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Queue-Size Distribution in a Discrete-Time Finite-Capacity Model with a Single Vacation Mechanism

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Abstract: In the paper a finite-capacity discrete-time queueing system with geometric interarrival times and generally distributed processing times is studied. Every time when the service station becomes idle it goes for a vacation of random duration that can be treated as a power-saving mechanism. Application of a single vacation policy is one way for the system to achieve symmetry in terms of system operating costs. A system of differential equations for the transient conditional queue-size distribution is established. The solution of the corresponding system written for double probability generating functions is found using the analytical method based on a linear algebraic approach. Moreover, the representation for the probability-generating function of the stationary queue-size distribution is obtained. Numerical study illustrating theoretical results is attached as well.

Keywords: discrete time; energy saving; finite capacity; queue-size distribution; single vacation

1. Introduction

Queueing models with a discrete-time parameter are an important alternative to the systems with continuous time which are used most often in modeling. In fact, continuous-time systems were initially proposed in the works of Erlang, a precursor of queueing theory, who assumed—in accordance with the nature of telecommunication traffic—a continuous distribution of the duration of a telephone call. Until 1958, the year of the publication of Torben Meisling's work [1], they were almost exclusively used and analyzed in queueing theory.

The special application of discrete-time models is associated with the modeling of processing (sending) packets in the nodes of digital telecommunication and computer networks. In discrete models time is divided into fixed-length intervals (called slots), which, in fact, play the role of the smallest undivided time units. Because, as a rule, time slots are short, it is not important for practice at what moment they are affected or terminated. We will refer, for clarity, to the start of the time slot. One can find the overview of main results related to discrete-time queueing models, e.g., in [2–4]. One of the first numerical analyses of such models was proposed in [5].

One of the key research areas in the field of design and development of wireless communication is the issue of energy saving. In the literature, different-type approaches and solutions are proposed. One of them is the exhaustive single vacation policy in that the service station (that can be identified as a network node) is switched off for a certain period (deterministic or random) whenever the system empties (the buffer does not contain waiting jobs and the service station is idle). The cost of maintaining a server in a ready-to-process state is generally significantly higher than the cost of mere job accumulation during the non-servicing period. Of course, suspension of service involves an increase in waiting times and the subsequent risk of losing jobs due to buffer overload. In consequence, such a policy is one way for the system to achieve symmetry in terms of system operating costs. Single vacation policy can be used in the modeling of the operation of mobile network nodes (e.g. working under the IEEE 802.16e Mobile WiMAX protocol) in which sleep and active

periods change cyclically. Moreover, in the automated production line, if the accumulating buffer becomes empty, machine conservation is sometimes planned that can be treated as a machine vacation.

In the paper, we study the queue-size distribution in a finite-buffer discrete-type model with a single vacation policy in the transient state. Non-stationary results for different-type queueing models are much rarer than those obtained for the steady state of the system. However, transient analysis is sometimes more desirable. For example, a system in which the traffic intensity is sparse (such a situation can be observed, e.g., in wireless sensor network node) or the traffic/processing rates change quite often, or a new control mechanism is implemented, requires rather time-dependent analysis than the stationary one. Applying the total probability law with respect to the first departure epoch after the opening of the system, we build a system of differential equations for the queue-size distribution at the fixed moment t conditioned by the buffer state at the starting time $t = 0$. The solution of the corresponding system written for double probability generating functions (PGFs for short) is derived in a compact form with the support of a linear algebraic approach.

The overview of different-type models with service interruptions can be found in [6]. A discrete-time model described by geometric distributions operating under a multiple vacation policy was investigated in [7]. Stationary distributions of the queue size and waiting time were obtained there using quasi-birth-and-death chain and matrix-geometric solution methods. In [8] an infinite-buffer discrete-time $GI/Geo/1$ model with multiple vacation policies was studied. By using the matrix-geometric approach stationary representations for the queue size and waiting time distributions were obtained. The same queueing system was investigated in [9] in the case of a single vacation scheme. A modification of the classical single/multiple vacation policy for the discrete-time model was considered in [10] where server vacations occur whenever the queue becomes empty or whenever a timer expires. In [11] a class of single-server discrete-time queueing models with server vacations, individual arrivals, and non-batch service was considered. It was shown that, under the assumption that interarrival, service, vacation, and server operational times can be cast with Markov-based representation, the class can be tractable as a matrix–geometric or a matrix-product problem. A $Geo/Geo/1$-type queue with a single vacation mechanism was studied in [12]. In [13] a more general $GI/Geo/1$-type model was investigated. Representations for stationary distributions of the queue size at arrivals epochs and of the waiting time for an arbitrary job were found there using the matrix-geometric solution method. The supplementary variable technique was used in [14] to find main performance measures in a discrete-time $Geo/G/1$ model operating under multiple vacations and with setup-closedown times. Discrete-time queueing model $GI/Geo/1$ with a single vacation policy was considered in [15]. The case of a multiple vacation policy was analyzed in [16]. The additional strategy of the N-policy was studied in [17]. One can find the solution to the problem of equilibrium strategies in discrete-time Markovian queues working under single and multiple vacations in [18]. In [19] (see also [20]) another mechanism limiting the access to the service station was considered, namely a dropping function "filtering" the arrival stream in dependence on the current queue size. An alternative strategy was studied in [21] where a flexible queueing system was analyzed that adapts to the system size by using a single or a bulk service discipline. Transient results for queueing models with vacation policies can be found e.g., in [22–25] (see also [26,27]). Moreover, one can find in [28–30] some new results for systems with different-type restrictions in the access to the service facility. In particular, in [28] a model with active queue management is studied.

