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Rogue Waves Generator and Chaotic and Fractal Behavior of the Maccari System with a Resonant Parametric Forcing

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Abstract: Using the Asymptotic Perturbation (AP) method we can find approximate solutions for the Maccari equation with a parametric resonant forcing acting over the frequency of a generic mode. Taking into account its nonlocal behavior and applying symmetry considerations, a system with two coupled equations for the phase and amplitude modulation can be obtained. The system can be solved, and we demonstrate the existence of a big modulation in the wave amplitude, producing a rogue waves train and, in this case, these waves are not isolated. We then obtain a rogue waves generator, being able of producing and controlling the rogue waves' amplitude. Another important finding is the existence of chaotic or fractal solutions, because of the presence of an arbitrary function in the solution.

Keywords: Maccari equation; fractal solutions; chaotic solutions; rogue waves; parametric excitation

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1. Introduction

Nonlinear partial differential equations (NPDEs) have been widely investigated and used in biology, chemistry, nonlinear mathematics, and, especially, in many branches of physics, such as quantum field theory, plasma physics, condensed matter physics, laser systems, fluid dynamics, and nonlinear optics [1,2]. Different nonlinear wave equations are important in physics and engineering research. Naively, we can think that integrable models are characterized by solitons and on the contrary the chaos and fractals are the predictable behavior of the nonintegrable nonlinear models. However, in higher dimensions the situation is quite different. First of all, we should understand the special meaning of integrability. For instance, we say a NPDE is Painlevè integrable if it possesses the Painlevè property and a NPDE is S-integrable if it has a spectral transform and then even an IST-inverse scattering transformation with its Lax pair. It is well known that nonlinear mathematical physics problems are more difficult than the linear ones. In linear physics, the Fourier transformation method is the most important approach to find the exact solutions for dispersive wave equations. From a mathematical point of view, the IST is a nontrivial extension of the Fourier transformation in nonlinear physics.

A lot of papers are devoted to NPDEs and their behaviors, integrability aspects, conservation laws, Painlevè analysis, numerical simulations, wave interactions, and various others. In particular, rogue waves have been extensively investigated in the last years [3–5]. In this paper, we consider the Maccari system [6–23] under a parametric excitation. We underline that the Maccari system usually describes isolated wave propagation in tiny spaces. In addition, we now demonstrate that with a parametric excitation a regular wave train can be generated. This method has been previously applied to the Hirota–Maccari equation [24] and now for the first time to the well-known Maccari system.

Let us consider the following Maccari system.

$$i\psi_t + \psi_{xx} + \psi\phi = 0, \tag{1}$$

$$\phi_t + \phi_y + (|\psi|^2)_x = 0, \tag{2}$$

where $\psi = \psi(x, y, t)$ is a differentiable complex function and $\phi = \phi(x, y, t)$ is a differentiable real function. We observe that the nonlocality is given by the real function $\phi = \phi(x, y, t)$; its values depend on the complex function $\psi = \psi(x, y, t)$ values all over the plane (x,y). The Maccari system overcomes the Kadomtsev–Petviashvili equation, when we study phenomena where the spatial scale in the y-direction is different from the spatial scale in the x-direction, and, in this case, the real part of the function ψ is the wave amplitude and the nonlocal effects are given by the real function ϕ [23].

We underline that the same happens for the Hirota–Maccari equation, but in this case a larger y-scale is necessary [24].

Now we consider a parametric forcing inducing a parametric resonance into Equation (1) and arrive at the following NPDE.

$$i\psi_t + \psi_{xx} + \psi\phi = f\widetilde{\psi}exp(iK_1x + iK_2y - i\Omega t), \tag{3}$$

$$\phi_t + \phi_y + (|\psi|^2)_x = 0, \tag{4}$$

where f is the parametric resonance amplitude, $\widetilde{\psi}$ is the complex conjugate of ψ, and the circular frequency of a given mode is

$$\omega = K_x^2 \tag{5}$$

The parametric forcing is in parametric resonance ($\Omega \sim 2\omega$) with the frequency of a generic mode.

