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Some Existence and Uniqueness Results for a Class of Fractional Stochastic Differential Equations

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Abstract: Many techniques have been recently used by various researchers to solve some types of symmetrical fractional differential equations. In this article, we show the existence and uniqueness to the solution of ς-Caputo stochastic fractional differential equations (CSFDE) using the Banach fixed point technique (BFPT). We analyze the Hyers–Ulam stability of CSFDE using the stochastic calculus techniques. We illustrate our results with three examples.

Keywords: fractional calculus; fixed-point theory

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1. Introduction

Fractional calculus is a mathematical axis studying the characterizations of non-integer order derivatives and integrals [1,2]. In fact, this field contains the methods and notions of solving symmetrical differential equations with fractional derivatives. The theory of fractional calculus began almost in the same decade as the definition of classical calculus was decided. It was first defined in Leibniz's letter to L'Hospital in 1695, where the notion of semi-derivative was presented. During this period, fractional derivative was founded by many famous scientists, e.g., Riemann, Lagrange, Liouville, Fourier, Grünwald, Euler, Heaviside, Abel, etc. The fractional calculus has been used to describe many real-world phenomena: control theory, electrical networks, fluid flow, optics and signal processing, dynamical processes, etc. (see [1,3–7]). Particularly, in [8], the authors analyzed a system of neural networks in the sense of fractional derivatives. In [4], some novel applications of the non-integer order operators in the theory of viscoelasticity were derived. The authors of ref. [9] have proposed a scheme of approximate non-integer order differentiation, including noise immunity. A fruitful discussion on the Adams method in the fractional-order sense was given in the ref. [10]. In the last few decades, some new fractional derivatives have been introduced by various researchers to improve the literature on fractional calculus. In [11], Almeida suggested a new fractional derivative with respect to a kernel function called ς-Caputo fractional derivative, and generalized the work of several researchers [1,12]. In this context, several research papers showed interest in the ς-Caputo fractional derivative; for instance, see [11,13,14]. In [15], a numerical study on the non-integer order relaxation–oscillation equations in terms of ς-Caputo fractional derivatives are proposed. In [16], a study on the Ulam stability for Langevin non-integer order differential equations in the sense of two different fractional orders of ς-Caputo derivative has been given. In [17], the authors explored an initial value problem for differential equations in the sense of ς-Caputo derivative via a monotone iterative approach.

Recently, the theory of Hyers–Ulam stability (HUS) has attracted the attention of several famous scientists due to its real-world applications in biology and fluid flow, where identifying the explicit solutions is a very hard task. Some novel research studies on this topic have been proposed in the following references [18–20]. In [21], the authors discussed the results regarding the existence and HUS of solutions for almost periodic stochastic differential equations in a fractional sense. In [22], some novel results on the existence and HUS of random stochastic impulsive functional differential equations with delay have been established. In [23], Ulam stability for partial integro-differential equations with uncertainty in a fractional-order sense has been explored. Most of the existing papers consider the Caputo fractional derivative for the existence, uniqueness and HUS of the solutions of fractional differential equations. There are a lot of papers which discuss the ψ-Caputo fractional derivative (see [24–26]) for the deterministic case. In this paper, we have studied this concept for the stochastic case.

In this work, the existence and uniqueness of CSFDE are provided. The HUS for the proposed problem with the help of the novel features of stochastic calculus is simulated.

This paper extends the work on [27–29] for the Caputo and Caputo–Hadamard fractional derivative.

We highlight the main advantages of our article as follows:

- To investigate the existence and uniqueness of the solution of CSFDE via BFPT.
- To investigate the HUS of CSFDE by using the stochastic calculus techniques.

We summarize the content of the article: Section 2 presents the basic definitions of ς-CFD and some fundamental notations. Section 3 investigates the global existence and uniqueness of the solution of CSFDE. In Section 4, we analyze the HUS of CSFDE. In Section 5, we give three illustrative examples.

2. Basic Notions

Denote by $\{\Sigma, \mathcal{F}, \mathbb{F}_\Pi, \mathbb{P}\}$, where $\mathbb{F}_\Pi = \{\mathbb{F}_\eta\}_{\eta\in[1,\Pi]}$ and $\Pi > 1$, the complete probability space; $W(\eta)$ is the standard Brownian motion.