The remaining part of the paper is organized as follows. The detailed mathematical description of the system is given in the next Section 2. The system of difference equations for the transient queue-size distribution conditioned by the initial buffer state is built in Section 3. In Section 4 we establish the corresponding system for double PGFs and write it in the specific form. Section 5 contains the main results: the solution of the last system in a compact form. Moreover, the appropriate result is obtained there for the equilibrium of

the system.Section 6 presents numerical study illustrating theoretical results and in the last Section 7 one can find a short conclusion.

2. Model Description

We deal with a discrete-time queueing model with the input stream of jobs governed by the binomial process (see e.g., [31,32]), i.e., successive interarrival times $\tau_1, \tau_2, \ldots$ are independent and geometrically distributed random variables as follows:

$$\mathbf{P}\{\tau_j = k\} = a_k = a(1-a)^{k-1}, \tag{1}$$

where $j, k \in \mathbb{N}$ and $a \in (0, 1)$ is a fixed parameter.

The processing of jobs is organized according to the natural FIFO service discipline with the general-type probability mass function (PMF for short), where b_k denotes the probability that the service time equals k time slots, $\sum_{k=1}^{\infty} b_k = 1$. An incoming job that finds the service station busy with processing is accumulated in the buffer of finite capacity $(B-1)$ places. Hence, the maximal system size (the number of jobs allowed to be present in the system simultaneously) equals B. Obviously, if the buffer is saturated on arrival, the arriving job is lost, i.e., it leaves the system without processing. If the arrival and departure occur at the same time slot, we assume that the arrival precedes the departure, so we accept the so-called Arrival First (AF) scheme.

Every time when the system becomes empty (the buffer does not contain any waiting job and the service station is free), a vacation period begins that can be treated as a kind of a power saving mechanism. More precisely, the service station goes for a vacation of random duration with a general-type PMF, whereby v_j we denote the probability that the vacation duration equals j time slots, $\sum_{j=1}^{\infty} v_j = 1$. During the vacation the processing is blocked completely. After finishing the vacation, if there is at least one job waiting in the accumulating buffer, the processing restarts immediately. Otherwise, the service station waits for the nearest arrival (being ready for service).

Because of the fact that interarrival times are geometrically distributed the probability $p(i, j)$ that during $i \geq 1$ time slots exactly $0 \leq j \leq i$ jobs will arrive can be easily expressed. Since the sum of independent random variables with the same geometric distribution has Pascal (negative binomial) distribution, we obtain

$$p(i, j) = a^j (1-a)^{i-j} \sum_{l=j}^{i} \binom{l-1}{j-1}. \tag{2}$$

where $i \geq 1$ and $0 \leq j \leq i$.

In the paper we use in some formulae the characteristic function (indicator), so let us put

$$\mathbf{1}_A(r) \overset{def}{=} \begin{cases} 1, & \text{if } r \in A, \\ 0, & \text{if } t \notin A, \end{cases} \tag{3}$$

where A is a certain set.

3. Transient Equations for the Queue-Size Distribution

Let us denote by X_r the number of jobs present in the system at the time (slot) $r \geq 0$, including the one being processed at this time, if any. In this section, we establish transient equations for the conditional queue-size distribution, where the condition is the state of the accumulating buffer (the number of jobs present) at the starting epoch $r = 0$.

Introduce the following notation:

$$Q_n(r, m) \overset{def}{=} \mathbf{P}\{X_r = m \mid X_0 = n\}, \tag{4}$$

where $m, n \in \{0, \ldots, B\}$ and $r \in \mathbb{N}$.

Suppose, firstly, that the buffer accumulating the incoming jobs is empty at the starting moment $r = 0$. In consequence, the vacation begins at this time. Let us fix the moment $r \in \mathbb{N}$. In relation to the first arrival, the following four different mutually excluding random events may occur:

- Event no. 1 ($E_1(r)$): the first job arrives before or at the moment r but after the completion of the vacation;
- Event no. 2 ($E_2(r)$): the vacation finishes before or at the moment r and during the vacation (maybe at the last time slot of the vacation duration) the first job enters the system;
- Event no. 3 ($E_3(r)$): the first job arrives before or at the moment r but the vacation finishes after r;
- Event no. 4 ($E_4(r)$): the first arrival occurs after time r.

We are interested in the probabilities of random events

$$(X_r = m) \cap E_i(r)$$

for $i = 1, 2, 3, 4$, conditioned by $X_0 = 0$.