In Section 2, the asymptotic perturbation (AP) method [11] is used to build an approximate solution to Equations (3) and (4) and obtain two coupled equations for the solution amplitude and phase modulations. Using symmetry considerations, we compare Maccari systems (3) and (4) with the Hirota–Maccari equation [24].

In Section 3, the model system is investigated together with its fixed points; it is solved, and the nonlinear behavior is carefully described, finding the existence of a wave train of arbitrary amplitude. We demonstrate the existence of a regular wave train. If the parametric strength increases from very small values, then the solution decreases, but it is proportional to the arbitrary initial condition on the amplitude. Suitable initial conditions can produce approximate solutions to Equations (3) and (4), characterized by a rogue waves train. This behavior has not been observed yet in experimental situations.

In Section 4, we find nonlinear solutions for the parametrically excited Maccari system because there is an arbitrary function in the approximate solution; then, we can obtain chaotic or fractal solutions [25,26].

In Section 5, we consider the existence of chaotic and fractal solutions for the parametrically excited Maccari system and see how there is a thin border between these solutions and the coherent ones.

In Section 6, we discuss the paper's main findings as well as directions and possible developments for future papers.

2. The Approximate Solution

The nonlocal Maccari systems (3) and (4) with a parametric forcing is investigated in this section and the parametric term f is scaled as $\varepsilon^2 f$, where ε is our small parameter, and we introduce two new variables.

$$\tau = \varepsilon^2 t,\ \xi = \varepsilon^{p_1} x \tag{6}$$

This in order to consider larger time and x-space scales and obtain the amplitude modulation response.

Taking into account the AP method [11], the approximate solution $\psi(x, y, t)$ of Equations (3) and (4) can be written in the following form:

$$\psi(x, y, t) = \varepsilon\Psi(\tau)exp(i\alpha), \tag{7}$$

$$\alpha = K_x x + K_y y - \frac{\Omega}{2} t, \tag{8}$$

$$\phi(x, y, t) = \varepsilon \Phi(x, y, t) \tag{9}$$

and τ in Equation (7) is given by (6).

Now, we study the boundary conditions for a square with a side L, in plane (x,y).

$$\begin{gathered} \psi(0, y, t) = \psi(L, y, t) \\ 0 \leqslant y \leqslant L, \\ K_x = \tfrac{2n_1\pi}{L},\ n_1 \text{ integer}, \end{gathered} \tag{10}$$

$$\begin{gathered} \psi(x, 0, t) = \psi(x, L, t) \\ 0 \leqslant x \leqslant L, \\ K_y = \tfrac{2n_2\pi}{L},\ n_2 \text{ integer} \end{gathered} \tag{11}$$

It is well known that the nearness of the excitation frequency Ω to the system frequency ω can be characterized through a detuning parameter σ,

$$\omega_{\underline{n}} = \frac{\Omega}{2} + \varepsilon^2 \sigma, \tag{12}$$

where

$$\underline{n} = (n_1, n_2), \tag{13}$$

$$\omega_{\underline{n}} = 4\left(\frac{n_1 n_2 \pi^2}{L^2}\right) \tag{14}$$

Only a single leading mode is involved in the approximate solution and we suppose that the system is excited near the natural frequency of a specific linear mode and that there are no other resonances with any other mode. It is quite obvious that the modes that are not directly excited by an external source or even through an internal resonance will decay with time, if friction terms are present. Note that the variable change (6) implies that

$$\frac{d}{dt}\left(\Psi \exp\left(-i\frac{\Omega}{2}t\right)\right) = \left(-i\frac{\Omega}{2}\Psi + \varepsilon^2 \frac{d\Psi}{d\tau}\right) exp\left(-i\frac{\Omega}{2}t\right) \tag{15}$$

$$\frac{d}{dx}(\Psi \exp(iK_x x)) = \left(iK_x \Psi + \varepsilon^p \frac{d\Psi}{d\xi}\right) exp(iK_x x) \tag{16}$$

The temporal rescaling (6) allow us to obtain the asymptotic behavior of the solution, when the nonlinear amplitude can be modified by the nonlinear effects.

The predominant linear part of Equations (3) and (4) can be eliminated by the assumed solution (7) and it allows to calculate the phase and amplitude modulation given by the parametric forcing.