Let $\mathcal{X}_\eta = L^2(\Sigma, \mathbb{F}_\eta, \mathbb{P})$ (for every $\eta \in [1,\Pi]$) be the family of all $\mathbb{F}_\eta$-measurable and mean square integrable functions $\lambda = (\lambda_1, \dots, \lambda_p)^T : \Sigma \to \mathbb{R}^p$ satisfies

$$||\lambda||_{ms} = \sqrt{\sum_{l=1}^{p} \mathbb{E}(|\lambda_l|^2)} = \sqrt{\mathbb{E}||\lambda||^2},$$

where $||\cdot||$ is the usual Euclidian norm.

Definition 1 ([14])**.** *Denote by $\varphi > 0$ and let $\varsigma \in C^1[c,b]$ the function satisfying $\varsigma'(\sigma) \neq 0$, $\forall\sigma \in [c,b]$. The ς-fractional integral of order φ for an integrable function g is defined as*

$$I^{\varphi,\varsigma}_{c^+} g(x) = \frac{1}{\Gamma(\varphi)} \int_c^x \varsigma'(\sigma)(\varsigma(x) - \varsigma(\sigma))^{\varphi-1} g(\sigma) d\sigma. \quad (1)$$

Definition 2 ([14])**.** *Denote by $\varphi > 0$ and let $\varsigma \in C^1[c,b]$ the function satisfying $\varsigma'(\sigma) \neq 0$, $\forall\sigma \in [c,b]$. The ς-Riemann–Liouville fractional derivative of order φ of a function g is defined by*

$$D^{\varphi,\varsigma}_{c^+} g(x) = \left(\frac{1}{\varsigma'(x)} \frac{d}{dx}\right) I^{1-\varphi,\varsigma}_{c^+} g(x). \quad (2)$$

Definition 3 ([14])**.** *Let $\varphi > 0$ and $\varsigma \in C^1[c,b]$ the functions satisfying $\varsigma'(\sigma) \neq 0$, $\forall \sigma \in [c,b]$. The ς-Caputo fractional derivative of order φ of a function g is defined by*

$$^{C}D^{\varphi,\varsigma}_{c^+} g(t) = D^{\varphi,\varsigma}_{c^+}\Big[g(t) - g(c)\Big]. \tag{3}$$

Definition 4 ([1])**.** *$E_{\rho,\kappa}(y)$ is called a Mittag–Leffler function with two parameters if:*

$$E_{\rho,\kappa}(y) = \sum_{m=0}^{+\infty} \frac{y^m}{\Gamma(m\rho + \kappa)},$$

where $\rho > 0$, $\kappa > 0$, $y \in \mathbb{C}$.

Theorem 1 ([30])**.** *Let $(\mathbb{E}, d)$ be a complete metric space and let $\mathcal{B} : \mathbb{E} \to \mathbb{E}$ (with $z \in [0,1)$) be a contraction. Assume that $j \in \mathbb{E}$, $d(j, \mathcal{B}(j)) \leq v$ and $v > 0$. Then, there is a unique $u \in \mathbb{E}$ such that $\mathcal{B}(u) = u$.*

Let the following CSFDE:

$$^{C}D^{\varphi,\varsigma}_{a^+} \xi(\eta) = f_1(\eta, \xi(\eta)) + f_2(\eta, \xi(\eta)) \frac{dW(\eta)}{d\eta}, \tag{4}$$

where the initial condition is $\xi(a) = \delta$, $\varsigma : [a,\Pi] \to \mathbb{R}$ be a C^1-increasing function with $\varsigma'(\eta) \neq 0$, $\forall \eta \in [a,\Pi]$, $0 < \varphi < 1$, $f_1 : [a,\Pi] \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ and $f_2 : [a,\Pi] \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ are measurable functions.

Let the following hypothesis:

$\mathcal{H}_1$: There is $L > 0$ satisfying

$$||f_1(\eta,\xi_1) - f_1(\eta,\xi_2)|| \vee ||f_2(\eta,\xi_1) - f_2(\eta,\xi_2)|| \leq L||\xi_1 - \xi_2||, \tag{5}$$

for all $(\eta, \xi_1, \xi_2) \in [a,\Pi] \times \mathbb{R}^p \times \mathbb{R}^p$.