Considering the random event $E_1(r)$, we obtain

$$\mathbf{P}\{(X_r = m) \cap E_1(r) \mid X_0 = 0\} = \sum_{i=1}^{r-1} a_i \sum_{j=1}^{i-1} v_j Q_1(r-i, m) + \mathbf{1}_{\{1\}}(m) a_r \sum_{j=1}^{r-1} v_j. \quad (5)$$

Indeed, the first summand on the right side of (5) relates to the situation in that the first arrival occurs before time r; in the second one the first job enters the system exactly at time r, so $m = 1$ with probability one.

In the case of $E_2(r)$ we get similarly

$$\mathbf{P}\{(X_r = m) \cap E_2(r) \mid X_0 = 0\} = \sum_{i=1}^{r-1} a_i \sum_{j=i+1}^{r-1} v_j \left[\sum_{k=0}^{B-2} p(j-i, k) Q_{k+1}(r-j, m) \right.$$

$$\left. + Q_B(r-j, m) \sum_{k=B-1}^{\infty} p(j-i, k) \right] + v_r \sum_{i=1}^{r-1} a_i \left[\mathbf{1}_{\{1,\dots,B-1\}}(m) p(r-i, m-1) \right. \quad (6)$$

$$\left. + \mathbf{1}_{\{B\}}(m) \sum_{k=B-1}^{\infty} p(r-i, k) \right] + \sum_{i=1}^{r-1} a_i v_i Q_1(r-i, m) + \mathbf{1}_{\{1\}}(m) a_r v_r.$$

The first summand on the right side of (6) refers to the situation in that the first arriving job enters the system before time r and the vacation finishes after this arrival but still before r. The second summand describes the case of the vacation completing at time r exactly and the arrival occurring earlier. The two last summands on the right side of (6) are related to the vacation completion time coinciding with the first arrival time: before and exactly at r, respectively.

For the random event $E_3(r)$ we obtain the following representation:

$$\mathbf{P}\{(X_r = m) \cap E_3(r) \mid X_0 = 0\} = \sum_{j=r+1}^{\infty} v_j \left[\sum_{i=1}^{r-1} a_i \left(\mathbf{1}_{\{1,\dots,B-1\}}(m) p(r-i, m-1) \right. \right.$$

$$\left. \left. + \mathbf{1}_{\{B\}}(m) \sum_{k=B-1}^{\infty} p(r-i, k) \right) + \mathbf{1}_{\{1\}}(m) a_r \right]. \quad (7)$$

Finally, considering $E_4(r)$, we have

$$\mathbf{P}\{(X_r = m) \cap E_4(r) \mid X_0 = 0\} = \mathbf{1}_{\{0\}}(m) \sum_{i=r+1}^{\infty} a_i. \quad (8)$$

The formula of total probability gives

$$Q_0(r,m) = \sum_{i=1}^{4} \mathbf{P}\{(X_r = m) \cap E_i(r) \mid X_0 = 0\}. \tag{9}$$

Investigate now the case of the system starting the operation with at least one job accumulated in the buffer. Due to the memoryless property of the geometric distribution of interarrival times, successive departure epochs are renewal moments in the evolution of the system. Applying the formula of total probability with respect to the first departure moment after the opening of the system ($r = 0$), we obtain

$$
\begin{aligned}
Q_n(r,m) = \ & \sum_{i=1}^{r-1} b_i \left[\sum_{k=0}^{B-n-1} p(i,k) Q_{n+k-1}(r-i,m) + Q_{B-1}(r-i,m) \sum_{k=B-n}^{\infty} p(i,k) \right] \\
& + b_r \mathbf{1}_{\{n-1,\ldots,B-1\}}(m) p(r,m-n) + \sum_{i=r+1}^{\infty} b_i \left(\mathbf{1}_{\{n,\ldots,B-1\}}(m) p(r,m-n) \right. \\
& \left. + \mathbf{1}_{\{B\}}(m) \sum_{k=B-n}^{\infty} p(r,k) \right),
\end{aligned}
\tag{10}
$$

where $n \in \{1,\ldots,B\}$. The interpretation of the right side of (10) is similar to those stated for (5) and (6).

In the next section, we collect the Formulae (5)–(8) and (10) and obtain the system of linear equations for double PGFs of conditional probabilities $Q_n(r,m)$.

4. System of Equations for PGFs

Let us define the double transform (PGF) of $Q_n(r,m)$ in the following way:

$$q_n(\theta,z) \overset{def}{=} \sum_{r=1}^{\infty} \theta^r \sum_{m=0}^{\infty} z^m Q_n(r,m), \tag{11}$$

where $|\theta| < 1$ and $|z| < 1$.

Moreover, introduce PGFs of sequences (a_k) and (v_k) as follows:

$$A(z) \overset{def}{=} \sum_{k=1}^{\infty} a_k z^k = \frac{az}{1 - z(1-a)}, \tag{12}$$

and

$$V(z) \overset{def}{=} \sum_{k=1}^{\infty} v_k z^k, \tag{13}$$

where $|z| < 1$.

To establish the system for PGFs in a compact form, we need some auxiliary calculations.