After using solutions (7)–(9) for the nonlinear systems (3) and (4), we get the linear equation.

$$(\omega_{\underline{n}} - K_1^2)\Psi = 0,\ (\text{order } O(\varepsilon)). \tag{17}$$

If we consider Equations (3) and (4) at the order ε^2, then we obtain the nonlinear differential equation (if $p_1 > 2$).

$$i\Psi_\tau - \sigma\Psi = f\widetilde{\Psi} - \Psi\Phi, \tag{18}$$

$$\Phi_y = 0. \tag{19}$$

3. The Model System

The complex-valued function Ψ can be obtained by its amplitude and phase:

$$\Psi = \rho exp(i\theta) \tag{20}$$

and the nonlinear system equations, Equations (18) and (19), yield the following nonlinear model system.

$$\frac{d\rho}{d\tau} + f\rho sin(2\theta) = 0, \tag{21}$$

$$\frac{d\theta}{d\tau} + fcos(2\theta) + \sigma - \Phi = 0, \tag{22}$$

$$\Phi_y = 0 \tag{23}$$

where Φ is a function of x and t, but for simplicity we consider it as a constant. Our starting assumptions lead us to the model systems (21)–(23), describing the phase and amplitude evolution.

As usual for the AP method, the validity of the approximate solutions (7)–(9) should be expected to be valid on the time scale,

$$t = O\left(\frac{1}{\varepsilon^2}\right) \tag{24}$$

and x-space scale,

$$X = O\left(\frac{1}{\varepsilon^{(p_1)}}\right) \tag{25}$$

on bounded intervals of the τ and x-variables. On the other side, the approximate solution (2.2) loses its validity, if we consider solutions on larger intervals, such that

$$\tau = x = O\left(\frac{1}{\varepsilon}\right). \tag{26}$$

In the next section we will demonstrate the existence of a train wave for the parametrically excited Maccari system. We underline an important difference compared to the parametrically excited Hirota–Maccari [24], where we need a different choice of the parameter p_2 for the y-scale with the variable change:

$$\eta = \varepsilon^{(p_2)} y \tag{27}$$

That means we need a larger y-space scale with respect to the Maccari system. That is why these two systems describes different physical situation: the Maccari one is suitable for rogue waves and the Hirota–Maccari equation for soliton propagation. Therefore, there is a symmetry change between the x- and y-variables.

The fixed points of model systems (21)–(23) correspond to periodic solutions of the starting systems (3) and (4), and we must consider the conditions $d\rho/dt = d\theta/dt = 0$. The fixed points for the nonlinear system models (21)–(23) are

$$\mathrm{P}_1 : \ \theta_E = 0, \ \Phi_E = \sigma + f, \ \rho_E = arbitraryvalue, \tag{28}$$

$$\mathrm{P}_2 : \ \theta_E = \frac{\pi}{2}, \ \Phi_E = \sigma - f, \ \rho_E = arbitraryvalue, \tag{29}$$

In the next section, we solve model systems (21)–(23) and describe the solution's behavior.

4. Solving the Model System

We can easily solve nonlinear systems (21) and (22), obtaining the following equation:

$$\frac{d\rho}{\rho} = \int_0^\tau fsin(2\theta)d\tau' = \int_0^\tau fsin(2\theta)\frac{d\tau'}{d\theta}d\theta, \tag{30}$$

where

$$\frac{d\tau'}{d\theta} = \frac{1}{\Phi - \sigma - f cos2\theta,} \tag{31}$$

The final solution is

$$\rho = \frac{\rho_0}{-2f} log|f cos2\theta + \sigma - \Phi| \tag{32}$$

Even Equation (22) can be easily integrated, obtaining

$$\theta - \theta_0 = arctan(tan(\tau_0 - \tau)(\Phi_E - \sigma + f)) \tag{33}$$

In Figures 1 and 2, we show the response amplitude ρ vs. the angular variable θ and observe the formation of a wave train of arbitrary amplitude. It can be described as a rogue wave train and we underline that this behavior has not been observed in experimental situations. We can consider the parametrically excited Maccari system as a rogue wave generator; indeed, we can manipulate the wave amplitude and control the rogue waves.