$\mathcal{H}_2$: $f_1(\cdot, 0)$ and $f_2(\cdot, 0)$ satisfying

$$||f_2(\cdot,0)||_\infty = \text{ess} \sup_{\eta \in [a,\Pi]} ||f_2(\eta,0)|| < \infty, \tag{6}$$

$$\int_a^\Pi ||f_1(\sigma,0)||^2 d\sigma < \infty.$$

3. Existence and Uniqueness of Solutions

Denote by $\mathbb{H}^2([a,\Pi])$ the family of all the processes ξ which are $\mathbb{F}_\Pi$-adapted, measurable such that

$$||\xi||_{\mathbb{H}^2} = \sup_{a \leq r \leq \Pi} ||\xi(r)||_{ms} < \infty.$$

It is not hard to prove that $(\mathbb{H}^2([a,\Pi]), || \cdot ||_{\mathbb{H}^2})$ is a Banach space. Let the operator $N_\delta : \mathbb{H}^2([a,\Pi]) \to \mathbb{H}^2([a,\Pi])$, for $\delta \in \mathcal{X}_a$, given by:

$$\begin{aligned} N_\delta y(\eta) &= \delta + \frac{1}{\Gamma(\varphi)}\left[\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta) - \varsigma(\sigma))^{\varphi-1} f_1(\sigma, y(\sigma)) d\sigma\right] \\ &+ \frac{1}{\Gamma(\varphi)}\left[\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta) - \varsigma(\sigma))^{\varphi-1} f_2(\sigma, y(\sigma)) dW(\sigma)\right]. \end{aligned} \tag{7}$$

Lemma 1. *N_δ, for every $\sigma \in \mathcal{X}_a$, is well defined.*

Proof. Let $q \in \mathbb{H}^2([a,\Pi])$. Then, one has

$$||N_\delta q(\eta)||_{ms}^2 \leq 3||\delta||_{ms}^2 + \frac{3}{\Gamma(\varphi)^2}\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1} f_1(\sigma,q(\sigma))d\sigma\right|\right|^2\right) + \frac{3}{\Gamma(\varphi)^2}\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1} f_2(\sigma,q(\sigma))dW(\sigma)\right|\right|^2\right). \quad (8)$$

Using the Cauchy–Schwartz inequality, one gets

$$\begin{aligned}
&\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1} f_1(\sigma,q(\sigma))d\sigma\right|\right|^2\right)\\
\leq\ & \left(\int_a^\eta (\varsigma'(\sigma))^2(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}d\sigma\right)\mathbb{E}\left(\int_a^\eta ||f_1(\sigma,q(\sigma))||^2 d\sigma\right)\\
\leq\ & M\left(\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}d\sigma\right)\mathbb{E}\left(\int_a^\eta ||f_1(\sigma,q(\sigma))||^2 d\sigma\right)\\
\leq\ & \frac{M}{2\varphi-1}(\varsigma(\eta)-\varsigma(a))^{2\varphi-1}E\left(\int_a^\eta ||f_1(\sigma,q(\sigma))||^2 d\sigma\right), \quad (9)
\end{aligned}$$

where $M = \sup\limits_{\sigma\in[a,\Pi]} \varsigma'(\sigma)$. By $\mathcal{H}_1$, one can derive that

$$||f_1(\sigma,q(\sigma))||^2 \leq 2L^2||q(\sigma)||^2 + 2||f_1(\sigma,0)||^2. \quad (10)$$

Thus,

$$E\left(\int_a^\eta ||f_1(\sigma,q(\sigma))||^2 d\sigma\right) \leq 2L^2(\Pi-a)\sup_{\sigma\in[a,\Pi]} E\left(||q(\sigma)||^2\right) + 2\int_a^\Pi ||f_1(\sigma,0)||^2 d\sigma. \quad (11)$$

Then,

$$\begin{aligned}
&\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1} f_1(\sigma,q(\sigma))d\sigma\right|\right|^2\right)\\
\leq\ & \frac{M(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}}{2\varphi-1}\left[2L^2(\Pi-a)\sup_{\sigma\in[a,\Pi]} E\left(||q(\sigma)||^2\right) + 2\int_a^\Pi ||f_1(\sigma,0)||^2 d\sigma\right]. \quad (12)
\end{aligned}$$