Let us start with the right side of (5). Double transform of the first summand (on arguments m and r) gives

$$
\begin{aligned}
& \sum_{m=0}^{\infty} z^m \sum_{r=2}^{\infty} \theta^r \sum_{i=1}^{r-1} a_i \sum_{j=1}^{i-1} v_j Q_1(r-i,m) \\
& = \sum_{m=0}^{\infty} z^m \sum_{i=1}^{\infty} a_i \theta^i \sum_{j=1}^{i-1} v_j \sum_{r-i=1}^{\infty} \theta^{r-i} Q_1(r-i,m) = \frac{a\theta}{1-\theta(1-a)} V(\theta(1-a)) q_1(\theta,z).
\end{aligned}
\tag{14}
$$

Similarly, taking the double generating function of the first summand on the right side of (6), we obtain

$$
\begin{aligned}
&\sum_{m=0}^{\infty} z^m \sum_{r=2}^{\infty} \theta^r \sum_{i=1}^{r-1} a_i \sum_{j=i+1}^{r-1} v_j \sum_{k=0}^{B-2} p(j-i,k) Q_{k+1}(r-j,m) \\
&= \sum_{m=0}^{\infty} z^m \sum_{k=0}^{B-2} \sum_{i=1}^{\infty} a_i \sum_{j=i+2}^{\infty} p(j-i,k) v_j \theta^j \sum_{r-j=1}^{\infty} \theta^{r-j} Q_{k+1}(r-j,m) \\
&= \sum_{k=0}^{B-2} q_{k+1}(\theta,z) \alpha_k(\theta),
\end{aligned}
\tag{15}
$$

where

$$
\alpha_k(\theta) \stackrel{def}{=} \sum_{i=1}^{\infty} a_i \sum_{j=i+2}^{\infty} v_j \theta^j p(j-i,k).
\tag{16}
$$

According to the second summand on the right side of (6), let us put

$$
\beta(\theta,z) \stackrel{def}{=} \sum_{m=1}^{B-1} z^m \sum_{r=1}^{\infty} v_r \theta^r \sum_{i=1}^{r-1} a_i p(r-i,m-1) + z^B \sum_{r=1}^{\infty} v_r \theta^r \sum_{i=1}^{r-1} a_i \sum_{k=B-1}^{\infty} p(r-i,k).
\tag{17}
$$

The third summand of (6), after transformation, leads to

$$
\begin{aligned}
&\sum_{m=0}^{\infty} z^m \sum_{r=1}^{\infty} \theta^r \sum_{i=1}^{r-i} a_i v_i Q_1(r-i,m) \\
&= \sum_{i=1}^{\infty} a_i \theta^i v_i \sum_{m=0}^{\infty} z^m \sum_{r=i+1}^{\infty} \theta^{r-i} Q_1(r-i,m) = \frac{a}{1-a} V(\theta(1-a)) q_1(\theta,z).
\end{aligned}
\tag{18}
$$

Next, let us define (compare the right side of (7))

$$
\gamma(\theta,z) \stackrel{def}{=} \sum_{r=1}^{\infty} \theta^r \sum_{j=r+1}^{\infty} v_j \left[\sum_{i=1}^{r-1} a_i \left(\sum_{m=1}^{B-1} z^m p(r-i,m-1) + z^B \sum_{k=B-1}^{\infty} p(r-i,k) \right) \right].
\tag{19}
$$

Summing up and transforming the last components on the right sides of (5)–(7), we get

$$
z \sum_{r=1}^{\infty} \theta^r a_r = z A(\theta) = \frac{az\theta}{1-\theta(1-a)}.
\tag{20}
$$

Lastly, double transform of the right side of (8) gives

$$
\sum_{r=1}^{\infty} \theta^r \sum_{i=r+1}^{\infty} a_i = \frac{(1-a)\theta}{1-(1-a)\theta}.
\tag{21}
$$

At this stage, referring to the representations (14)–(21), after simplification, we obtain from (9)

$$
\begin{aligned}
q_0(\theta,z) = \quad &\sum_{k=0}^{B-2} \alpha_k(\theta) q_{k+1}(\theta,z) + q_B(\theta,z) \sum_{k=B-1}^{\infty} \alpha_k(\theta) \\
&+ q_1(\theta,z) V(\theta(1-a)) \frac{a}{(1-\theta(1-a))(1-a)} \\
&+ \beta(\theta,z) + \gamma(\theta,z) + \frac{\theta(1-a+az)}{1-\theta(1-a)}.
\end{aligned}
\tag{22}
$$

Putting

$$
g(\theta) \stackrel{def}{=} \frac{aV(\theta(1-a))}{(1-\theta(1-a))(1-a)}
\tag{23}
$$

and

$$h(\theta, z) \overset{def}{=} \beta(\theta, z) + \gamma(\theta, z) + \frac{\theta(1 - a + az)}{1 - \theta(1 - a)}, \tag{24}$$

we get, finally,

$$q_0(\theta, z) = \sum_{k=0}^{B-2} \alpha_k(\theta) q_{k+1}(\theta, z) + q_B(\theta, z) \sum_{k=B-1}^{\infty} \alpha_k(\theta) + q_1(\theta, z) g(\theta) + h(\theta, z). \tag{25}$$