If at time $t = 0$ we turn on the parametric scenario and look at Equations (13) and (14), we can see that the rogue wave generator starts, and these nonlinear waves propagate in a square with side L, and the group velocity is

$$V_j = \frac{d\omega}{dK_j}, \ j = x, y \tag{34}$$

where ω is given by Equation (14). We can look at a few examples in Figures 3–6. We observe that, solving the Maccari system, we can obtain an approximate solution for the real associated scalar field $u(x, y, t)$ in the following form [23].

$$u(x, y, t) = 2\rho(x, y, t) cos(\theta(x, y, t) - \Omega t) + \phi(x) \tag{35}$$

The behavior of the Maccari system in an unbounded scenario, towards infinity both in the x- and y-direction, needs further investigation.

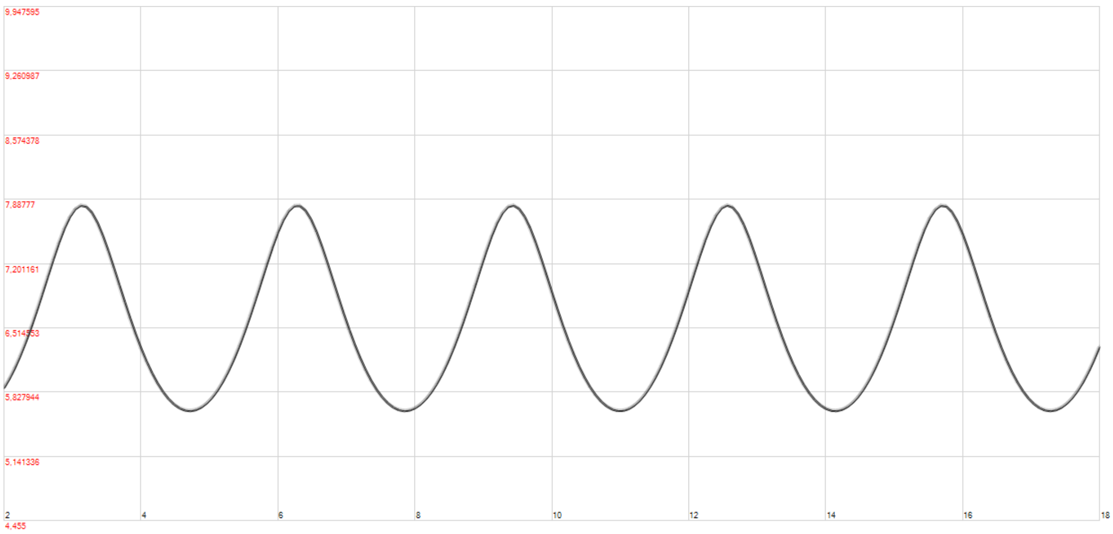

Figure 1. The response amplitude ρ (y-axis) vs. the angular variable θ (x-axis), where $\rho_0 = 0.08$, $f = -0.02$, $\sigma = 0.06$, and $\phi = 0.02$. Note that we can obtain a wave train and modulate their amplitude, $\theta_0 = 0$.