Using Itô's isometry formula, one gets

$$\begin{aligned}
&\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1} f_2(\sigma,q(\sigma))dW(\sigma)\right|\right|^2\right)\\
=\ & \mathbb{E}\left(\int_a^\eta (\varsigma'(\sigma))^2(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}||f_2(\sigma,q(\sigma))||^2 d\sigma\right). \quad (13)
\end{aligned}$$

Using $\mathcal{H}_1$, one has

$$||f_2(\sigma,q(\sigma))||^2 \leq 2L^2||q(\sigma)||^2 + 2||f_2(\cdot,0)||_\infty^2. \quad (14)$$

Hence,

$$\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1} f_2(\sigma,q(\sigma))dW(\sigma)\right|\right|^2\right)$$

$$
\begin{aligned}
&\leq \; 2ML^2\mathbb{E}\left(\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}||q(\sigma)||^2 d\sigma\right)\\
&+ \; 2M||f_2(\cdot,0)||_\infty^2 \int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2} d\sigma\\
&\leq \; \frac{2ML^2}{2\varphi-1}(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}||q||_{\mathbb{H}_2}^2 + \frac{2M}{2\varphi-1}(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}||f_2(\cdot,0)||_\infty^2. \quad (15)
\end{aligned}
$$

Therefore, N_δ is well defined. □

Theorem 2. *Under $\mathcal{H}_1$ and $\mathcal{H}_2$, for every $\sigma \in \mathcal{X}_a$, Equation (4) has a unique global solution $\xi(\cdot,\sigma)$ on $[a,\Pi]$.*

Proof. Let $\Pi > a$ be arbitrary. Let $\theta > 0$, such that $\theta^{2\varphi-1} > 2L^2M(\Pi+1)\frac{\Gamma(2\varphi-1)}{\Gamma(\varphi)^2}$. We define a norm $||\cdot||$ on the space $\mathbb{H}^2([a,\Pi])$ by

$$
||\xi||_\theta = \sup_{\eta\in[a,\Pi]} \sqrt{\frac{\mathbb{E}\left(||\xi(\eta)||^2\right)}{e^{\theta(\varsigma(\eta)-\varsigma(a))}}}, \quad \forall \xi \in \mathbb{H}^2([a,\Pi]). \tag{16}
$$

It is not hard to show that $||\cdot||_{\mathbb{H}^2}$ and $||\cdot||_\theta$ are equivalent. Consequently, $(\mathbb{H}^2([a,\Pi]), ||\cdot||_\theta)$ is a Banach space.

Let $\xi_1, \xi_2 \in \mathbb{H}^2([a,\Pi])$. Using (7), we get $\forall \eta \in [a,\Pi]$

$$
\mathbb{E}\left(||N_\delta\xi_1(\eta) - N_\delta\xi_2(\eta)||^2\right)
$$

$$
\begin{aligned}
&\leq \; \frac{2}{\Gamma(\varphi)^2}\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_1(\sigma,\xi_1(\sigma)) - f_1(\sigma,\xi_2(\sigma)))d\sigma\right|\right|^2\right)\\
&+ \; \frac{2}{\Gamma(\varphi)^2}\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_2(\sigma,\xi_1(\sigma)) - f_2(\sigma,\xi_2(\sigma)))dW(\sigma)\right|\right|^2\right).
\end{aligned}
$$

Using Hölder inequality, one has

$$
\begin{aligned}
&\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_1(\sigma,\xi_1(\sigma)) - f_1(\sigma,\xi_2(\sigma)))d\sigma\right|\right|^2\right)\\
&\leq L^2M(\eta-a)\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}\mathbb{E}\left(||\xi_1(\sigma)-\xi_2(\sigma)||^2\right)d\sigma.
\end{aligned}
$$

Moreover, using Itô isometry, we have

$$
\begin{aligned}
&\mathbb{E}\left(\left|\left|\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_2(\sigma,\xi_1(\sigma)) - f_2(\sigma,\xi_2(\sigma)))dW(\sigma)\right|\right|^2\right)\\
&= \; \mathbb{E}\left(\int_a^\eta (\varsigma'(\sigma))^2(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}||f_2(\sigma,\xi_1(\sigma)) - f_2(\sigma,\xi_2(\sigma))||^2 d\sigma\right)\\
&\leq \; L^2M\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}\mathbb{E}\left(||\xi_1(\sigma)-\xi_2(\sigma)||^2\right)d\sigma. \quad (17)
\end{aligned}
$$