Let us now analyze Equation (10). Observe that

$$
\begin{aligned}
&\sum_{m=0}^{\infty} z^m \sum_{r=1}^{\infty} \theta^r \sum_{i=1}^{r-1} b_i \sum_{k=0}^{B-n-1} p(i, k) Q_{n+k-1}(r - i, m) \\
&= \sum_{k=0}^{B-n-1} \sum_{i=1}^{\infty} b_i \theta^i p(i, k) \sum_{m=0}^{\infty} z^m \sum_{r=i+1}^{\infty} \theta^{r-i} Q_{n+k-1}(r - i, m) \\
&= \sum_{k=0}^{B-n-1} \tau_k(\theta) q_{n+k-1}(\theta, z),
\end{aligned} \tag{26}
$$

where

$$\tau_k(\theta) \overset{def}{=} \sum_{i=1}^{\infty} b_i \theta^i p(i, k), \quad k \in \{0, \dots, i\}. \tag{27}$$

Moreover, denoting

$$\eta_n(\theta, z) \overset{def}{=} \sum_{r=1}^{\infty} \theta^r \left(\sum_{i=r}^{\infty} b_i \sum_{m=n}^{B-1} z^m p(r, m - n) + z^B \sum_{i=r+1}^{\infty} b_i \sum_{k=B-n}^{\infty} p(r, k) \right), \tag{28}$$

we obtain from (10)

$$q_n(\theta, z) = \sum_{k=0}^{B-n-1} \tau_k(\theta) q_{n+k-1}(\theta, z) + q_{B-1}(\theta, z) \sum_{k=B-n}^{\infty} \tau_k(\theta) + \eta_n(\theta, z), \tag{29}$$

where $n \in \{1, \dots, B\}$.

Let us transform Equations (25) and (29) applying the following interchanging:

$$x_n(\theta, z) \overset{def}{=} q_{B-n}(\theta, z) \quad n \in \{0, \dots, B\}. \tag{30}$$

We obtain from (25)

$$x_B(\theta, z) = \sum_{k=1}^{B-1} \alpha_{B-k-1}(\theta) x_k(\theta, z) + x_1(\theta, z) g(\theta) + x_0(\theta, z) \sum_{k=B-1}^{\infty} \alpha_k(\theta) + h(\theta, z). \tag{31}$$

Similarly, from (29) we get

$$\sum_{k=-1}^{n-1} \tau_{k+1}(\theta) x_{n-k}(\theta, z) - x_n(\theta, z) = \phi_n(\theta, z), \tag{32}$$

where $n \in \{0, \dots, B - 1\}$, and

$$\phi_n(\theta, z) \overset{def}{=} -x_1(\theta, z) \sum_{k=n+1}^{\infty} \tau_k(\theta) - \eta_{B-n}(\theta, z). \tag{33}$$

5. Representation for Solution

In this section, we obtain the solution of the system (31)–(32) in a closed form. It is proved in [33] that each solution of the following system of infinitely many linear equations:

$$\sum_{k=-1}^{n-1} \tau_{k+1} x_{n-k} - x_n = \phi_n, \quad n \geq 1, \tag{34}$$

where (τ_k) and (ϕ_k) are known sequences of coefficients and free terms, respectively, and (x_k) is the sequence of unknowns, can be represented in the form

$$x_n = LR_n + \sum_{k=1}^{n} R_{n-k}\phi_k, \quad n \geq 1, \tag{35}$$

where L is a certain constant and the sequence (R_k) can be defined recursively by successive terms of the sequence of coefficients (τ_k) in the following way:

$$R_0 = 0, \; R_1 = (\tau_0)^{-1}, \\ R_k = R_1 \left(R_{k-1} - \sum_{j=0}^{k-1} \tau_{j+1} R_{k-1-i} \right), \tag{36}$$

where $k \geq 2$. Alternatively, the sequence (R_k) can be obtained by using generating functions. Indeed, defining

$$T(z) \stackrel{def}{=} \sum_{k=1}^{\infty} \tau_k z^k, \quad |z| < 1, \tag{37}$$

it can be easily shown that

$$\sum_{k=0}^{\infty} R_k z^k = \frac{z}{T(z) - z}. \tag{38}$$

In consequence, expanding the right side of (38) in the Maclaurin series and comparing the coefficients at successive powers of z, we have

$$R_k = \frac{1}{k!} \frac{d^k}{dz^k} \left(\frac{z}{T(z) - z} \right) \Big|_{z=0}. \tag{39}$$

Comparing systems of Equations (32) and (34) we can observe the following two essential differences. The first one is the dependence of unknowns, coefficients, and free terms on arguments θ and z in (32). The next is the number of equations that is finite in (30) (where $n \in \{0, \ldots, B-1\}$) and infinite in (34) (where $n \geq 1$). Moreover, we have an additional Equation (31) written for $n = B$.