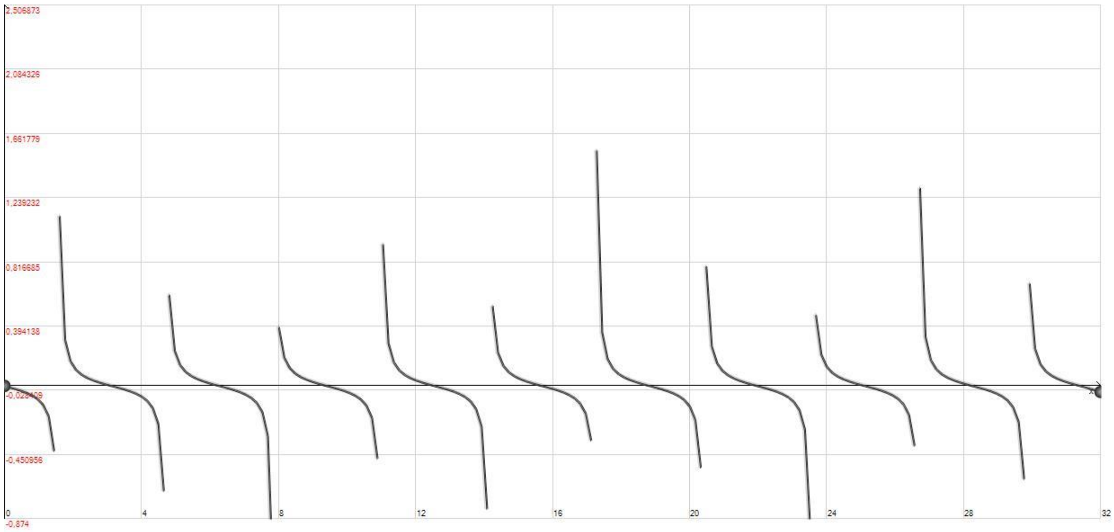

Figure 2. The angular amplitude θ (y-axis) vs. the time variable τ (x-axis). Note that the angular variable changes periodically, describing the wave train modulation, where ρ_0 = 0.08, $f = -0.02$, σ = 0.06, ϕ = 0.02, and $\tau_0 = \theta_0 = 0$.

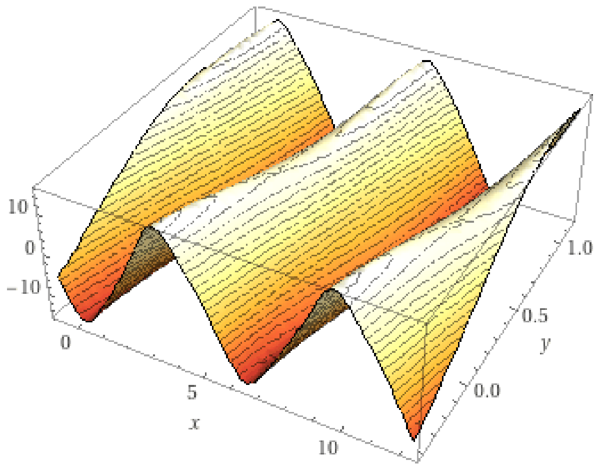

Figure 3. The approximate solution (35) for the parametrically excited Maccari system, where ρ_0 = 0.08, $f = -0.02$, σ = 0.06, $\phi = 0.02$, $y = 5$, and $\tau_0 = \theta_0 = 0$. Note that in the figure y stands for time t.

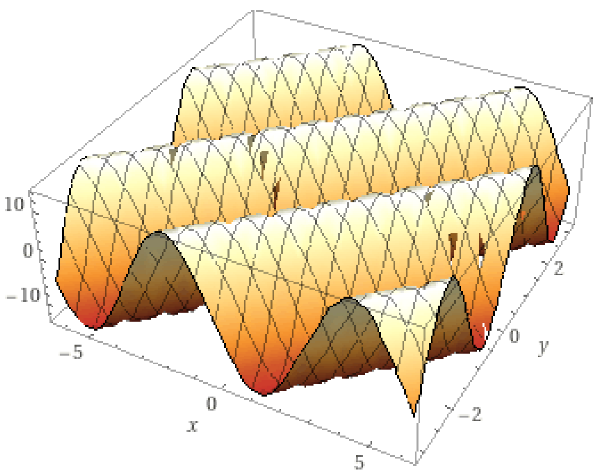

Figure 4. The approximate solution (35) for the parametrically excited Maccari system, where ρ_0 = 0.1, f = 0.01, $\sigma = -0.06$, $\phi = 0.02$, $t = time = 10$, and $\tau_0 = \theta_0 = 0$. Note that in the figure x stands for the x-variable and y stands for the t-variable.