Then,

$$
\mathbb{E}\left(||N_\delta\xi_1(\eta) - N_\delta\xi_2(\eta)||^2\right)
$$

$$\leq \frac{2L^2M}{\Gamma(\varphi)^2}(\Pi+1)\int_a^{\eta}\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}\mathbb{E}\Big(||\xi_1(\sigma)-\xi_2(\sigma)||^2\Big)d\sigma$$
$$= \frac{2L^2M}{\Gamma(\varphi)^2}(\Pi+1)\int_a^{\eta}\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}\frac{\mathbb{E}\Big(||\xi_1(\sigma)-\xi_2(\sigma)||^2\Big)}{e^{\theta(\varsigma(\sigma)-\varsigma(a))}}e^{\theta(\varsigma(\sigma)-\varsigma(a))}d\sigma$$
$$\leq \frac{2L^2M}{\Gamma(\varphi)^2}(\Pi+1)||\xi_1-\xi_2||_\theta^2\int_a^{\eta}\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}e^{\theta(\varsigma(\sigma)-\varsigma(a))}d\sigma. \tag{18}$$

Set $J=\int_a^{\eta}\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}e^{\theta(\varsigma(\sigma)-\varsigma(a))}d\sigma$. Thus, by using Lemma 2.6 in [16], we get

$$J \leq \frac{\Gamma(2\varphi-1)}{\theta^{2\varphi-1}}e^{\theta(\varsigma(\eta)-\varsigma(a))}. \tag{19}$$

Therefore, we have

$$\frac{\mathbb{E}\Big(||N_\delta\xi_1(\eta)-N_\delta\xi_2(\eta)||^2\Big)}{e^{\theta(\varsigma(\eta)-\varsigma(a))}} \leq \frac{2L^2M}{\Gamma(\varphi)^2}(\Pi+1)\frac{\Gamma(2\varphi-1)}{\theta^{2\varphi-1}}||\xi_1-\xi_2||_\theta^2. \tag{20}$$

Hence,

$$||N_\delta\xi_1-N_\delta\xi_2||_\theta \leq C||\xi_1-\xi_2||_\theta, \tag{21}$$

where $C=\sqrt{\frac{2L^2M}{\Gamma(\varphi)^2}(\Pi+1)\frac{\Gamma(2\varphi-1)}{\theta^{2\varphi-1}}}$. Therefore, there is a unique solution of (4) such that $\xi(a)=\delta$. □

4. Hyers–Ulam Stability

In this section, we study the Hyers–Ulam stability of Equation (4) using the generalized Gronwall inequality and the stochastic calculus techniques.

Definition 5. *Equation (4) is Hyers–Ulam stable with respect to ϵ if there is a number $M_1>0$ satisfying for each $\epsilon>0$, and for each solution $y\in\mathbb{H}^2([a,\Pi])$, with $y(a)=\delta$, of the following inequality:*

$$\mathbb{E}\left|\left|y(\eta)-y(a)-\left(\int_a^{\eta}\frac{\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}}{\Gamma(\varphi)}(f_1(\sigma,y(\sigma))d\sigma+f_2(\sigma,y(\sigma))dW(\sigma))\right)\right|\right|^2 \leq \epsilon, \tag{22}$$

for all $\eta\in[a,\Pi]$, there exists a solution $\xi\in\mathbb{H}^2([a,\Pi])$ of (4), with $\xi(a)=\delta$, such that

$$\mathbb{E}||y(\eta)-\xi(\eta)||^2 \leq M_1\epsilon,\ \forall\eta\in[a,\Pi].$$

Theorem 3. *Under Assumptions $\mathcal{H}_1$-$\mathcal{H}_2$, the ς-Caputo stochastic fractional differential Equation (4) are Hyers–Ulam stable with respect to ϵ on $[a,\Pi]$.*

Proof. Let $\epsilon>0$ and $y\in\mathbb{H}^2([a,\Pi])$ be a function satisfying (22) and denote by $\xi\in\mathbb{H}^2([a,\Pi])$ the solution of (4) with initial data $y(a)$; thus

$$\xi(\eta)=y(a)+\frac{1}{\Gamma(\varphi)}\left[\int_a^{\eta}\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_1(\sigma,\xi(\sigma))d\sigma+f_2(\sigma,\xi(\sigma))dW(\sigma))\right]. \tag{23}$$