Referring to (35), we can write

$$x_n(\theta, z) = L(\theta, z)R_n(\theta) + \sum_{k=1}^{n} R_{n-k}(\theta)\phi_k(\theta), \tag{40}$$

where $n \geq 1$, and (compare (36))

$$R_0(\theta) = 0, \; R_1(\theta) = (\tau_0(\theta))^{-1}, \\ R_k(\theta) = R_1(\theta) \left(R_{k-1}(\theta) - \sum_{j=0}^{k-1} \tau_{j+1}(\theta) R_{k-1-i}(\theta) \right) \tag{41}$$

for $k \geq 2$.

Note that to obtain the explicit-form solution of the system (31)–(32) we need the representation not only for $L(\theta, z)$ but also for $x_0(\theta, z)$ (since the Formula (40) is valid

for $n \geq 1$) and for $x_1(\theta, z)$ (occurring in the definition (33) of $\phi_n(\theta, z)$). As it turns out, the Equation (31), treated as a kind of a boundary condition, will play a special role here.

Substituting $n = 0$ into (32) and applying (33), we get

$$\tau_0(\theta)x_1(\theta, z) - x_0(\theta, z) = -x_1(\theta, z) \sum_{k=1}^{\infty} \tau_k(\theta) - \eta_B(\theta, z). \tag{42}$$

Hence we obtain the formula linking $x_0(\theta, z)$ with $x_1(\theta, z)$, namely

$$x_0(\theta, z) = x_1(\theta, z) \sum_{k=0}^{\infty} \tau_k(\theta, z) + \eta_B(\theta, z). \tag{43}$$

Simultaneously, from the other side, $x_1(\theta, z)$ can be expressed using (40) written for $n = 1$. Indeed, we have

$$x_1(\theta, z) = L(\theta, z)R_1(\theta). \tag{44}$$

Substituting (44) into (43), we obtain

$$x_0(\theta, z) = L(\theta, z)R_1(\theta) \sum_{k=0}^{\infty} \tau_k(\theta) + \eta_B(\theta, z). \tag{45}$$

So, having the Formulae (44)–(45), the remaining task is to find the representation for $L(\theta, z)$.

Using (40), (44) and (45) on the right side of (31), we obtain

$$\begin{aligned}
x_B(\theta, z) = \quad &\sum_{k=1}^{B-1} \alpha_{B-k-1}(\theta) \Big[L(\theta, z)R_k(\theta) + \sum_{i=1}^{k} R_{k-i}(\theta) \\
&\times \Big(-L(\theta, z)R_1(\theta) \sum_{j=i+1}^{\infty} \tau_j(\theta) - \eta_{B-i}(\theta, z) \Big) \Big] \\
&+ L(\theta, z)R_1(\theta)g(\theta) + \Big(L(\theta, z)R_1(\theta) \sum_{i=0}^{\infty} \tau_i(\theta) \\
&+ \eta_B(\theta, z) \Big) \sum_{k=B-1}^{\infty} \alpha_k(\theta) + h(\theta, z).
\end{aligned} \tag{46}$$

Similarly, substituting $n = B$ into (40) and referring to (33), we have

$$\begin{aligned}
x_B(\theta, z) = \quad &L(\theta, z)R_B(\theta) + \sum_{k=1}^{B} R_{B-k}(\theta) \\
&\times \Big(-L(\theta, z)R_1(\theta) \sum_{i=k+1}^{\infty} \tau_i(\theta) - \eta_{B-k}(\theta, z) \Big).
\end{aligned} \tag{47}$$

Comparing the right sides of (46) and (47), we get the following formula for $L(\theta, z)$:

$$L(\theta, z) = \frac{N(\theta, z)}{D(\theta)}, \tag{48}$$

where

$$N(\theta, z) \stackrel{def}{=} \sum_{k=1}^{B} R_{B-k}(\theta)\eta_{B-k}(\theta, z) - \sum_{k=1}^{B-1} \alpha_{B-k-1}(\theta) \sum_{i=1}^{k} R_{k-i}(\theta)\eta_{B-i}(\theta, z) \\ + \eta_B(\theta, z) \sum_{k=B-1}^{\infty} \alpha_k(\theta) + h(\theta, z) \tag{49}$$

and

$$\begin{aligned}
D(\theta) \stackrel{def}{=} \quad &R_B(\theta) - R_1(\theta) \sum_{k=1}^{B} R_{B-k}(\theta) \sum_{i=k+1}^{\infty} \tau_i(\theta) \\
&- \sum_{k=1}^{B-1} \alpha_{B-k-1}(\theta) \Big(R_k(\theta) - R_1(\theta) \sum_{i=1}^{k} R_{k-i}(\theta) \sum_{j=i+1}^{\infty} \tau_j(\theta) \Big) \\
&- R_1(\theta)g(\theta) - R_1(\theta) \sum_{i=0}^{\infty} \tau_i(\theta) \sum_{k=B-1}^{\infty} \alpha_k(\theta).
\end{aligned} \tag{50}$$

Finally, referring to Formulae (30), (33), (40), (44), (45) and (48), we can state the following main result:

Theorem 1. *The double PGF $q_n(\cdot, \cdot)$ of the queue-size conditional distribution in the considered discrete-time Geo/G/1/B-type queueing model with single vacation policy can be represented in the following way:*

$$q_n(\theta, z) = \frac{N(\theta, z)}{D(\theta)} R_{B-n}(\theta) - \sum_{k=1}^{B-n} R_{B-n-k}(\theta)$$
$$\times \left(\frac{N(\theta, z)}{D(\theta)} R_1(\theta) \sum_{i=k+1}^{\infty} \tau_i(\theta) + \eta_{B-k}(\theta, z) \right), \tag{51}$$

where $n \in \{0, \dots, B-1\}$, and

$$q_B(\theta, z) = \frac{N(\theta, z)}{D(\theta)} R_1(\theta) \sum_{k=0}^{\infty} \tau_k(\theta) + \eta_B(\theta, z). \tag{52}$$

The formulae for $\tau_k(\theta)$, $\eta_k(\theta, z)$, $R_k(\theta)$, $N(\theta, z)$ and $D(\theta)$ are given in (27), (28), (41), (49) and (50), respectively.

It is possible to obtain from Theorem 1 a useful corollary. Let us put

$$P_X(z) \stackrel{def}{=} \sum_{m=0}^{\infty} z^m \mathbf{P}\{X = m\}, \tag{53}$$

where $|z| < 1$ and X stands for the stationary number of jobs present in the system. Obviously, the stationary distribution is independent of the initial buffer state. The well-known Abelian theorem leads to the following result:

Corollary 1. *The PGF $P_X(\cdot)$ of the number of jobs X present in the system in the equilibrium (the stationary state) can be computed from the formula*

$$P_X(z) = \lim_{\theta \uparrow 1} (1 - \theta) \sum_{r=1}^{\infty} \theta^r \sum_{m=0}^{\infty} z^m \mathbf{P}\{X_r = m \mid X_0 = B\}$$
$$= \lim_{\theta \uparrow 1} (1 - \theta) q_B(\theta, z)$$
$$= \lim_{\theta \uparrow 1} (1 - \theta) \left(\frac{N(\theta, z)}{D(\theta)} R_1(\theta) \sum_{k=0}^{\infty} \tau_k(\theta) + \eta_B(\theta, z) \right), \tag{54}$$

where $|z| < 1$.

Obviously, we also have

$$\mathbf{E}(X) = P_X'(1). \tag{55}$$

6. Numerical Study

In this section, we illustrate theoretical results obtained in the previous section via numerical examples. Let us study the model in which the parameter $a \in (0, 1)$ of the geometric distribution of interarrival times takes on values from the set

$$0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.$$

Investigate three different types of the service (processing) time distribution, namely

- deterministic: $b_2 = 1$, $b_k = 0$, $k = 1, k \geq 3$;
- finite discrete: $b_1 = 0.25$, $b_2 = 0.50$, $b_3 = 0.25$, $b_k = 0$, $k \geq 4$;
- geometric : $b_k = (0.5)^k$, $k \geq 1$.

Let us note that for all these distributions the mean value equals 2 while variances, however, are different: 0 for the deterministic case, 4.5 for the finite discrete one, and 2 for the geometric distribution.

Similarly, let us consider three different probability distributions of the single vacation duration, namely

- deterministic: $v_2 = 1$, $v_k = 0$, $k = 1$, $k \geq 3$;
- finite discrete: $v_1 = 0.25$, $v_2 = 0.50$, $v_3 = 0.25$, $v_k = 0$, $k \geq 4$;
- geometric : $v_k = (0.5)^k$, $k \geq 1$.

As in the case of the service time distributions, for all distributions the mean value is the same and equal to 2. The variance equals 0, 4.5, and 2 for deterministic, finite discrete, and geometric cases, respectively.

In Table 1 and Figure 1, the values of the mean stationary queue size are presented for $B = 3$ in a function of the parameter a for deterministic service time distribution and three different probability distributions of the single vacation duration. Let us observe that, obviously, mean queue sizes increase with increasing a (or, equivalently, with decreasing mean interarrival times). Note that the impact of the service type distribution (keeping the same means) and its variance seems to be slight.

Table 1. Mean queue size for deterministic service time distribution.

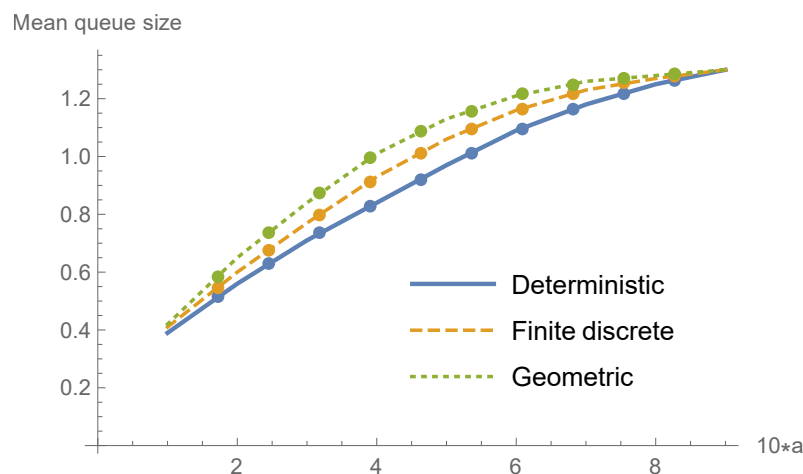| Parameter a	Deterministic SV	Finite Discrete SV	Geometric SV
0.1	0.3921	0.4052	0.4197
0.2	0.5635	0.6037	0.6464
0.3	0.7051	0.7742	0.8438
0.4	0.8371	0.9253	1.0077
0.5	0.9667	1.0560	1.1318
0.6	1.0866	1.1583	1.2126
0.7	1.1835	1.2277	1.2575
0.8	1.2515	1.2711	1.2829
0.9	1.2974	1.3021	1.3047