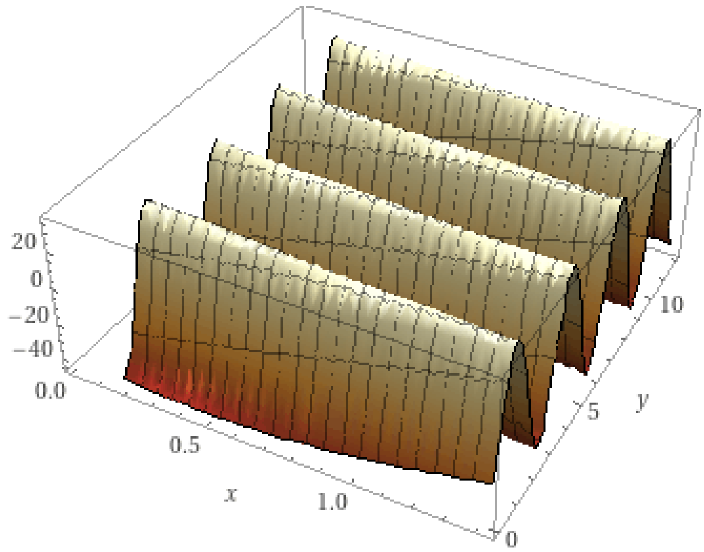

Figure 5. The approximate solution (35) for the parametrically excited Maccari system, where ρ_0 = 0.2, f = 0.01, σ = −0.04, $\phi = 0.02$, y = 5, and $\tau_0 = \theta_0 = 0$. Note that in the figure y stands for the τ-variable.

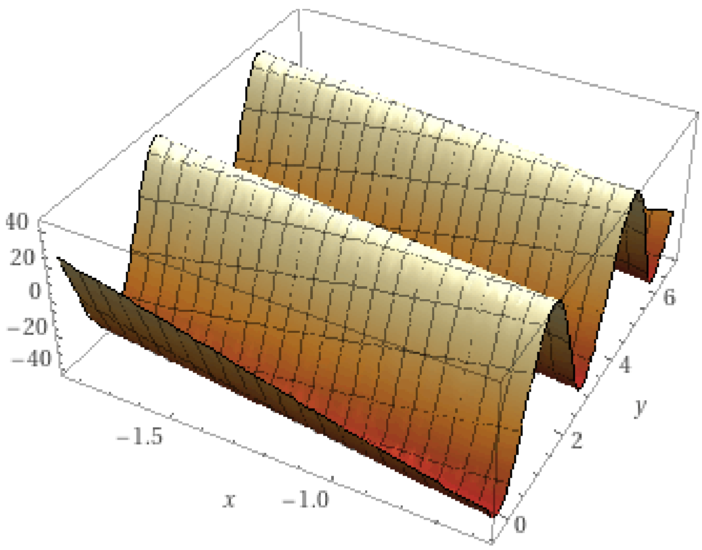

Figure 6. The approximate solution (35) for the parametrically excited Maccari system, where ρ_0 = 0.2, f = 0.01, σ = −0.04, $\phi = 0.02$, t = 10, and $\tau_0 = 0, \theta_0 = 0$. Note that in the figure x stands for the x-variable and y stands for the y-variable.

5. Chaotic and Fractal Solutions

We can assume that the function $\phi(x)$ as an arbitrary function of (x) (see (9) and (22)), from where we can then produce a fractal or chaotic solution. This topic has been widely considered in the general sense in [26]; here, we describe only fractal solutions.

We use the fractal property of the continuous but nowhere differentiable Weierstrass function.

$$W(x) = \sum_{k=1}^{N} (c_1)^k sin\left((c_2)^k x\right),\ N \to +\infty \tag{36}$$

with c_2 odd and

$$c_1 c_2 > \frac{3\pi}{2} \tag{37}$$

A fractal solution for the associated real scalar field (35) of the parametrically excited Maccari system is (see Figures 7 and 8 for two examples)

$$u(x,y,t) = 2\rho(x,y,t)cos\left(\theta(x,y,t) + K_x x + K_y y - \frac{\Omega}{2}t\right) + W(x) \tag{38}$$

where

$$\theta - \theta_0 = arctan(tan(\tau_0 - \tau)(W(x) - \sigma + f)) \tag{39}$$

$$\rho(x,y,t) = \frac{\rho_0}{-2f} log|f cos 2\theta(x,y,t) + \sigma - W(x)| \tag{40}$$