Thus,

$$\mathbb{E}||y(\eta)-\xi(\eta)||^2$$
$$\leq 2\mathbb{E}||y(\eta)-y(a)-\frac{1}{\Gamma(\varphi)}(\int_a^{\eta}\varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}[f_1(\sigma,y(\sigma))d\sigma$$
$$+f_2(\sigma,y(\sigma))dW(\sigma)])||^2$$

$$+2\mathbb{E}||\frac{1}{\Gamma(\varphi)}(\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}[(f_1(\sigma,y(\sigma))-f_1(\sigma,\xi(\sigma)))d\sigma$$
$$+(f_2(\sigma,y(\sigma))-f_2(\sigma,\xi(\sigma)))dW(\sigma)])||^2.$$

Then, applying assumptions $\mathcal{H}_1$-$\mathcal{H}_2$ and Cauchy–Schwartz inequality, we have

$$\mathbb{E}||y(\eta)-\xi(\eta)||^2$$

$$\begin{aligned}
&\leq 2\epsilon+4\mathbb{E}\left|\left|\frac{1}{\Gamma(\varphi)}\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_1(\sigma,y(\sigma))-f_1(\sigma,\xi(\sigma)))d\sigma\right|\right|^2\\
&+ 4\mathbb{E}\left|\left|\frac{1}{\Gamma(\varphi)}\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{\varphi-1}(f_2(\sigma,y(\sigma))-f_2(\sigma,\xi(\sigma)))dW(\sigma)\right|\right|^2\\
&\leq 2\epsilon+\frac{4L^2M(\varsigma(\eta)-\varsigma(a))^{2\varphi-1}}{(2\varphi-1)\Gamma(\varphi)^2}\mathbb{E}\left(\int_a^\eta ||y(\sigma)-\xi(\sigma)||^2d\sigma\right)\\
&+ \frac{4L^2M}{\Gamma(\varphi)^2}\mathbb{E}\left(\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}||y(\sigma)-\xi(\sigma)||^2d\sigma\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}||y(\eta)-\xi(\eta)||^2 &\leq 2\epsilon+\frac{4L^2M(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}}{(2\varphi-1)\Gamma(\varphi)^2}\int_a^\eta \mathbb{E}||y(\sigma)-\xi(\sigma)||^2d\sigma\\
&+ \frac{4L^2M}{\Gamma(\varphi)^2}\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}\mathbb{E}||y(\sigma)-\xi(\sigma)||^2d\sigma. \quad (24)
\end{aligned}$$

Set $z(\eta)=\mathbb{E}||y(\eta)-\xi(\eta)||^2$. Thus, one gets

$$z(\eta)\leq \alpha_1+\alpha_2\int_a^\eta z(\sigma)d\sigma+\alpha_3\int_a^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{2\varphi-2}z(\sigma)d\sigma, \quad (25)$$

where $\alpha_1=2\epsilon$, $\alpha_2=\dfrac{4L^2M(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}}{(2\varphi-1)\Gamma(\varphi)^2}$ and $\alpha_3=\dfrac{4L^2M}{\Gamma(\varphi)^2}$.

Applying the generalized Gronwall inequality (see [31]), we have

$$\begin{aligned}
z(\eta) &\leq \left[\alpha_1+\alpha_2\int_a^\eta z(\sigma)d\sigma\right]E_{2\varphi-1}\left(\alpha_3\Gamma(2\varphi-1)(\varsigma(\eta)-\varsigma(a))^{2\varphi-1}\right)\\
&\leq \alpha_4+\alpha_5\int_a^\eta z(\sigma)d\sigma, \quad (26)
\end{aligned}$$

where $\alpha_4=2\epsilon E_{2\varphi-1}\left(\alpha_3\Gamma(2\varphi-1)(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}\right)$ and $\alpha_5=\dfrac{4L^2M(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}}{(2\varphi-1)\Gamma(\varphi)^2}$ $E_{2\varphi-1}\left(\alpha_3\Gamma(2\varphi-1)(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}\right)$.