Figure 1. Mean queue size for deterministic service time distribution.

A similar conclusion can be formulated in the case of finite discrete service time distribution for which appropriate results and presented and visualized in Table 2 and in Figure 2, and for geometric service time distribution (see Table 3 and Figure 3).

Table 2. Mean queue size for finite discrete service time distribution.

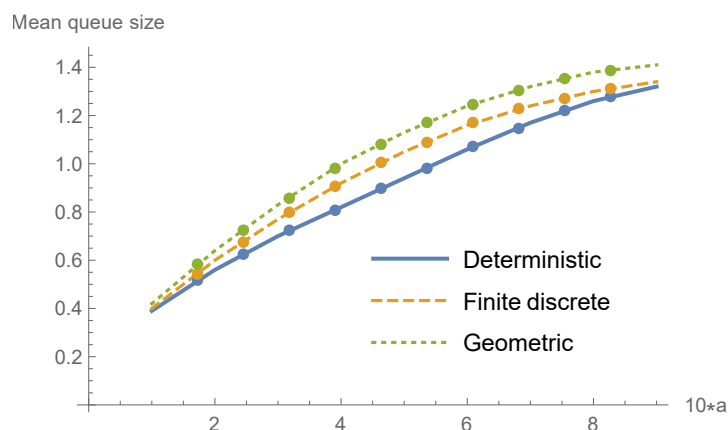| Parameter a	Deterministic SV	Finite Discrete SV	Geometric SV
0.1	0.3902	0.4033	0.4179
0.2	0.5609	0.6015	0.6448
0.3	0.6988	0.7698	0.8417
0.4	0.8236	0.9174	1.0061
0.5	0.9448	1.0473	1.1356
0.6	1.0619	1.1563	1.2295
0.7	1.1679	1.2408	1.2915
0.8	1.2558	1.3014	1.3297
0.9	1.3231	1.3431	1.3542

Figure 2. Mean queue size for finite discrete service time distribution.

Table 3. Mean queue size for geometric service time distribution.

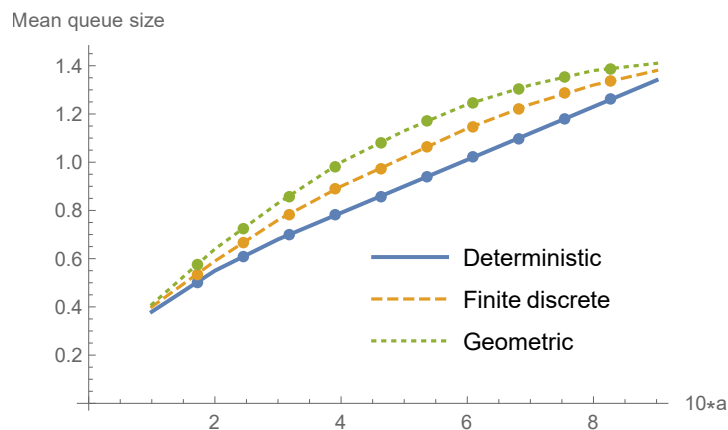| Parameter a	Deterministic SV	Finite Discrete SV	Geometric SV
0.1	0.3830	0.3961	0.4108
0.2	0.5509	0.5923	0.6366
0.3	0.6810	0.7555	0.8318
0.4	0.7931	0.8965	0.9963
0.5	0.9001	1.0229	1.1324
0.6	1.0088	1.1371	1.2414
0.7	1.1206	1.2381	1.3242
0.8	1.2225	1.3222	1.3810
0.9	1.3365	1.3845	1.4124

Figure 3. Mean queue size for geometric service time distribution.

7. Conclusions

In this article, the queue-size distribution is studied in a finite-buffer discrete-time queueing model in which a single vacation policy is implemented. Identifying renewal moments in the evolution of the system and applying the formula of total probability and linear algebra, the representation for the double probability generating function of the transient queue-size distribution is found in a compact form. From the formula, it is possible to obtain the mean stationary queue size just by using the Abelian theorem and differentiation. A numerical study illustrating analytical results is attached. The considered queueing system has potential applications in modeling energy-saving modes or maintenance periods in wireless communication networks, production lines, etc.

Conflicts of Interest: The authors declare no conflict of interest.

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