 K(t,s)\Big[f\Big(s, I_{0^+}^{\beta_{n-1}}[v_4(s)-\omega(s)], I_{0^+}^{\beta_{n-1}-\beta_1}[v_4(s)-\omega(s)],\dots,[v_4(s)-\omega(s)]\Big)+M\Big]ds \\ &\quad+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t) \\ &\leq \int_0^1 K(t,s)\Big(\delta_n I_{0^+}^{\beta_{n-1}}[v_4(s)-\omega(s)]+\delta_{n-1}I_{0^+}^{\beta_{n-1}-\beta_1}[v_4(s)-\omega(s)]+\cdots+\delta_1[v_4(s)-\omega(s)]\Big)ds \\ &\quad+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t)+c_3\int_0^1 K(1,s)ds \\ &\leq \int_0^1[\delta_nK_n(t,\tau)+\delta_{n-1}K_{n-1}(t,\tau)+\cdots+\delta_1K(t,\tau)]v(\tau)d\tau+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t)+c_3\int_0^1 K(1,s)ds \\ &= \int_0^1 K_{\delta_1,\delta_2,\dots,\delta_n}(t,\tau)v(\tau)d\tau+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}\int_0^1 v(t)dA(t)+c_3\int_0^1 K(1,s)ds.\end{aligned} \tag{29}$$

Multiply both sides of (29) by $d\psi_{\delta_1,\delta_2,\dots,\delta_n}(t)$ and integrate over $[0,1]$ and use (11) to obtain

$$\begin{aligned}\int_0^1 v_4(t)d\psi_{\delta_1,\delta_2,\dots,\delta_n}(t) &\leq \int_0^1 (B_{\delta_1,\delta_2,\dots,\delta_n}v_4)(t)d\psi_{\delta_1,\delta_2,\dots,\delta_n}(t)+c_3\int_0^1 K(1,s)ds \\ &= \int_0^1 v_4(s)d\Big(\int_0^s d\tau\int_0^1 K_{\delta_1,\delta_2,\dots,\delta_n}(t,\tau)d\psi_{\delta_1,\delta_2,\dots,\delta_n}(\tau)\Big) \\ &\quad+\int_0^1 v_4(s)d\Big(A(\tau)\int_0^1\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}t^{\alpha-\beta_{n-1}-1}d\psi_{\delta_1,\delta_2,\dots,\delta_n}(t)\Big)+c_3\int_0^1 K(1,s)ds \\ &= \Big\langle B^*_{\delta_1,\delta_2,\dots,\delta_n}\psi_{\delta_1,\delta_2,\dots,\delta_n}, v_4\Big\rangle(t)+c_3\int_0^1 K(1,s)ds \\ &= r(B_{\delta_1,\delta_2,\dots,\delta_n})\int_0^1 v_4(t)d\psi_{\delta_1,\delta_2,\dots,\delta_n}(t)+c_3\int_0^1 K(1,s)ds.\end{aligned}$$

Consequently, we have

$$\int_0^1 v_4(t)d\psi_{\delta_1,\delta_2,\ldots,\delta_n}(t) \leq \frac{c_3\int_0^1 K(1,s)ds}{1-r(B_{\delta_1,\delta_2,\ldots,\delta_n})}.$$

Note that $v_4 \in P_0$, and, thus,

$$\|v_4\| \leq \frac{c_3\int_0^1 K(1,s)ds}{1-r(B_{\delta_1,\delta_2,\ldots,\delta_n})}\left[\int_0^1 t^{\alpha-\beta_{n-1}-1}d\psi_{\delta_1,\delta_2,\ldots,\delta_n}(t)\right]^{-1}.$$