Applying the classical Gronwall inequality, we can derive that

$$z(\eta)\leq \alpha_4 e^{\alpha_5(\eta-a)}\leq \alpha_4 e^{\alpha_5(\Pi-a)}. \quad (27)$$

Hence,

$$z(\eta)\leq M_1\epsilon, \quad (28)$$

where $M_1=2E_{2\varphi-1}\left(\alpha_3\Gamma(2\varphi-1)(\varsigma(\Pi)-\varsigma(a))^{2\varphi-1}\right)e^{\alpha_5(\Pi-a)}$.

Therefore, Equation (4) is Hyers–Ulam stable with respect to ϵ. □

5. Examples

This section is devoted to show our results in three examples.

Example 1. *Let the CSFDE for each $\epsilon > 0$ and for $\eta \in [1, e^2]$, given by*

$$\begin{cases} {}^{C}D_{1^+}^{\frac{2}{3},\varsigma}\xi(\eta) = f_1(\eta,\xi(\eta)) + f_2(\eta,\xi(\eta))\frac{dW(\eta)}{d\eta}, \\ \mathbb{E}\Big|y(\eta) - y(1) - \frac{1}{\Gamma(\varphi)}\Big(\int_1^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{-\frac{1}{3}}(f_1(\sigma,y(\sigma))d\sigma + f_2(\sigma,y(\sigma))dW(\sigma))\Big)\Big|^2 \leq \epsilon, \\ y(1) = \delta, \end{cases} \tag{29}$$

where $\varphi = \frac{2}{3}$, $\varsigma(\eta) = \ln(\eta)$ and

$$\begin{aligned} \xi(\eta) &\in \mathbb{H}^2([1,e^2],\mathbb{R}) \\ f_1(\eta,\xi(\eta)) &= \sqrt{\ln(\eta)}(\arctan(\xi(\eta)) + \cos(\xi(\eta))) \\ f_2(\eta,\xi(\eta)) &= \sqrt{\eta}\cos(\xi(\eta)). \end{aligned}$$

We will prove that Equation (29) is Hyers–Ulam stable with respect to ϵ.

Let $(\eta,\xi_1,\xi_2) \in [1,e^2]\times\mathbb{R}\times\mathbb{R}$, thus

$$|f_1(\eta,\xi_1) - f_1(\eta,\xi_2)| \leq 4|\xi_1 - \xi_2|,$$

and

$$|f_2(\eta,\xi_1) - f_2(\eta,\xi_2)| \leq e|\xi_1 - \xi_2|.$$

Hence, assumption $\mathcal{H}_1$ fulfilled. Moreover,

$$||f_2(\cdot,0)||_\infty = \text{ess} \sup_{\eta\in[1,e^2]} |f_2(\eta,0)| \leq e,$$

and

$$\int_1^{e^2} |f_1(\eta,0)|^2 d\eta \leq 2(e^2+1).$$

Thus, assumptions $\mathcal{H}_1$-$\mathcal{H}_2$ fulfilled. Hence, applying Theorem 3, Equation (29) has a unique solution, and it is Hyers–Ulam stable with respect to ϵ on $[1,e^2]$.

Example 2. *Let the CSFDE for each $\epsilon > 0$ and for $\eta \in [0.5, 6]$, given by*

$$\begin{cases} {}^{C}D_{1^+}^{\frac{3}{4},\varsigma}\xi(\eta) = f_1(\eta,\xi(\eta)) + f_2(\eta,\xi(\eta))\frac{dW(\eta)}{d\eta}, \\ \mathbb{E}\Big|y(\eta) - y(0.5) - \frac{1}{\Gamma(\varphi)}\Big(\int_{0.5}^\eta \varsigma'(\sigma)(\varsigma(\eta)-\varsigma(\sigma))^{-\frac{1}{4}}(f_1(\sigma,y(\sigma))d\sigma + f_2(\sigma,y(\sigma))dW(\sigma))\Big)\Big|^2 \leq \epsilon, \\ y(0.5) = \delta, \end{cases} \tag{30}$$

where $\varphi = \frac{3}{4}$, $\varsigma(\eta) = \sqrt{\eta}$ and

$$\begin{aligned} \xi(\eta) &\in \mathbb{H}^2([0.5,6],\mathbb{R}) \\ f_1(\eta,\xi(\eta)) &= \frac{e^\eta}{1+e^\eta}(1+\xi(\eta)) \\ f_2(\eta,\xi(\eta)) &= \frac{1+\sin(\xi(\eta))}{(1+\eta)^2}. \end{aligned}$$

We will prove that Equation (31) is Hyers–Ulam stable with respect to ϵ.