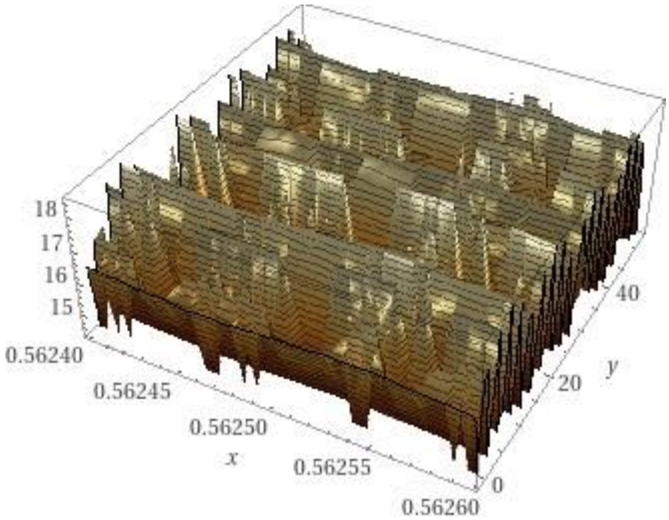

Figure 7. The approximate fractal solution (35) for the parametrically excited Maccari system, where ρ_0 = 0.08, f = −0.02, σ = 0.06, y = 0.7, ϕ is the Weierstrass function with c_1 = 0.02 and c_2 = 251, see Equation (36), and $\tau_0 = \theta_0 = 0$. Note that in the figure y stands for time τ.

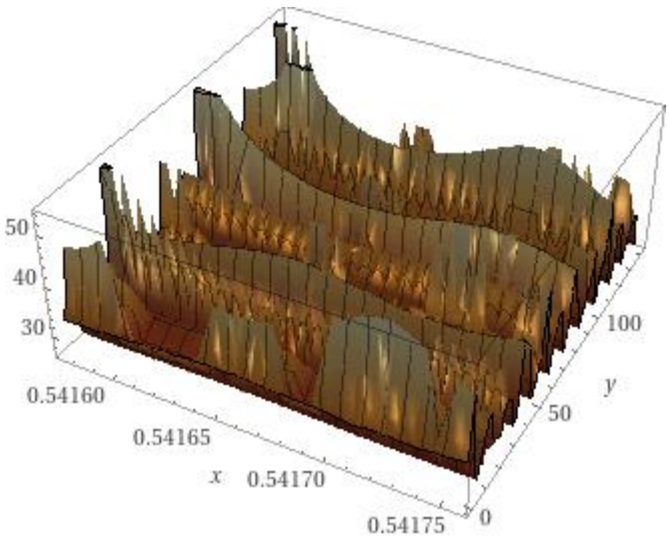

Figure 8. The approximate fractal solution (35) for the parametrically excited Maccari system, where ρ_0 = 0.24, f = −0.02, σ = 0.05, ϕ is the Weierstrass function (36) with c_1 = 0.02 and c_2 = 251, and $\tau_0 = \theta_0 = 0$. Note that in the figure y stands for y-variable.

6. Conclusions

The Maccari system in parametric resonance with a generic mode was studied. The well-known AP perturbation method, previously used for other nonlinear partial differential equations, was used to obtain a model system formed by two coupled nonlinear equations, describing the temporal evolution for the solution phase and amplitude.

We investigated the model system and described the solution behavior, demonstrating the existence of a wave train of arbitrary amplitude. If the parametric strength increases from very small values, then the solution decreases, but it is proportional to the arbitrary initial condition on the solution amplitude.

Carefully chosen initial conditions produce a rogue waves train for the approximate solution of the nonlinear Equations (3) and (4). This behavior has not been observed yet in experimental situations. We conclude that the parametrically excited Maccari system is a rogue wave generator and we can choose the wave amplitude.

Moreover, if we carefully choose the initial conditions, the function $\phi = \phi(x)$ being independent from the y-coordinate and time t (see (9) and (23)), then we can obtain fractal or chaotic solutions.

This AP method could be applied to other resonances of NPDEs in nonlinear physics.

We demonstrated that a linearized perturbation is not adequate to describe the system's behavior. Therefore, a careful nonlinear analysis is necessary, done in such a way to design a control using the relevant pulses without studying an integro-differential equation.

We conclude that a sinusoidal pulse control is able to produce rogue waves; however, in the future, other pulse control functions should be investigated.

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