If we choose

$$R_2 > \max\left\{\Theta_{M,K,A}, \frac{c_3\int_0^1 K(1,s)ds}{1-r(B_{\delta_1,\delta_2,\ldots,\delta_n})}\left[\int_0^1 t^{\alpha-\beta_{n-1}-1}d\psi_{\delta_1,\delta_2,\ldots,\delta_n}(t)\right]^{-1}\right\},$$

then (28) is false. Hence, we obtain (27), and Lemma 8 implies that

$$i(T, B_{R_2}\cap P, P) = 1. \tag{30}$$

Combining (26) and (30), we obtain

$$i(T,(B_{R_2}\backslash\overline{B_{\Theta_{M,K,A}}})\cap P,P) = i(T,B_{R_2}\cap P,P) - i(T,B_{\Theta_{M,K,A}}\cap P,P) = 1-0=1.$$

Then, T has a fixed point in $(B_{R_2}\backslash\overline{B_{\Theta_{M,K,A}}})\cap P$, i.e., there exists $v^{**}\in(B_{R_2}\backslash\overline{B_{\Theta_{M,K,A}}})\cap P$ such that $Tv^{**}=v^{**}$, and then $u^{**}(t) = I_{0^+}^{\beta_{n-1}}[v^{**}(t)-\omega(t)]$ is a positive solution for (1). This completes the proof. □

In what follows, we provide two examples to illustrate our main theorems.

Example 1. *Let $\alpha = 2.9, n = 3, \beta_1 = 0.5, \beta_2 = 1.5, \beta = 1.6, A(t) = t, t\in[0,1]$. $\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} = \frac{\Gamma(1.4)}{\Gamma(1.3)} = 0.99 > \int_0^1 t^{\alpha-\beta_{n-1}-1}dA(t)$, and (**H2**) holds. From Remark 2, we have*

$$r(B_{\gamma_1,\gamma_2,\gamma_3}) \geq \int_0^1 K_{\gamma_1,\gamma_2,\gamma_3}(1,s)s^{\alpha-\beta_{n-1}-1}ds.$$

Thus, there exist $\gamma_i \geq$ with $\sum_{i=1}^3\gamma_i^2\neq 0$, such that $r(B_{\gamma_1,\gamma_2,\gamma_3})>1$.

Let $f(t,x_3,x_2,x_1) = \frac{\Theta_{M,K,A}^{1-\chi_1}}{\int_0^1 G(1,t)dt}(\gamma_1+\gamma_2+\gamma_3)^{-\chi_1}e^{-1-t}(\gamma_1x_1+\gamma_2x_2+\gamma_3x_3)^{\chi_1} - M, \chi_1 > 1, t\in[0,1], x_i\in\mathbb{R}_+, i=1,2,3$. Then, when $t\in[0,1], x_i\in[0,\Theta_{M,K,A}]$, we have

$$f(t,x_3,x_2,x_1)+M \leq \frac{e^{-1}\Theta_{M,K,A}^{1-\chi_1}}{\int_0^1 G(1,t)dt}(\gamma_1+\gamma_2+\gamma_3)^{-\chi_1}\Theta_{M,K,A}^{\chi_1}(\gamma_1+\gamma_2+\gamma_3)^{\chi_1} = \frac{e^{-1}\Theta_{M,K,A}}{\int_0^1 G(1,t)dt} :\equiv Q(t), t\in[0,1].$$

Moreover,

$$\begin{aligned}&\liminf_{\gamma_1x_1+\gamma_2x_2+\gamma_3x_3\to+\infty}\frac{f(t,x_3,x_3,x_1)+M}{\gamma_1x_1+\gamma_2x_2+\gamma_3x_3}\\ &= \liminf_{\gamma_1x_1+\gamma_2x_2+\gamma_3x_3\to+\infty}\frac{\frac{\Theta_{M,K,A}^{1-\chi_1}}{\int_0^1 G(1,t)dt}(\gamma_1+\gamma_2+\gamma_3)^{-\chi_1}e^{-1-t}(\gamma_1x_1+\gamma_2x_2+\gamma_3x_3)^{\chi_1}}{\gamma_1x_1+\gamma_2x_2+\gamma_3x_3} = +\infty, \text{ uniformly on } t\in[0,1].\end{aligned}$$