$(\eta, \xi_1, \xi_2) \in [0.5, 6] \times \mathbb{R} \times \mathbb{R}$, then

$$|f_1(\eta, \xi_1) - f_1(\eta, \xi_2)| \leq |\xi_1 - \xi_2|,$$

and

$$|f_2(\eta, \xi_1) - f_2(\eta, \xi_2)| \leq |\xi_1 - \xi_2|.$$

Thus, assumption $\mathcal{H}_1$ holds. On the other hand,

$$||f_2(\cdot, 0)||_\infty = \text{ess} \sup_{\eta \in [0.5,6]} |f_2(\eta, 0)| \leq 1,$$

and

$$\int_{0.5}^{6} |f_1(\eta, 0)|^2 d\eta \leq \ln(1 + e^6).$$

Then, assumptions $\mathcal{H}_1$-$\mathcal{H}_2$ are fulfilled. Hence, applying Theorem 3, Equation (31) has a unique solution, and it is Hyers–Ulam stable with respect to ϵ on $[0.5, 6]$.

Example 3. *Let the CSFDE, for each $\epsilon > 0$ and for $\eta \in [0, 5]$, given by*

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{5}, \varsigma} \xi(\eta) = f_1(\eta, \xi(\eta)) + f_2(\eta, \xi(\eta)) \frac{dW(\eta)}{d\eta}, \\ \mathbb{E}\Big|y(\eta) - y(0) - \frac{1}{\Gamma(\varphi)} \Big(\int_0^\eta \varsigma'(\sigma)(\varsigma(\eta) - \varsigma(\sigma))^{-\frac{4}{5}} (f_1(\sigma, y(\sigma))d\sigma + f_2(\sigma, y(\sigma))dW(\sigma)) \Big) \Big|^2 \leq \epsilon, \\ y(0) = \delta, \end{cases} \tag{31}$$

where $\varphi = \frac{1}{5}$, $\varsigma(\eta) = \eta$ and

$$\begin{aligned} \xi(\eta) &\in \mathbb{H}^2([0, 5], \mathbb{R}) \\ f_1(\eta, \xi(\eta)) &= 2e^{-\eta} \xi(\eta) \\ f_2(\eta, \xi(\eta)) &= 3 \sin(\xi(\eta)). \end{aligned}$$

We will prove that Equation (31) is Hyers–Ulam stable with respect to ϵ.

$(\eta, \xi_1, \xi_2) \in [0, 5] \times \mathbb{R} \times \mathbb{R}$, then

$$|f_1(\eta, \xi_1) - f_1(\eta, \xi_2)| \leq 2|\xi_1 - \xi_2|,$$

and

$$|f_2(\eta, \xi_1) - f_2(\eta, \xi_2)| \leq 3|\xi_1 - \xi_2|.$$

Thus, assumption $\mathcal{H}_1$ hold. On the other hand,

$$||f_2(\cdot, 0)||_\infty = \text{ess} \sup_{\eta \in [0,5]} |f_2(\eta, 0)| = 0,$$

and

$$\int_0^5 |f_1(\eta, 0)|^2 d\eta = 0.$$

Then, assumptions $\mathcal{H}_1$-$\mathcal{H}_2$ are fulfilled. Hence, applying Theorem 3, Equation (31) has a unique solution, and it is Hyers–Ulam stable with respect to ϵ on $[0, 5]$.

6. Conclusions

In this research paper, we have proved the existence and uniqueness of CSFDE. We have simulated the HUS for the proposed problem with the help of the novel features of stochastic calculus. We have illustrated three examples to justify the correctness and applicability of the proposed results. The applications of some well-known terms of functional analysis, such as the Cauchy–Schwarz inequality, properties of measurable functions, supremum norm, Itô's isometry formula, Hölder inequality, and generalized Gronwall inequality make the study more visible to the literature. The proposed results will be very useful to prove the existence of a unique solution and Hyers–Ulam stability of ς-Caputo type fractional stochastic differential equations.

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