*Therefore, (**H1**), (**H3**)–(**H4**) hold. From Theorem 1, (1) has at least one positive solution.*

Example 2. *Let* $\alpha = 1.9, n = 2, \beta_1 = 0.2, \beta = 0.8$, $A(t) = t, t \in [0,1]$. *Then,* $\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} = \frac{\Gamma(1.7)}{\Gamma(1.1)} = 0.96 > \int_0^1 t^{\alpha-\beta_{n-1}-1} dA(t)$, *and* ***(H2)*** *holds. Note that in Remark 2 we have*

$$r(B_{\delta_1,\delta_2}) \le \int_0^1 K_{\delta_1,\delta_2}(1,s)ds + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta_{n-1})}\int_0^1 dA(t).$$

If $\int_0^1 K_{\delta_1,\delta_2}(1,s)ds < 0.04$, *and then* $r(B_{\delta_1,\delta_2}) < 1$. *Therefore, there exist* $\delta_1, \delta_2 \ge 0(\delta_1^2 + \delta_2^2 \ne 0)$ *to ensure* $r(B_{\delta_1,\delta_2}) < 1$.

Let $f(t,x_2,x_1) = \frac{\zeta_1\Theta_{M,K,A}}{\int_0^1 G(1,t)dt} e^{\Theta_{M,K,A}(\delta_1+\delta_2)} e^{-(\delta_1 x_1+\delta_2 x_2)+t} - M, \zeta_1 > 1, t \in [0,1], x_1, x_2 \in \mathbb{R}_+$. *Then, when* $t \in [0,1], x_i \in [0, \Theta_{M,K,A}]$, *we have*

$$f(t,x_2,x_1) + M \ge \frac{\zeta_1\Theta_{M,K,A}}{\int_0^1 G(1,t)dt} e^{\Theta_{M,K,A}(\delta_1+\delta_2)} e^{-\Theta_{M,K,A}(\delta_1+\delta_2)} = \frac{\zeta_1\Theta_{M,K,A}}{\int_0^1 G(1,t)dt} :\equiv \widetilde{Q}(t), t \in [0,1].$$

Moreover,

$$\limsup_{\delta_1 x_1+\delta_2 x_2 \to +\infty} \frac{f(t,x_2,x_1)+M}{\delta_1 x_1 + \delta_2 x_2} = \limsup_{\delta_1 x_1+\delta_2 x_2 \to +\infty} \frac{\frac{\zeta_1\Theta_{M,K,A}}{\int_0^1 G(1,t)dt} e^{\Theta_{M,K,A}(\delta_1+\delta_2)} e^{-(\delta_1 x_1+\delta_2 x_2)+t}}{\delta_1 x_1 + \delta_2 x_2} = 0, \text{ uniformly on } t \in [0,1].$$

Hence, ***(H1)***, ***(H5)***, *and* ***(H6)*** *hold. From Theorem 2, (1) has at least one positive solution.*

4. Conclusions

In this paper, we use the fixed point index to investigate the existence of positive solutions for the higher-order Riemann–Liouville type fractional integral boundary value problem (1) with fractional derivatives and a semi-positone non-linearity. We note that in most integral boundary value problems, the usual approach in the literature is to incorporate integral boundary conditions into their Green functions (see Remark 1), so there has been no real improvement in the approach. However, in this paper we consider a linear operator $B_{\mu_1,\mu_2,\ldots,\mu_n}$, and investigate the effects of integral boundary conditions on its eigenvalues. Then, by using Gelfand's formula and the Krein–Rutman theorem, we present some properties of its first eigenvalue, and obtain our existence theorems for the considered problem under conditions concerning the first eigenvalue of the linear operator $B_{\mu_1,\mu_2,\ldots,\mu_n}$. The results obtained here improve some results in the literature.

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